Mutually Disjoint Designs and New 5-Designs Derived from Groups and Codes

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Abstract

The paper gives constructions of disjoint 5-designs obtained from permutation groups and extremal self-dual codes. Several new simple 5-designs are found with parameters that were left open in the table of 5-designs given in [10], namely, 5-(v,k,λ) designs with $(v,k,\lambda) = (18,8,2m)$ $(m=6,9)$, $(19,9,7m)$ $(m=6,9)$, $(24,9,6m)$ $(m=3,4,5)$, $(25,9,30)$, $(25,10,24m)$ $(m=4,5)$, $(26,10,126)$, $(30,12,440)$, $(32,6,3m)$ $(m=2,3,4)$, $(33,7,84)$, and $(36,12,45n)$ for $2 \le n \le 17$. These results imply that a simple 5-(v,k,λ) design with $(v,k) = (24,9)$, $(25,9)$, $(26,10)$, $(32,6)$, or $(33,7)$ exists for all admissible values of λ.

1 Introduction

A t-(v,k,λ) *design* D is a pair $(X,\mathcal{B})$ where X is a set of v *points* and a collection $\mathcal{B}$ of k-subsets of X called *blocks* such that every t-subset of X is

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contained in exactly λ blocks (see [3] and [6]). A design with no repeated block is called *simple*. All designs in this paper are simple. A *Steiner system* $S(t,k,v)$ is a t-(v,k,λ) design with $\lambda = 1$. An automorphism of a t-design D is any permutation of the points that preserves the collection of blocks, and the *automorphism group* $\mathrm{Aut}(D)$ of D is the group consisting of all automorphisms of D.

The main goal of this paper is to determine the value of λ for which there exists a 5-(v,k,λ) design with

$$(v,k) = (24,9), (25,9), (26,10), (32,6) \text{ and } (33,7). \tag{1}$$

Given a v-set X, we denote by $\binom{X}{k}$ the collection of all k-subsets of X. Since $D = (X, \binom{X}{k})$ is a t-$(v,k,\binom{v-t}{k-t})$ design, we have $\lambda \le \binom{v-t}{k-t}$ for any simple t-(v,k,λ) design. If $D = (X,\mathcal{B})$ is a simple t-(v,k,λ) design then $(X, \binom{X}{k} \setminus \mathcal{B})$ is a simple t-$(v,k,\binom{v-t}{k-t} - \lambda)$ design. Hence, it is sufficient to determine the existence of t-(v,k,λ) designs with $\lambda \le \lfloor \binom{v-t}{k-t}/2 \rfloor$. If a t-(v,k,λ) design exists then $\lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$ is an integer for every integer s in the range $0 \le s \le t$. This often implies that λ is divisible by some positive integer u. For the values (v,k) listed in (1), we have $\lambda = 6m, 15m, 63m, 3m$ and $43m$, respectively, where m is a positive integer. The largest m satisfying these conditions for a given (v,k) from (1) is listed in the last column of Table 1, and the values of m for which a 5-(v,k,mu) design is known to exist are listed in the third column of Table 1 (see Table 4.46 in [10]).

Table 1: Parameters of some 5-designs

(v,k,λ)	New	Known (Table 4.46 in [10])	Max
$(24,9,6m)$	$3,4,5$	$1,2,6,\ldots,323$	323
$(25,9,15m)$	2	$1,3,4,\ldots,161$	161
$(26,10,63m)$	2	$1,3,4,\ldots,161$	161
$(32,6,3m)$	$2,3,4$	1	4
$(33,7,42m)$	2	$1,3,4$	4

Kramer and Mesner [13] introduced a method for the construction of t-designs with prescribed automorphism group. Using this approach, a software package DISCRETA [4] was developed that has already led to the discovery of several new t-designs with large t. Two t-(v,k,λ) designs with the same point set are said to be *disjoint* if they have no blocks in common. By finding permutations such that all images of the block set of a t-design under these permutations are mutually disjoint, mutually disjoint t-designs can be constructed. Moreover, the union of s mutually disjoint t-(v,k,λ) designs gives a simple t-$(v,k,s\lambda)$ design. In Sections 3 and 4, we construct 5-(v,k,λ) designs with parameters $(v,k,\lambda) = (25,9,30)$, $(26,10,126)$, $(32,6,3m)$ $(m = 2,3,4)$ and $(33,7,84)$ where the values of m are listed in the second column of Table 1. These 5-designs are constructed by considering mutually disjoint designs of known 5-designs and using DISCRETA, along with Proposition 2.1.

It is well known that a Steiner system $S(5,8,24)$ and a 5-$(24,12,48)$ design can be constructed by taking as blocks the supports of codewords of weights 8 and 12 in the binary extended Golay $[24,12,8]$ code. Similarly, a number of other 5-designs are constructed from self-dual codes (see [16, Table 1.61]). Mutually disjoint Steiner systems $S(5,8,24)$ and 5-$(24,12,48)$ designs were constructed [1], [2], [9], [12]. In Sections 5 and 6, we investigate mutually disjoint designs for 5-designs which are obtained from extremal self-dual codes. In particular, 5-$(24,9,6m)$ designs with $m = 3,4,5$ are constructed from mutually disjoint 5-$(24,9,6)$ designs. Thus, we determine all values of λ for which there exists a 5-(v,k,λ) design with (v,k) as in (1). These values are listed in Table 1. Some other new 5-designs with parameters that were not known to exist before are also constructed in Section 6, namely, 5-(v,k,λ) designs with $(v,k,\lambda) = (18,8,2m)$ $(m = 6,9)$, $(19,9,7m)$ $(m = 6,9)$, $(25,10,24m)$ $(m = 4,5)$, $(30,12,440)$, and $(36,12,45n)$ for $2 \le n \le 17$.

