Classification of Circulant D-Optimal Designs of Orders 62 and 74

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Abstract

We give a classification of circulant D-optimal designs of orders 62 and 74. We also give a revised classification of circulant D-optimal designs of orders $26, 50$ and 66.

1 Introduction

A *D-optimal design* of order n is a square $(1, -1)$-matrix of order n having maximum determinant. Ehlich [2] showed that for $n \equiv 2 \pmod 4$ and $n > 2$, any square $(1, -1)$-matrix M of order n satisfies

$$(1) \qquad \det M \leq (2n-2)(n-2)^{(n-2)/2}$$

and that equality is possible only if $2n - 2$ is a sum of two perfect squares. Moreover, if $n = 2v \equiv 2 \pmod 4$ and A, B are commutative $(1, -1)$-matrices of order v such that

$$AA^T + BB^T = (2v-2)I + 2J \tag{2}$$

then

$$X(A,B) = \begin{pmatrix} A & B \\ B^T & -A^T \end{pmatrix}$$

is a D-optimal design meeting the above bound (1), where I and J are the identity and all-one's matrices, respectively, and A^T is the transposed matrix of A [2]. If the matrices A and B are circulant then $X(A,B)$ is called a *circulant D-optimal design* [3]. Most of the known D-optimal designs are circulant (cf. [1], [3]).

For orders $n \leq 58$ and $n = 66$, all circulant D-optimal designs meeting the bound (1) were classified in [3]. The aim of this note is to extend the classification to orders up to 74, under the equivalence defined in [3].

Theorem 1. *There are exactly* 155 *circulant D-optimal designs of order* 62*, up to equivalence. There are exactly* 537 *circulant D-optimal designs of order* 74*, up to equivalence.*

Remark 2. For order $n = 70$, there is no circulant D-optimal design meeting the bound (1), since $2n-2$ is not a sum of two perfect squares. Hence the classification has been complete for orders up to 74.

We also give a revised classification of circulant D-optimal designs of orders $26, 50$ and 66.

2 Preliminaries

From now on, we suppose that $n = 2v > 2$ where v is odd. Let $\mathbb{Z}_v = \{0, 1, \ldots, v-1\}$ be the ring of integers modulo v. For $A \subset \mathbb{Z}_v$ and $i \in \mathbb{Z}_v$, define

$$P_A(i) = |\{(x,y) \in A \times A \mid y - x = i\}| \text{ and}$$
$$P_A = (P_A(1), P_A(2), \ldots, P_A(v-1)).$$

Let A and B be a k-subset and an r-subset of $\mathbb{Z}_v$, respectively. If a pair (A,B) satisfies

$$P_A + P_B = (\lambda, \lambda, \ldots, \lambda)$$

then it is called a 2-$\{v; k, r; \lambda\}$ *supplementary difference set*. It follows from the definition that $\lambda(v-1) = k(k-1) + r(r-1)$. If $X(A, B)$ is a circulant D-optimal design meeting the bound (1), then it was shown in [1] that

$$(v-2k)^2 + (v-2r)^2 = 2n-2 \tag{3}$$

where k and r are the numbers of -1's in the first rows of A and B, respectively. It was also shown in [1] that circulant matrices A and B with first rows which have k -1's and r -1's, respectively, satisfying (2) correspond to a 2-$\{v; k, r; \lambda\}$ supplementary difference set satisfying that $\lambda = k + r - (v-1)/2$. More precisely, we identify the pair of the sets of the positions of -1's in the first rows of A and B with the corresponding supplementary difference set. Hence circulant D-optimal designs are often considered in this note using the corresponding supplementary difference sets.