2 Preliminaries

Throughout this paper, X_v will denote the set $\{1,2,\dots,v\}$. Let $D = (X_v,\mathcal{B})$ be a t-(v,k,λ) design. We denote $\{B^\sigma \mid B \in \mathcal{B}\}$ by $\mathcal{B}^\sigma$, where B^σ is the image of a block B under a permutation σ on X_v. Disjoint t-(v,k,λ) designs $(X_v,\mathcal{B})$ and $(X_v,\mathcal{B}^\sigma)$ are constructed by finding a permutation σ such that $\mathcal{B}$ and $\mathcal{B}^\sigma$ have no blocks in common. Let $(X_v,\mathcal{B}_1),(X_v,\mathcal{B}_2),\dots,(X_v,\mathcal{B}_m)$ be mutually disjoint t-(v,k,λ) designs. Then $(X_v,\cup_{i=1}^m \mathcal{B}_i)$ is a simple t-$(v,k,m\lambda)$ design. In this paper, we use this approach to find simple 5-designs with parameters that were not known to exist before.

Some of the 5-designs constructed in this paper are found using DISCRETA [4]. If G is a permutation group on X_v and $B_1,B_2,\dots,B_s$ are k-

subsets of X such that $(X_v, \mathcal{B})$ is a t-(v, k, λ) design D where $\mathcal{B} = \{B_i^\sigma \mid \sigma \in G, i = 1, 2, \ldots, s\}$, we call the blocks $B_1, B_2, \ldots, B_s$ *base blocks* of D (with respect to G). For description of designs constructed using this approach, we give a permutation group G along with base blocks.

We will need also the following auxiliary construction due to Tran van Trung [17].

Proposition 2.1 ([17]). *If there is a t-(v, k, λ_1) design $D_1 = (X_v, \mathcal{B}_1)$ and there is a t-$(v, k+1, \lambda_2)$ design $D_2 = (X_v, \mathcal{B}_2)$ such that $\lambda_1(v - t + 1)/(k - t + 1) = \lambda_1 + \lambda_2$, then*

$$D_{new} = (X_v \cup \{v + 1\}, \mathrm{app}_{v+1}(\mathcal{B}_1) \cup \mathcal{B}_2)$$

is a t-$(v+1, k+1, \lambda_1+\lambda_2)$ design, where $\mathrm{app}_{v+1}(\mathcal{B}_1) = \{B \cup \{v+1\} \mid B \in \mathcal{B}_1\}$.

3 5-(25, 9, 15m) designs and 5-(26, 10, 63m) designs

In order to construct a 5-$(25, 9, 15)$ design using Proposition 2.1, we first construct a 5-$(24, 8, 3)$ design and a 5-$(24, 9, 12)$ design. A Steiner system $S(5, 8, 24)$ $(X_{24}, \mathcal{E}_1)$ can be constructed from the group generated by the following permutations

$(1, 3, 14, 10, 8, 16, 6, 12, 5, 20, 9, 23, 4, 18, 7, 22, 15, 21, 11, 19, 17, 13, 24)$,
$(1, 2, 24)(3, 13, 23)(4, 17, 12)(5, 19, 16)(6, 11, 18)(7, 21, 10)(8, 15, 20)(9, 22, 14)$,

which is isomorphic to $\mathrm{PSL}(2, 23)$, along with a base block $\{1, 2, 3, 4, 5, 7, 16, 19\}$. Define permutations on X_{24}

$$\alpha_1 = (2, 5, 3, 19, 8, 4, 11, 12, 9, 15)(6, 16, 14, 18, 20, 10)(7, 17),$$
$$\alpha_2 = (2, 20, 17, 6, 19, 14, 12, 8, 7, 16, 5, 4, 11, 13, 9)(3, 10, 15, 18).$$

Then $\{(X_{24}, \mathcal{E}_1), (X_{24}, \mathcal{E}_1^{\alpha_1}), (X_{24}, \mathcal{E}_1^{\alpha_2})\}$ are three mutually disjoint Steiner systems $S(5, 8, 24)$ (see [12]). Hence $(X_{24}, \mathcal{E}_1 \cup \mathcal{E}_1^{\alpha_1} \cup \mathcal{E}_1^{\alpha_2})$ is a simple 5-$(24, 8, 3)$ design $E_{24,1}$. A 5-$(24, 9, 12)$ design $E_{24,2}$ can be constructed from the above group $\mathrm{PSL}(2, 23)$ along with base blocks

$$\{1, 2, 3, 4, 5, 6, 8, 10, 13\}, \{1, 2, 3, 4, 5, 6, 12, 16, 17\}.$$

By Proposition 2.1, a 5-$(25, 9, 15)$ design $(X_{25}, \mathcal{B}_{25})$ can be constructed from $E_{24,1}$ and $E_{24,2}$.

Moreover we have verified that $\{(X_{25}, \mathcal{B}_{25}), (X_{25}, \mathcal{B}_{25}^{\sigma_{25}})\}$ are two disjoint 5-$(25, 9, 15)$ designs, where

$$\sigma_{25} = (1, 13, 2, 24)(3, 22, 6, 23, 16, 10, 21, 17, 4, 8)(5, 15, 12, 14, 19, 18, 9, 7)(11, 20).$$

Hence $D_{25} = (X_{25}, \mathcal{B}_{25} \cup \mathcal{B}_{25}^{\sigma_{25}})$ is a simple 5-$(25, 9, 30)$ design, and we have the following.