If (A, B) is a supplementary difference set then the following pairs:

(E0) $(\mathbb{Z}_v \backslash A, B)$ and $(A, \mathbb{Z}_v \backslash B)$,

(E1) (B, A),

(E2) $(\pm A + a, \pm B + b)$ for any $a, b \in \mathbb{Z}_v$,

(E3) (dA, dB) for any $d \in U(\mathbb{Z}_v)$

are also supplementary difference sets, where $U(\mathbb{Z}_v) = \{d \mid 1 \leq d \leq v-1,\ \gcd(d, v) = 1\}$ and d is regarded as an integer for $\gcd(d, v) = 1$. These supplementary difference sets are called equivalent [3]. According to [3], we say that circulant D-optimal designs are *equivalent* if the corresponding supplementary difference sets are equivalent.

If $X(A, B)$ is a circulant D-optimal design then $X(\pm A, \pm B)$ and $X(\pm B, \pm A)$ are also circulant D-optimal designs which are equivalent to $X(A, B)$ by (E0) and (E1). Hence we may assume that

$$1 \leq k \leq r \leq \frac{n-2}{4}. \tag{4}$$

Therefore, in order to complete the classification of circulant D-optimal designs of order $n = 2v$ meeting the bound (1), we classify 2-$\{v; k, r; k+r-\frac{v-1}{2}\}$ supplementary difference sets for all possible (k, r) satisfying (3) and (4), by the method given in the next section.

3 Classification results

Lemma 3. *Let A and B be a k-subset and an r-subset of $\mathbb{Z}_v$ with the characteristic vectors of the following form*

$$(5) \quad (\underbrace{0,\dots,0}_{\alpha_1^+},\underbrace{1,\dots,1}_{\alpha_1^-},\dots,\underbrace{0,\dots,0}_{\alpha_s^+},\underbrace{1,\dots,1}_{\alpha_s^-}) \text{ and }$$

$$(\underbrace{0,\dots,0}_{\beta_1^+},\underbrace{1,\dots,1}_{\beta_1^-},\dots,\underbrace{0,\dots,0}_{\beta_t^+},\underbrace{1,\dots,1}_{\beta_t^-})$$

respectively, where $\sum_{i=1}^s \alpha_i^+ = v-k$, $\sum_{i=1}^t \beta_i^+ = v-r$, $\sum_{i=1}^s \alpha_i^- = k$, $\sum_{i=1}^t \beta_i^- = r$, and $\alpha_i^\pm, \beta_i^\pm > 0$. Then the following statements hold.

(i) *$P_A(1) = k - s$ and $P_B(1) = r - t$,*

(ii) *if $i \geq 2$, then*

$$P_A(i) \geq |\{\, j \mid \alpha_j^+ = i-1 \,\}| \; + \sum_{\alpha_j^- > i} (\alpha_j^- - i)$$

$$P_B(i) \geq |\{\, j \mid \beta_j^+ = i-1 \,\}| \; + \sum_{\beta_j^- > i} (\beta_j^- - i)$$

and equality holds for $i = 2$.

Proof. The tedious but straightforward proof is omitted. □

Lemma 4. *Let (A, B) be a 2-$\{v; k, r; k + r - \frac{v-1}{2}\}$ supplementary difference set. Suppose that A and B have the characteristic vectors of form (5). Then*

$$(6) \qquad s + t = \frac{v-1}{2}.$$

Proof. From Lemma 3 (i), $P_A(1) = k - s$ and $P_B(1) = r - t$. Since $P_A(1) + P_B(1) = k + r - (v-1)/2$, we obtain $s + t = (v-1)/2$. □