Lemma 3.1. (i) *There are at least two disjoint* 5-$(25, 9, 15)$ *designs.*

(ii) *There is a simple* 5-$(25, 9, 30)$ *design.*

We have verified by MAGMA [5] that D_{25} has a trivial automorphism group.

From Table 1, it is sufficient to consider whether a 5-$(25, 9, 15m)$ design exists for $m \leq 161$, and it is known that there is a 5-$(25, 9, 15m)$ design for $m = 1, 3, 4, \ldots, 161$. Hence, using Lemma 3.1, we can determine all λ for which there exists a simple 5-$(25, 9, \lambda)$ design.

Theorem 3.2. *A simple* 5-$(25, 9, 15m)$ *design exists for all* $m \leq 161$.

Using DISCRETA [4], a 5-$(25, 10, 96)$ design D'_{25} can be constructed from the group generated by the following two permutations

$$(2, 4, 15, 11, 9, 17, 7, 13, 6, 21, 10, 24, 5, 19, 8, 23, 16, 22, 12, 20, 18, 14, 25),$$
$$(2, 3, 25)(4, 14, 24)(5, 18, 13)(6, 20, 17)(7, 12, 19)(8, 22, 11)(9, 16, 21)(10, 23, 15),$$

which is isomorphic to PSL$(2, 23)$, along with base blocks

$$\begin{array}{ll} \{1, 2, 3, 4, 5, 6, 7, 8, 15, 19\}, & \{1, 2, 3, 4, 5, 6, 7, 10, 19, 22\}, \\ \{2, 3, 4, 5, 6, 7, 8, 9, 17, 22\}, & \{2, 3, 4, 5, 6, 7, 8, 10, 11, 15\}, \\ \{2, 3, 4, 5, 6, 7, 11, 14, 15, 18\}. & \end{array}$$

By Proposition 2.1, a 5-$(26, 10, 126)$ design D_{26} is constructed from the 5-$(25, 9, 30)$ design D_{25} and the 5-$(25, 10, 96)$ design D'_{25}. Thus, we have the following.

Lemma 3.3. *There exists a simple* 5-$(26, 10, 126)$ *design.*

We have verified by MAGMA [5] that D_{26} has a trivial automorphism group.

From Table 1, it is sufficient to determine whether a 5-$(26, 10, 63m)$ design exists for $m \leq 161$. It is known that there exists a 5-$(26, 10, 63m)$ design for $m = 1, 3, 4, \ldots, 161$. Hence, by Lemma 3.3, we have the following result.

Theorem 3.4. *A simple* 5-$(26, 10, 63m)$ *design exists for all* $m \leq 161$*.*

Remark 3.5. Alternative proofs of Theorems 3.2 and 3.4 are given in Section 5.

4 5-(32, 6, 3m) designs and 5-(33, 7, 42m) designs

In this section, using DISCRETA [4], we construct 5-$(32, 6, 3m)$ designs for $m = 1, 2, 3, 4$ and a 5-$(33, 7, 84)$ design.

Let $G_{32,1}$ be the group generated by the following two permutations

$$\begin{aligned}&(1,10,23,28,13,18,31,4,9,22,27,16,17,30,3,12,\\&\qquad 21,26,15,20,29,2,11,24,25,14,19,32)(5,6,7,8),\\&(1,6,29,2,5,30)(3,7,31)(4,8,32)(9,18,25,10,17,26)\\&\qquad (11,19,27)(12,20,28)(13,14)(21,22).\end{aligned}$$

Let $G_{32,2}$ be the group generated by the following two permutations

$$\begin{aligned}&(1,3,18,23,10,27,28,11,6,9,30,19,15,14,22,31,\\&\qquad 4,13,21,20,16,5,26,29,24,7,8,25,12,17,32),\\&(1,2,32)(3,17,31)(4,12,16)(5,25,11)(6,8,24)(9,29,23)\\&\qquad (10,26,28)(13,20,15)(14,21,19)(18,22,30),\end{aligned}$$

and let G_{33} be the group generated by the following two permutations

$$\begin{aligned}&(1,3)(4,31)(5,21)(6,24)(7,10)(8,15)(9,17)(11,25)(12,19)\\&\qquad (13,26)(14,16)(18,27)(20,30)(22,29)(23,33)(28,32),\\&(1,20,29,7,18,23,5,13,33,10,16,15,25,27,31,\\&\qquad 3,2,30,26,24,14,17,11,32,12,4,22,19,6,28,21).\end{aligned}$$

Table 2: Groups and base blocks of $D_{32,m}$ $(m = 1, 2, 3, 4)$ and D_{33}

Designs D		Groups	Base blocks	Aut(D)
$D_{32,1}$	5-$(32, 6, 3)$ design	$G_{32,1}$	Table 3	$G_{32,1}$
$D_{32,2}$	5-$(32, 6, 6)$ design	$G_{32,2}$	Table 4	$G_{32,2}$
$D_{32,3}$	5-$(32, 6, 9)$ design	$G_{32,1}$	Table 5	$G_{32,1}$
$D_{32,4}$	5-$(32, 6, 12)$ design	$G_{32,2}$	Table 6	$G_{32,2}$
D_{33}	5-$(33, 7, 84)$ design	G_{33}	Table 7	G_{33}

The groups are isomorphic to $\mathrm{PSL}(2, 7) \times S_4$, $\mathrm{PSL}(2, 31)$ and $\mathrm{PSL}(2, 32)$, respectively.