Now let A and A' be k-subsets of $\mathbb{Z}_v$. Suppose that A and A' have the characteristic vectors of the following form

$$(\underbrace{0,\dots,0}_{\alpha_1^+},\underbrace{1,\dots,1}_{\alpha_1^-},\dots,\underbrace{0,\dots,0}_{\alpha_s^+},\underbrace{1,\dots,1}_{\alpha_s^-}) \text{ and}$$

$$(\underbrace{0,\dots,0}_{\alpha'^+_1},\underbrace{1,\dots,1}_{\alpha'^-_1},\dots,\underbrace{0,\dots,0}_{\alpha'^+_s},\underbrace{1,\dots,1}_{\alpha'^-_s})$$

respectively, where $\sum_{i=1}^{s} \alpha_i^+ = \sum_{i=1}^{s} {\alpha'}_i^+ = v - k$, $\sum_{i=1}^{s} \alpha_i^- = \sum_{i=1}^{s} {\alpha'}_i^- = k$, and $\alpha_i^\pm, {\alpha'}_i^\pm > 0$. We write $A \preceq A'$ if $({\alpha'}_1^+, {\alpha'}_2^+, \ldots, {\alpha'}_s^+)$ is greater than or equal to $(\alpha_1^+, \alpha_2^+, \ldots, \alpha_s^+)$ in lexicographic order. For r-subsets B and B', $B \preceq B'$ are defined similarly.

We construct all 2-$\{v; k, r; k + r - \frac{v-1}{2}\}$ supplementary difference sets (A, B) satisfying the following conditions:

(i) the characteristic vectors of A and B are of form (5),

(ii) $A \preceq (\pm A + a)$ and $B \preceq (\pm B + b)$ for $a, b \in \mathbb{Z}_v$ such that $(\pm A + a)$ and $(\pm B + b)$ have the characteristic vectors of form (5), the numbers s for A and $(\pm A + a)$ are the same, and the numbers t for B and $(\pm B + b)$ are the same,

(iii) $P_A(1) \geq P_A(2)$ and $P_B(1) \leq P_B(2)$,

(iv) $P_A \geq P_{dA}$ and $P_B \leq P_{dB}$ for all $d \in U(\mathbb{Z}_v)$,

where $P_A \geq P_{dA}$ means that P_A is greater than or equal to P_{dA} in lexicographic order. From (E2) and (E3), without loss of generality, we may assume that the conditions (i), (ii) and (iv) hold. Although (iii) is an additional condition which can be easily obtained from (iv), the condition (iii) is useful to reduce the number of possible A and B since it can be directly checked by Lemma 3 without examination of (iv). Hence all 2-$\{v; k, r; k + r - \frac{v-1}{2}\}$ supplementary difference sets which must be checked further for equivalence to complete the classification can be obtained by considering all pairs (s, t) satisfying (6). Then we divide the supplementary difference sets into equivalence classes and take one from every class. This completes the classification of 2-$\{v; k, r; k+r-\frac{v-1}{2}\}$ supplementary difference sets. For all possible (k, r) satisfying (3) and (4), by completing the classification of 2-$\{v; k, r; k+r-\frac{v-1}{2}\}$ supplementary difference sets, we complete the classification of circulant D-optimal designs of order $n = 2v$ meeting the bound (1).

For orders $n = 62$ and 74, pairs (k, r) satisfying (3) and (4) are $(10, 15)$ and $(13, 16)$, respectively. Hence by classifying 2-$\{v; k, r; k + r - \frac{v-1}{2}\}$ supplementary difference sets with $(v, k, r) = (31, 10, 15)$ and $(37, 13, 16)$, we completed the classification of circulant D-optimal designs of orders 62 and 74, respectively. All the inequivalent circulant D-optimal designs of order 62 are presented as supplementary difference sets in Appendix. To save space, we give only 20 circulant D-optimal designs of order 74 in Table 1.

Table 1: Examples of circulant D-optimal designs of order 74

A	B
$\{2, 3, 4, 7, 8, 13, 18, 19, 20, 21, 27, 28, 36\}$	$\{1, 2, 4, 5, 7, 9, 11, 13, 14, 15, 17, 20, 24, 27, 31, 36\}$
$\{2, 3, 6, 7, 12, 18, 25, 26, 27, 28, 29, 35, 36\}$	$\{1, 3, 5, 7, 10, 12, 14, 15, 17, 18, 20, 23, 24, 30, 35, 36\}$
$\{2, 3, 4, 5, 8, 9, 12, 13, 19, 20, 21, 26, 36\}$	$\{1, 2, 4, 7, 9, 13, 15, 17, 20, 24, 26, 27, 28, 29, 32, 36\}$
$\{3, 7, 13, 14, 15, 16, 17, 18, 22, 23, 28, 35, 36\}$	$\{1, 3, 5, 8, 11, 12, 14, 15, 17, 21, 22, 23, 25, 28, 33, 36\}$
$\{2, 6, 7, 12, 19, 20, 21, 22, 23, 24, 30, 31, 36\}$	$\{1, 3, 4, 6, 8, 9, 12, 14, 18, 20, 24, 27, 31, 34, 35, 36\}$
$\{1, 2, 3, 4, 5, 10, 11, 19, 24, 25, 29, 35, 36\}$	$\{1, 3, 6, 8, 10, 11, 14, 16, 18, 19, 22, 28, 29, 31, 32, 36\}$
$\{1, 2, 6, 7, 10, 11, 12, 13, 14, 26, 29, 30, 36\}$	$\{1, 2, 4, 8, 10, 13, 15, 19, 21, 23, 24, 28, 29, 31, 35, 36\}$
$\{1, 2, 3, 5, 6, 12, 13, 22, 23, 27, 34, 35, 36\}$	$\{1, 4, 6, 9, 10, 11, 13, 16, 19, 22, 24, 28, 29, 30, 32, 36\}$
$\{2, 3, 4, 5, 6, 10, 15, 16, 17, 23, 31, 32, 36\}$	$\{1, 3, 4, 6, 7, 8, 11, 13, 17, 19, 21, 22, 25, 28, 31, 36\}$
$\{1, 2, 5, 10, 11, 12, 15, 16, 17, 18, 26, 35, 36\}$	$\{1, 3, 4, 6, 7, 11, 13, 14, 18, 19, 21, 23, 27, 29, 32, 36\}$
$\{1, 2, 5, 6, 7, 8, 11, 19, 20, 21, 31, 35, 36\}$	$\{1, 3, 6, 9, 12, 14, 18, 19, 20, 22, 24, 28, 29, 31, 35, 36\}$
$\{2, 7, 10, 18, 19, 20, 21, 24, 25, 26, 34, 35, 36\}$	$\{1, 3, 5, 6, 9, 11, 13, 18, 22, 25, 28, 29, 31, 34, 35, 36\}$
$\{1, 2, 6, 11, 20, 21, 22, 25, 26, 27, 28, 35, 36\}$	$\{1, 2, 3, 5, 8, 9, 10, 13, 15, 18, 21, 25, 27, 29, 33, 36\}$
$\{1, 2, 3, 4, 7, 8, 15, 19, 25, 26, 27, 35, 36\}$	$\{1, 3, 5, 7, 8, 10, 13, 15, 16, 21, 25, 28, 29, 31, 35, 36\}$
$\{2, 3, 4, 5, 8, 9, 13, 21, 27, 28, 29, 30, 36\}$	$\{1, 3, 5, 7, 8, 10, 13, 15, 18, 21, 22, 25, 29, 34, 35, 36\}$
$\{1, 2, 6, 7, 8, 17, 20, 21, 28, 33, 34, 35, 36\}$	$\{1, 2, 4, 5, 8, 9, 11, 13, 16, 18, 22, 24, 26, 29, 30, 36\}$
$\{1, 2, 5, 6, 7, 10, 22, 23, 24, 31, 32, 33, 36\}$	$\{1, 2, 4, 6, 8, 12, 16, 17, 19, 20, 23, 25, 26, 29, 31, 36\}$
$\{1, 5, 6, 7, 8, 12, 17, 18, 21, 22, 34, 35, 36\}$	$\{1, 2, 4, 5, 7, 8, 11, 13, 15, 19, 23, 24, 26, 31, 33, 36\}$
$\{2, 7, 10, 11, 12, 18, 19, 20, 21, 24, 25, 26, 36\}$	$\{1, 3, 5, 7, 10, 12, 15, 21, 22, 24, 25, 27, 28, 31, 32, 36\}$
$\{1, 2, 3, 4, 5, 6, 10, 19, 20, 27, 28, 31, 36\}$	$\{1, 3, 6, 8, 9, 12, 14, 18, 19, 23, 24, 26, 30, 32, 33, 36\}$