We list in Table 2 the groups and base blocks for constructing 5-$(32, 6, 3m)$ designs $D_{32,m}$ $(m = 1, 2, 3, 4)$ and a 5-$(33, 7, 84)$ design D_{33}. The automorphism group Aut(D) for each design D is also given in the table where these automorphism groups are determined by MAGMA [5].

Table 4.46 in [10] claims the existence of a 5-$(32, 6, 3)$ design with automorphism group $\mathrm{PSL}(2, 31)$. This could not be verified with DISCRETA, but there exists a 5-$(32, 6, 3)$ design having $\mathrm{PSL}(2, 7) \times S_4$ as automorphism group. The base blocks of this design are listed in Table 3. Thus we have the following.

Lemma 4.1. *There is a simple* 5-$(32, 6, 3m)$ *design for* $m = 1, 2, 3, 4$ *and there is a simple* 5-$(33, 7, 84)$ *design.*

From Table 1, it is sufficient to determine whether a 5-$(32, 6, 3m)$ design exists for $m \leq 4$, and it is known that there is a 5-$(32, 6, 3)$ design. Hence, we can determine all λ for which there is a simple 5-$(32, 6, \lambda)$ design.

Theorem 4.2. *A simple* 5-$(32, 6, 3m)$ *design exists for all* $m \leq 4$.

From Table 1, it is sufficient to determine whether a 5-$(33, 7, 42m)$ design exists for $m \leq 4$, and it is known that there is a 5-$(33, 7, 42m)$ design for $m = 1, 3, 4$. Hence, we determined all λ for which there is a simple 5-$(33, 7, \lambda)$ design.

Theorem 4.3. *A simple* 5-$(33, 7, 42m)$ *design exists for all* $m \leq 4$.

Table 3: Base blocks of $D_{32,1}$

$\{1,2,3,4,5,6\}$,	$\{1,2,3,5,6,7\}$,	$\{1,2,3,5,6,8\}$,	$\{1,2,3,5,9,18\}$,
$\{1,2,3,5,9,20\}$,	$\{1,2,3,5,10,28\}$,	$\{1,2,3,8,12,16\}$,	$\{1,2,5,6,9,10\}$,
$\{1,2,5,6,9,25\}$,	$\{1,2,5,6,9,26\}$,	$\{1,2,5,6,11,12\}$,	$\{1,2,5,6,11,19\}$,
$\{1,2,5,6,11,20\}$,	$\{1,2,5,7,9,14\}$,	$\{1,2,5,7,9,24\}$,	$\{1,2,5,7,9,30\}$,
$\{1,2,5,7,10,16\}$,	$\{1,2,5,7,10,30\}$,	$\{1,2,5,7,11,26\}$,	$\{1,2,5,7,12,28\}$,
$\{1,2,5,9,13,19\}$,	$\{1,2,5,9,13,22\}$,	$\{1,2,5,9,15,28\}$,	$\{1,2,5,9,17,30\}$,
$\{1,2,5,9,17,31\}$,	$\{1,2,5,9,27,32\}$,	$\{1,2,5,10,15,24\}$,	$\{1,2,5,10,15,31\}$,
$\{1,2,5,10,19,20\}$,	$\{1,2,5,10,19,27\}$,	$\{1,2,5,10,19,31\}$,	$\{1,2,5,10,23,24\}$,
$\{1,2,5,11,16,28\}$,	$\{1,2,5,19,27,31\}$,	$\{1,2,7,11,16,19\}$,	$\{1,2,7,11,16,27\}$,
$\{1,5,9,13,17,21\}$,	$\{1,5,9,14,18,22\}$,	$\{1,5,9,14,18,25\}$,	$\{1,5,9,14,19,24\}$,
$\{1,5,9,14,22,30\}$,	$\{1,5,9,14,23,26\}$,	$\{1,5,10,14,19,27\}$,	$\{1,5,10,15,18,32\}$,
$\{1,5,10,15,19,26\}$			

Table 4: Base blocks of $D_{32,2}$

$\{1,2,3,4,5,9\}$,	$\{1,2,3,4,5,17\}$,	$\{1,2,3,4,5,22\}$,	$\{1,2,3,4,5,26\}$,
$\{1,2,3,4,5,28\}$,	$\{1,2,3,4,5,30\}$,	$\{1,2,3,4,7,12\}$,	$\{1,2,3,4,7,17\}$,
$\{1,2,3,4,7,30\}$,	$\{1,2,3,4,8,20\}$,	$\{1,2,3,4,8,25\}$,	$\{1,2,3,4,10,28\}$,
$\{1,2,3,4,11,23\}$,	$\{1,2,3,4,11,30\}$,	$\{1,2,3,4,20,23\}$,	$\{1,2,3,5,6,28\}$,
$\{1,2,3,5,9,22\}$,	$\{1,2,3,5,10,14\}$,	$\{1,2,3,5,10,16\}$,	$\{1,2,3,5,12,20\}$,
$\{1,2,3,7,9,31\}$			

5 5-(24, 9, $6m$) designs

Recently, Jimbo and Shiromoto [9] have found 22 mutually disjoint Steiner systems $S(5,8,24)$ by considering the binary extended Golay $[24,12,8]$ code as a bordered double circulant code and using some permutations of special type. Inspired by this result, we give in this section 11 mutually disjoint 5-$(24,9,6)$ designs obtained from the Pless symmetry code P_{24} of length 24 (see [15] for the Pless symmetry codes).