4 Remarks on the known classification

For orders $n \leq 58$ and $n = 66$, all circulant D-optimal designs meeting the bound (1) were classified in [3]. By the method given in Section 3, we completed the classification for these orders alternatively. Comparing our classification with the known classification in [3], we obtained the following revised classification of circulant D-optimal designs for only orders $n = 26, 50$ and 66.

- $n = 26$: According to [3, Table I] the authors in [3] claimed that there are two 2-$\{13; 3, 6; 3\}$ supplementary difference sets and there is a 2-$\{13; 4, 4; 2\}$ supplementary difference set, up to equivalence. Here we correct an omission by [3] in their classification of 2-$\{13; 4, 4; 2\}$ supplementary difference sets and we claim that the following two 2-$\{13; 4, 4; 2\}$ supplementary difference sets defined by (A, A) and (A, B)

are inequivalent where

$$A = \{1, 4, 5, 12\} \text{ and } B = \{1, 2, 6, 12\}.$$

In order to show this, it is enough to check the conditions (E2) and (E3). We verified that

$$\{(d, a) \in U(\mathbb{Z}_{13}) \times \mathbb{Z}_{13} | d(\pm A + a) = A\} = \{(1, 0), (3, 5), (4, 2), (9, 11), (10, 8), (12, 0)\}.$$

But for each of the above elements d, there is no element $b \in \mathbb{Z}_{13}$ with $d(\pm A + b) = B$. This means that (A, A) and (A, B) are inequivalent.

- $n = 50$: According to [3, Table I] the authors in [3] claimed that there are 39 inequivalent 2-$\{25; 9, 9; 6\}$ supplementary difference sets. However, our computer search shows that there are only 30 inequivalent 2-$\{25; 9, 9; 6\}$ supplementary difference sets. We have verified that the authors in [3] do not apply (E1) in the definition of equivalent supplementary difference sets. In fact, by (E1) the following pairs of supplementary difference sets are equivalent:

$$\begin{aligned}
&(D_1, D_{34}), (D_2, D_{32}), (D_3, D_{26}), (D_4, D_{29}), (D_5, D_{31}), (D_6, D_{16}),\\
&(D_7, D_{33}), (D_8, D_{10}), (D_9, D_{22}), (D_{11}, D_{15}), (D_{12}, D_{37}), (D_{13}, D_{24}),\\
&(D_{17}, D_{18}), (D_{19}, D_{28}), (D_{20}, D_{27}), (D_{21}, D_{23}), (D_{30}, D_{35}), (D_{38}, D_{39}),
\end{aligned}$$

where D_i denotes the supplementary difference set (A_i, B_i) in [3, Table I]. In Table 2, we give nine missing supplementary difference sets.

- $n = 66$: The authors in [3] claimed that there are 509 and 516 inequivalent 2-$\{33; 12, 13; 9\}$ and 2-$\{33; 11, 15; 10\}$ supplementary difference sets, respectively. However, our computer search shows that there are 576 and 537 inequivalent 2-$\{33; 12, 13; 9\}$ and 2-$\{33; 11, 15; 10\}$ supplementary difference sets, respectively. In [3, Table I], the authors only gave some supplementary difference sets. Hence we do not compare the classification in [3] with our classification.

Hence we have the following revised classification.