The code P_{24} is an extremal ternary bordered double circulant self-dual $[24,12,9]$ code with the following generator matrix

$$\left(\begin{array}{ccccc} & & & & 1 \\ I_{12} & & R_{11} & & \vdots \\ & & & & 1 \\ & 2 & \cdots & 2 & 0 \end{array}\right),$$

where R_{11} is the circulant matrix with first row $(0,2,1,2,2,2,1,1,1,2,1)$ and I_{12} is the identity matrix of order 12. The codewords of weight 9 in P_{24} support a 5-$(24,9,6)$ design $D_{24}=(X_{24},\mathcal{B}_{24})$ [15].

Table 5: Base blocks of $D_{32,3}$

$\{1,2,3,4,5,6\}$,	$\{1,2,3,4,5,9\}$,	$\{1,2,3,5,6,7\}$,	$\{1,2,3,5,6,8\}$,
$\{1,2,3,5,6,19\}$,	$\{1,2,3,5,6,20\}$,	$\{1,2,3,5,8,9\}$,	$\{1,2,3,5,8,17\}$,
$\{1,2,3,5,9,16\}$,	$\{1,2,3,5,9,17\}$,	$\{1,2,3,5,9,25\}$,	$\{1,2,3,5,9,28\}$,
$\{1,2,3,5,9,30\}$,	$\{1,2,3,5,10,15\}$,	$\{1,2,3,5,10,20\}$,	$\{1,2,3,5,10,24\}$,
$\{1,2,3,5,10,27\}$,	$\{1,2,3,5,12,20\}$,	$\{1,2,3,8,12,16\}$,	$\{1,2,3,8,12,20\}$,
$\{1,2,3,8,12,28\}$,	$\{1,2,5,6,9,10\}$,	$\{1,2,5,6,9,13\}$,	$\{1,2,5,6,9,17\}$,
$\{1,2,5,6,9,18\}$,	$\{1,2,5,6,9,23\}$,	$\{1,2,5,6,9,26\}$,	$\{1,2,5,6,9,30\}$,
$\{1,2,5,6,11,12\}$,	$\{1,2,5,6,11,15\}$,	$\{1,2,5,6,11,27\}$,	$\{1,2,5,6,11,28\}$,
$\{1,2,5,7,9,14\}$,	$\{1,2,5,7,9,15\}$,	$\{1,2,5,7,9,19\}$,	$\{1,2,5,7,9,24\}$,
$\{1,2,5,7,9,25\}$,	$\{1,2,5,7,9,27\}$,	$\{1,2,5,7,9,28\}$,	$\{1,2,5,7,10,16\}$,
$\{1,2,5,7,10,18\}$,	$\{1,2,5,7,10,20\}$,	$\{1,2,5,7,10,28\}$,	$\{1,2,5,7,10,30\}$,
$\{1,2,5,7,10,31\}$,	$\{1,2,5,7,11,16\}$,	$\{1,2,5,7,11,18\}$,	$\{1,2,5,7,11,26\}$,
$\{1,2,5,7,11,30\}$,	$\{1,2,5,7,11,32\}$,	$\{1,2,5,7,12,32\}$,	$\{1,2,5,9,13,18\}$,
$\{1,2,5,9,13,21\}$,	$\{1,2,5,9,13,23\}$,	$\{1,2,5,9,13,25\}$,	$\{1,2,5,9,13,31\}$,
$\{1,2,5,9,14,19\}$,	$\{1,2,5,9,14,26\}$,	$\{1,2,5,9,14,30\}$,	$\{1,2,5,9,15,16\}$,
$\{1,2,5,9,15,19\}$,	$\{1,2,5,9,15,22\}$,	$\{1,2,5,9,15,24\}$,	$\{1,2,5,9,15,28\}$,
$\{1,2,5,9,15,31\}$,	$\{1,2,5,9,17,30\}$,	$\{1,2,5,9,17,31\}$,	$\{1,2,5,9,19,22\}$,
$\{1,2,5,9,19,30\}$,	$\{1,2,5,9,19,32\}$,	$\{1,2,5,9,23,27\}$,	$\{1,2,5,9,23,28\}$,
$\{1,2,5,9,23,31\}$,	$\{1,2,5,9,26,31\}$,	$\{1,2,5,10,15,20\}$,	$\{1,2,5,10,15,23\}$,
$\{1,2,5,10,19,28\}$,	$\{1,2,5,10,19,31\}$,	$\{1,2,5,10,23,31\}$,	$\{1,2,5,10,27,28\}$,
$\{1,2,5,10,27,31\}$,	$\{1,2,5,10,31,32\}$,	$\{1,2,5,11,12,23\}$,	$\{1,2,5,11,12,31\}$,
$\{1,2,5,11,15,27\}$,	$\{1,2,5,11,15,32\}$,	$\{1,2,5,11,19,28\}$,	$\{1,2,5,11,20,27\}$,
$\{1,2,5,11,20,28\}$,	$\{1,2,5,11,28,31\}$,	$\{1,2,5,19,27,31\}$,	$\{1,2,7,11,15,20\}$,
$\{1,2,7,11,15,23\}$,	$\{1,2,7,11,15,27\}$,	$\{1,2,7,11,16,24\}$,	$\{1,2,7,11,16,27\}$,
$\{1,2,7,11,16,32\}$,	$\{1,2,7,11,20,27\}$,	$\{1,5,9,13,17,21\}$,	$\{1,5,9,13,18,27\}$,
$\{1,5,9,14,17,22\}$,	$\{1,5,9,14,18,22\}$,	$\{1,5,9,14,18,23\}$,	$\{1,5,9,14,18,26\}$,
$\{1,5,9,14,19,25\}$,	$\{1,5,9,14,19,26\}$,	$\{1,5,9,14,23,26\}$,	$\{1,5,9,14,23,28\}$,
$\{1,5,10,14,19,28\}$,	$\{1,5,10,14,27,31\}$,	$\{1,5,10,14,27,32\}$,	$\{1,5,10,15,18,24\}$,
$\{1,5,10,15,18,27\}$,	$\{1,5,10,15,18,31\}$,	$\{1,5,10,15,19,24\}$,	$\{1,5,10,15,19,28\}$,
$\{1,5,10,15,19,32\}$			