Proposition 5. *Up to equivalence, there are exactly four,* 30 *and* 1113 *circulant D-optimal designs of orders* 26, 50 *and* 66*, respectively.*

Table 2: Nine missing 2-$\{25; 9, 9; 6\}$ supplementary difference sets

A	B
$\{1, 3, 4, 5, 7, 13, 14, 23, 24\}$	$\{1, 4, 6, 7, 11, 15, 18, 19, 24\}$
$\{1, 2, 6, 11, 16, 17, 18, 23, 24\}$	$\{1, 3, 4, 7, 10, 11, 13, 15, 24\}$
$\{1, 2, 6, 9, 10, 11, 12, 16, 24\}$	$\{1, 3, 6, 12, 15, 18, 19, 20, 24\}$
$\{1, 2, 3, 7, 8, 9, 12, 19, 24\}$	$\{1, 3, 4, 7, 14, 15, 17, 20, 24\}$
$\{1, 2, 5, 7, 12, 13, 22, 23, 24\}$	$\{1, 4, 6, 10, 14, 17, 22, 23, 24\}$
$\{1, 2, 3, 4, 7, 8, 14, 16, 24\}$	$\{1, 4, 8, 10, 15, 16, 19, 23, 24\}$
$\{1, 2, 3, 6, 7, 8, 14, 16, 24\}$	$\{1, 4, 8, 9, 11, 15, 20, 23, 24\}$
$\{1, 2, 3, 7, 8, 9, 13, 16, 24\}$	$\{1, 3, 4, 8, 11, 12, 14, 17, 24\}$
$\{1, 5, 6, 9, 12, 13, 14, 15, 24\}$	$\{1, 4, 8, 13, 16, 18, 19, 20, 24\}$

References

[1] Th. Chadjipantelis and S. Kounias, Supplementary difference sets and D-optimal designs for $n \equiv 2 \bmod 4$, *Discrete Math.* **57** (1985), 211–216.

[2] H. Ehlich, Determinantenabschätzungen für binäre Matrizen, *Math. Z.* **83** (1964), 123–132.

[3] S. Kounias, C. Koukouvinos, N. Nikolaou and A. Kakos, The nonequivalent circulant D-optimal designs for $n \equiv 2 \text{ mod } 4, n \leq 54, n = 66$, *J. Combin. Theory Ser. A* **65** (1994), 26–38.

Appendix

In Appendix, we give all the inequivalent circulant D-optimal designs of order 62 as supplementary difference sets (A, B) where the sets A and B are written for each supplementary difference set.