Table 6: Base blocks of $D_{32,4}$

$\{1,2,3,4,5,6\}$,	$\{1,2,3,4,5,9\}$,	$\{1,2,3,4,5,11\}$,	$\{1,2,3,4,5,17\}$,
$\{1,2,3,4,5,19\}$,	$\{1,2,3,4,5,20\}$,	$\{1,2,3,4,5,22\}$,	$\{1,2,3,4,5,23\}$,
$\{1,2,3,4,5,27\}$,	$\{1,2,3,4,5,28\}$,	$\{1,2,3,4,5,29\}$,	$\{1,2,3,4,7,12\}$,
$\{1,2,3,4,7,15\}$,	$\{1,2,3,4,7,16\}$,	$\{1,2,3,4,7,17\}$,	$\{1,2,3,4,7,20\}$,
$\{1,2,3,4,7,27\}$,	$\{1,2,3,4,7,30\}$,	$\{1,2,3,4,8,10\}$,	$\{1,2,3,4,8,12\}$,
$\{1,2,3,4,8,20\}$,	$\{1,2,3,4,8,25\}$,	$\{1,2,3,4,8,28\}$,	$\{1,2,3,4,8,29\}$,
$\{1,2,3,4,8,30\}$,	$\{1,2,3,4,10,12\}$,	$\{1,2,3,4,11,15\}$,	$\{1,2,3,4,11,23\}$,
$\{1,2,3,4,11,30\}$,	$\{1,2,3,5,6,12\}$,	$\{1,2,3,5,6,25\}$,	$\{1,2,3,5,6,28\}$,
$\{1,2,3,5,9,17\}$,	$\{1,2,3,5,9,22\}$,	$\{1,2,3,5,10,14\}$,	$\{1,2,3,5,10,24\}$,
$\{1,2,3,5,12,20\}$,	$\{1,2,3,7,9,25\}$,	$\{1,2,3,7,9,31\}$	

Table 7: Base blocks of D_{33}

$\{1,2,3,4,5,6,9\}$,	$\{1,2,3,4,5,6,10\}$,	$\{1,2,3,4,5,6,22\}$,	$\{1,2,3,4,5,10,12\}$,
$\{1,2,3,4,5,10,13\}$,	$\{1,2,3,4,5,10,17\}$,	$\{1,2,3,4,5,10,23\}$,	$\{1,2,3,4,5,10,26\}$,
$\{1,2,3,4,5,10,29\}$,	$\{1,2,3,4,5,14,15\}$,	$\{1,2,3,4,5,14,17\}$,	$\{1,2,3,4,5,14,18\}$,
$\{1,2,3,4,5,14,20\}$,	$\{1,2,3,4,5,14,23\}$,	$\{1,2,3,4,5,18,23\}$,	$\{1,2,3,4,5,22,26\}$,
$\{1,2,3,4,5,26,29\}$,	$\{1,2,3,4,7,8,27\}$,	$\{1,2,3,4,7,8,29\}$,	$\{1,2,3,4,7,9,11\}$,
$\{1,2,3,4,7,9,14\}$,	$\{1,2,3,4,7,9,16\}$,	$\{1,2,3,4,7,9,17\}$,	$\{1,2,3,4,7,9,21\}$,
$\{1,2,3,4,7,10,15\}$,	$\{1,2,3,4,7,11,17\}$,	$\{1,2,3,4,7,12,13\}$,	$\{1,2,3,4,7,12,14\}$,
$\{1,2,3,4,7,14,25\}$,	$\{1,2,3,4,7,14,27\}$,	$\{1,2,3,4,7,16,27\}$,	$\{1,2,3,4,7,19,29\}$,
$\{1,2,3,4,7,25,33\}$			

Let G_{24} be the group of order 11 generated by the following permutation

$$\sigma_{24} = (13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23). \tag{2}$$

We have verified that $\{(X_{24}, \mathcal{B}_{24}^{\sigma}) \mid \sigma \in G_{24}\}$ gives 11 mutually disjoint 5-$(24, 9, 6)$ designs. For any non-empty subset $S \subset G_{24}$, $(X_{24}, \cup_{\sigma\in S}\mathcal{B}_{24}^{\sigma})$ is a simple 5-$(24, 9, 6|S|)$ design. Hence, we have the following result.

Lemma 5.1. (i) *There exist at least* 11 *mutually disjoint* 5-$(24, 9, 6)$ *designs.*

(ii) *A simple* 5-$(24, 9, 6m)$ *design exists for* $m = 1, 2, \dots, 11$.

We have verified by Magma [5] that the 5-$(24, 9, 6m)$ designs $(X_{24}, \cup_{i=0}^{m-1}\mathcal{B}_{24}^{(\sigma_{24}^i)})$ have automorphism groups of orders 44 ($m = 2, 3, \dots, 9$), 220 ($m = 10$) and 2420 ($m = 11$).

From Table 1, it is sufficient to determine whether a 5-$(24, 9, 6m)$ design exists for $m \le 323$, and it is known that there is a 5-$(24, 9, 6m)$ design for $m = 1, 2, 6, \dots, 323$. Hence, using Lemma 5.1, we can determine all λ for which there is a simple 5-$(24, 9, \lambda)$ design.