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{2,5,8,9,13,14,27,28,29,30},{1,3,5,7,10,12,13,15,16,17,20,23,24,25,30}
{2,8,9,12,13,14,15,21,29,30},{1,3,6,7,8,9,11,12,14,18,20,22,25,29,30}
{2,3,4,5,10,11,19,20,23,30},{1,3,4,6,7,8,10,12,14,17,18,20,25,29,30}
{1,2,9,10,15,16,17,20,21,30},{1,3,5,7,8,9,10,12,13,15,18,21,22,25,30}
{1,2,3,7,8,14,21,22,29,30},{1,2,3,4,6,8,10,13,16,19,20,22,24,29,30}
{2,6,7,8,9,12,13,21,22,30},{1,3,5,6,8,11,13,14,15,17,19,25,26,29,30}
{2,5,6,10,11,18,27,28,29,30},{1,3,5,6,8,9,10,13,15,20,21,23,24,26,30}
{2,3,4,5,9,13,21,22,29,30},{1,2,3,5,7,8,10,13,16,18,20,22,23,29,30}
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{1,5,6,9,10,15,16,17,29,30},{1,2,4,5,7,9,10,13,14,16,18,20,22,23,30}
{2,3,4,7,8,9,15,16,19,30},{1,2,4,6,8,12,14,15,16,17,20,23,25,26,30}
{2,5,6,7,13,18,19,28,29,30},{1,3,4,5,6,7,9,11,14,17,18,21,23,27,30}
{1,2,4,5,10,11,18,19,29,30},{1,3,5,7,8,9,12,14,16,17,19,20,24,29,30}
{1,2,3,8,9,16,19,28,29,30},{1,3,5,6,7,11,13,14,16,17,20,22,25,29,30}
{2,5,6,7,8,9,13,20,21,30},{1,3,5,8,10,11,14,16,20,21,22,24,25,29,30}
{2,6,15,16,21,26,27,28,29,30},{1,3,4,6,9,11,15,17,18,21,22,24,25,26,30}
{2,3,7,8,9,10,11,21,24,30},{1,2,4,6,8,9,10,14,16,19,21,22,25,26,30}
{1,2,3,7,13,14,21,28,29,30},{1,3,5,8,10,11,15,18,19,21,24,27,28,29,30}
{3,4,9,13,14,15,16,17,23,30},{1,2,3,5,6,8,10,11,14,16,18,21,24,25,30}
{3,8,14,19,20,26,27,28,29,30},{1,2,4,5,6,8,10,12,15,18,19,20,23,27,30}
{3,8,15,19,20,21,22,23,29,30},{1,2,3,5,7,8,11,12,14,17,18,20,22,25,30}
{1,2,3,6,7,8,17,18,26,30},{1,2,3,5,8,11,13,16,19,20,21,23,25,26,30}
{2,7,8,9,10,11,15,23,29,30},{1,3,4,5,7,10,11,15,18,20,21,23,25,29,30}
{2,6,7,12,13,14,15,16,25,30},{1,2,4,5,6,8,10,12,15,16,20,21,23,27,30}
{1,2,7,13,20,21,27,28,29,30},{1,3,4,7,9,11,15,16,18,19,20,22,25,29,30}
{1,2,3,4,8,15,16,24,29,30},{1,3,5,6,7,10,13,14,15,17,20,23,25,26,30}
{1,2,3,4,9,14,20,21,29,30},{1,3,4,6,7,8,11,14,15,17,21,23,25,29,30}
{1,4,5,8,9,10,11,23,24,30},{1,2,3,5,8,10,13,15,18,19,22,23,24,26,30}
{1,2,3,9,10,15,19,28,29,30},{1,2,3,4,6,9,11,12,15,18,20,22,26,27,30}
{1,2,3,4,8,12,13,19,20,30},{1,2,4,6,7,9,12,13,15,19,21,25,28,29,30}
{1,2,6,11,12,13,21,28,29,30},{1,3,4,5,7,9,11,12,14,17,18,21,24,29,30}
{1,2,5,6,7,10,22,23,24,30},{1,2,3,5,7,10,12,13,17,19,22,23,25,29,30}
{1,2,3,7,8,9,19,20,23,30},{1,3,4,6,7,8,10,12,15,16,19,22,24,29,30}
{1,2,3,9,10,16,19,28,29,30},{1,3,5,6,9,10,12,14,15,20,22,25,28,29,30}
{1,4,12,13,22,23,24,28,29,30},{1,2,4,5,6,8,10,12,15,18,19,20,23,27,30}
{1,2,8,9,12,13,14,15,23,30},{1,3,6,7,8,10,11,12,14,17,20,22,25,26,30}
{1,2,3,4,10,11,15,16,24,30},{1,2,4,5,6,8,10,12,15,16,17,20,23,27,30}
{1,2,4,5,6,10,11,19,20,30},{1,3,5,8,10,11,13,14,17,21,22,24,28,29,30}
{1,2,3,7,12,13,14,20,29,30},{1,2,3,5,7,9,10,12,15,18,22,24,25,29,30}
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{1,2,6,7,8,9,15,18,19,30},{1,3,5,6,7,9,10,12,15,17,20,21,24,25,30}
{1,2,6,7,19,21,22,28,29,30},{1,3,4,5,8,10,14,16,17,20,21,22,24,27,30}
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