Theorem 5.2. *A simple* 5-$(24, 9, 6m)$ *design exists for all* $m \le 323$.

In Table 8, we list the parameters (v, k, λ) of 5-(v, k, λ) designs D_{new} constructed by Proposition 2.1 where the parameters of D_1 and D_2 in Proposition 2.1 and their references are also listed.

Lemma 5.3. *There exists a simple* 5-$(25, 10, 96)$ *design and there exists a simple* 5-$(25, 10, 120)$ *design.*

Table 8: 5-designs by Proposition 2.1 from designs in Lemma 5.1

	D_{new}	D_1		D_2	
N_1	$(25, 10, 96)$	$(24, 9, 24)$	Lemma 5.1	$(24, 10, 72)$	[10, Table 4.46]
N_2	$(25, 9, 30)$	$(24, 8, 6)$	[10, Table 4.46]	$(24, 9, 24)$	Lemma 5.1
N_3	$(25, 10, 120)$	$(24, 9, 30)$	Lemma 5.1	$(24, 10, 90)$	[10, Table 4.46]
N_4	$(26, 10, 126)$	$(25, 9, 30)$	Lemma 5.1	$(25, 10, 96)$	N_1

Remark 5.4. The designs N_2 and N_4 give alternative proofs of Theorems 3.2 and 3.4, respectively.

Moreover, from [10, Table 4.46] it is sufficient to consider whether a 5-$(25, 10, 24m)$ design exists for $m \leq 323$, and it is known that there is a 5-$(25, 10, 24m)$ design for $m = 2, 6, \ldots, 323$. Therefore, we have the following result.

Corollary 5.5. *There exists a simple* 5-$(25, 10, 24m)$ *design for* $m = 2, 4, \ldots, 323$.

6 Other 5-designs derived from self-dual codes

In this section, we investigate mutually disjoint designs for 5-designs which are constructed from some self-dual codes [16, Table 1.61].

6.1 5-(36, 12, 15m) designs

The Pless symmetry code P_{36} of length 36 is an extremal ternary bordered double circulant self-dual code of length 36 with the following generator matrix

$$\left(\begin{array}{ccccc} & & & & 1 \\ I_{18} & & R_{17} & & \vdots \\ & & & & 1 \\ & 2 & \cdots & 2 & 0 \end{array}\right),$$

where R_{17} is the circulant matrix with first row

$$(0, 2, 2, 1, 2, 1, 1, 1, 2, 2, 1, 1, 1, 2, 1, 2, 2).$$

A 5-$(36, 12, 45)$ design $D_{36} = (X_{36}, \mathcal{B}_{36})$ is constructed by taking as blocks the supports of codewords of weight 12 in P_{36} [15].

Let G_{36} be the group of order 17 generated by the following permutation

$$\sigma_{36} = (19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35).$$

We have verified that $\{(X_{36}, \mathcal{B}_{36}^{\sigma}) \mid \sigma \in G_{36}\}$ gives 17 mutually disjoint 5-$(36, 12, 45)$ designs.

Lemma 6.1. *There are at least* 17 *mutually disjoint* 5-$(36, 12, 45)$ *designs.*

Remark 6.2. Jimbo [8] pointed out that Lemmas 5.1 and 6.1 have been proved by Angata and Shiromoto independently.

From [10, Table 4.46], it is sufficient to determine whether a 5-$(36, 12, 15m)$ design exists for $m \leq 87652$, and it is known that there exists a 5-$(36, 12, 15m)$ design for only $m = 3$. Our mutually disjoint designs imply the following existence result for 5-$(36, 12, 15m)$ designs.

Theorem 6.3. *There exists a simple* 5-$(36, 12, 15m)$ *design with* $m = 3n$ *for all* $1 \leq n \leq 17$.

We have verified by Magma [5] that the 5-$(36, 12, 45m)$ designs $(X_{36}, \cup_{i=0}^{m-1} \mathcal{B}_{36}^{(\sigma_{36}^i)})$ have automorphism groups of orders 68 ($m = 2, 3, \ldots, 15$), 544 ($m = 16$) and 9248 ($m = 17$).

6.2 5-(18, 8, 2m) designs

Let $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$ be the finite field of order 4, where $\bar{\omega} = \omega^2 = \omega + 1$. An extremal Hermitian self-dual $\mathbb{F}_4$-code S_{18} of length 18 was given in [14] and is generated by

$$\left(\begin{array}{cc} & 1 \\ R'_{17} & \vdots \\ & 1 \end{array} \right),$$

where R'_{17} is the circulant matrix with first row

$$(1, \omega, \bar{\omega}, \omega, \omega, \omega, \bar{\omega}, \bar{\omega}, \bar{\omega}, \bar{\omega}, \bar{\omega}, \bar{\omega}, \omega, \omega, \omega, \bar{\omega}, \omega).$$

A 5-$(18, 8, 6)$ design $D_{18} = (X_{18}, \mathcal{B}_{18})$ is constructed by taking as blocks the supports of codewords of weight 8 in S_{18} [14].

We have verified that $\{(X_{18},\mathcal{B}_{18}),(X_{18},\mathcal{B}_{18}^{\sigma_{18,1}}),(X_{18},\mathcal{B}_{18}^{\sigma_{18,2}})\}$ are mutually disjoint 5-$(18,8,6)$ designs, where

$$\sigma_{18,1} =(3,16,5,12,6,14,9,11)(4,7,8)(10,18,15,17,13),$$
$$\sigma_{18,2} =(2,17,4,16,11,18,8,13,9,3,10)(5,12,7,6)(14,15).$$

Lemma 6.4. *There are at least three mutually disjoint* 5-$(18,8,6)$ *designs.*

From [10, Table 4.46], it is sufficient to determine whether a 5-$(18,8,2m)$ design exists for $m \le 71$, and it is known that there is a 5-$(18,8,2m)$ design for $m = 3,7,8,15,16,20,22,23,24,\ 30,\dots,33,\ 38,\dots,41,\ 46,\dots,49,\ 52,54,\dots,57,\ 62,\dots,65,\ 70$ and 71. Our mutually disjoint designs determine the existence of a 5-$(18,8,6m)$ design ($m = 2,3$). We have verified by Magma [5] that the 5-$(18,8,6m)$ designs ($m = 2,3$) $(X_{18},\mathcal{B}_{18}\cup\mathcal{B}_{18}^{\sigma_{18,1}})$ and $(X_{18},\mathcal{B}_{18}\cup\mathcal{B}_{18}^{\sigma_{18,1}}\cup\mathcal{B}_{18}^{\sigma_{18,2}})$ have automorphism groups of orders 1 and 2, respectively. Moreover, since a simple 5-$(18,9,5m)$ design is known for $m = 6,9$ (see Table 4.46 in [10]), by Proposition 2.1, a 5-$(19,9,7m)$ design is constructed for the first time for $m = 6,9$.

Proposition 6.5. *There exists a simple* 5-$(18,8,6m)$ *design for* $m = 2,3$*, and there exists a simple* 5-$(19,9,7m)$ *design for* $m = 6,9$.

6.3 5-(30, 12, 220m) designs

Let C_{30} be the bordered double circulant $\mathbb{F}_4$-code with the following generator matrix

$$\left(\begin{array}{ccc} & & 1\\ I_{15} & R_{14} & \vdots\\ & & 1\\ & 1\ \cdots\ 1 & 1\end{array}\right),$$

where R_{14} is the circulant matrix with first row

$$(\bar{\omega},\omega,0,1,\omega,\bar{\omega},0,1,\bar{\omega},\omega,1,1,0,0).$$

The code C_{30} is an extremal Hermitian self-dual code of length 30 and it is equivalent to the extended quadratic residue code of length 30 in [14]. A 5-$(30,12,220)$ design $D_{30} = (X_{30},\mathcal{B}_{30})$ is constructed by taking as blocks the supports of codewords of weight 12 in C_{30} [14].

We have verified that $\{(X_{30}, \mathcal{B}_{30}), (X_{30}, \mathcal{B}_{30}^{(\sigma_{30}^{12})})\}$ are mutually disjoint 5-$(30, 12, 220)$ designs, where

$$\sigma_{30} = (16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29).$$

Lemma 6.6. *There exist at least two disjoint* 5-$(30, 12, 220)$ *designs.*

From [10, Table 4.46], it is sufficient to determine whether a 5-$(30, 12, 220m)$ design exists for $m \leq 1092$, and it is known that there is a 5-$(30, 12, 220m)$ design for $m = 1, 345, 760, 805, 920$. Our mutually disjoint designs imply the existence of a 5-$(30, 12, 440)$ design. We have verified by Magma [5] that the 5-$(30, 12, 440)$ design $(X_{30}, \mathcal{B}_{30} \cup \mathcal{B}_{30}^{(\sigma_{30}^{12})})$ has automorphism group of order 56.

Proposition 6.7. *There exists a simple* 5-$(30, 12, 440)$ *design.*

6.4 Mutually disjoint 5-(24, 10, 36) designs

Let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ be the ring of integers modulo 4. Let C_{24} be the bordered double circulant $\mathbb{Z}_4$-code with the following generator matrix

$$\left(\begin{array}{ccccc} & & & & 1 \\ & I_{12} & & R'_{11} & \vdots \\ & & & & 1 \\ & & 3 & \cdots \; 3 & 2 \end{array}\right),$$

where R'_{11} is the circulant matrix with first row $(2, 3, 3, 1, 2, 1, 1, 0, 1, 0, 0)$. The code C_{24} is an extremal Type II $\mathbb{Z}_4$-code of length 24 and it is equivalent to the code $C_{1,11}$ given in [7, Table 1]. A 5-$(24, 10, 36)$ design $D'_{24} = (X_{24}, \mathcal{B}'_{24})$ is constructed by taking as blocks the supports of codewords of weight 10 in C_{24} [7].

We have verified that $\{(X_{24}, {\mathcal{B}'}_{24}^{(\sigma_{24}^{i})}) \mid i = 0, 1, 2, 3, 4\}$ are five mutually disjoint 5-$(24, 10, 36)$ designs, where σ_{24} is given in (2).

Proposition 6.8. *There are at least* 5 *mutually disjoint* 5-$(24, 10, 36)$ *designs.*

From [10, Table 4.46], it is sufficient to determine whether a 5-$(24, 10, 18m)$ design exists for $m \leq 323$, and it is known that there is a 5-$(24, 10, 18m)$ design for $m = 2, 4, 5, \ldots, 323$. Our mutually disjoint 5-$(24, 10, 36)$ designs do not give simple designs with new parameters, although the resulting designs are non-isomorphic to the 5-$(24, 10, m36)$ designs ($m = 2, 3, 4, 5$) invariant under PSL$(2, 23)$ in [11] since we have verified by MAGMA [5] that the 5-$(24, 10, 36m)$ designs $(X_{24}, \cup_{i=0}^{m-1} \mathcal{B}'^{(\sigma_{24}^i)}_{24})$ have automorphism groups of orders 22 ($m = 2, 3, 4, 5$).

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