Classification of Quaternary Hermitian Self-Dual Codes of Length 20

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November 4, 2010

Abstract

A classification of quaternary Hermitian self-dual codes of length 20 is given. Using this classification, a classification of extremal quaternary Hermitian self-dual codes of length 22 is also given.

Index Terms: Self-dual code, quaternary code, Hermitian self-dual code, extremal code, mass formula.

1 Introduction

Let $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$ be the finite field of order four, where $\bar{\omega} = \omega^2 = \omega + 1$. Codes over $\mathbb{F}_4$ are often called quaternary. All codes in this note are quaternary. The *Hermitian dual code* $C^\perp$ of a code C of length n is defined as $C^\perp = \{x \in \mathbb{F}_4^n \mid x \cdot c = 0 \text{ for all } c \in C\}$, where $x \cdot y = \sum_{i=1}^n x_i y_i^2$ for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{F}_4^n$, which is known as the Hermitian inner product. A code C is called *Hermitian self-orthogonal* if $C \subset C^\perp$, and C is called *Hermitian self-dual* if $C = C^\perp$. It is known that an $[n, k]$ code C is Hermitian self-dual if and only if the weights of all codewords of C are even, that is, C is even, and $n = 2k$ [7]. It was shown in [7] that the

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minimum weight d of a Hermitian self-dual code of length n is bounded by $d \leq 2\lfloor n/6 \rfloor + 2$. A Hermitian self-dual code of length n and minimum weight $d = 2\lfloor n/6 \rfloor + 2$ is called *extremal*.

Two codes C and C' are *equivalent* if there is some monomial matrix M over $\mathbb{F}_4$ such that $C' = CM = \{cM \mid c \in C\}$ [7]. A monomial matrix which maps C to itself is called an automorphism of C, and the set of all automorphisms of C forms the automorphism group $\mathrm{Aut}(C)$ of C. Clearly, ωI and $\bar{\omega} I$ are elements of $\mathrm{Aut}(C)$, where I denotes the identity matrix, so $\{I, \omega I, \bar{\omega} I\}$ is the smallest possible automorphism group. Such an automorphism group is called *trivial*.

The classification of Hermitian self-dual codes was begun by [7] and the classification for lengths up to 14 was done in [7]. The classification is extended to length 18 [2, 3]. At length 20, a classification of extremal Hermitian self-dual codes was completed [5] under a weaker equivalence (see Subsection 3.3 for the definition). At length 22, at least 46 inequivalent extremal self-dual codes are known [6]. In Table 1, the numbers $\#_d$ of inequivalent Hermitian self-dual codes with minimum weight d are given along with references.

Table 1: Hermitian self-dual codes

| n | Indecomposable | | | | Decomposable | | Total | References |
	$\#_2$	$\#_4$	$\#_6$	$\#_8$	$\#_2$	$\#_4$		
2	1						1	[7]
4	0				1		1	[7]
6	0	1			1		2	[7]
8	0	1			2		3	[7]
10	0	2			3		5	[7]
12	0	4			5	1	10	[7]
14	0	9	1		10	1	21	[7]
16	0	27	4		21	3	55	[2]
18	0	152	30	1	55	7	245	[3, 5]
20	0	2163	999	2	245	18	3427	[5], Section 3
22	0	?	?	723	3427	52	?	Section 4

The main aim of this note is to give a complete classification of Hermitian self-dual codes of length 20.

Theorem 1. *There are* 3427 *inequivalent Hermitian self-dual codes of length* 20. *Of these two are extremal,* 999 *have minimum weight* 6, 2181 *have minimum weight* 4, *and* 245 *have minimum weight* 2.

The method used, which is similar to that given in [5], is described in Subsection 2.1.

From the classification of Hermitian self-dual $[20, 10, 6]$ codes, we also classify extremal Hermitian self-dual codes of length 22.

It has been a question to determine if there is a Hermitian self-dual code with a trivial automorphism group (see [7, Open Problem (4)]). There is no Hermitian self-dual code with a trivial automorphism group for lengths up to 18 (see [2, 3, 7]). From our classification given in Section 3, we see that such a code exists at length 20.

Generator matrices of all Hermitian self-dual codes of length 20 and all extremal Hermitian self-dual codes of length 22 can be obtained electronically from [4]. All computer calculations in this note were done by MAGMA [1].

2 Preliminaries

2.1 Classification method

Here we describe a method for classifying Hermitian self-dual codes. This method is similar to that given in [5].

Suppose that C is a Hermitian self-dual $[n, n/2, d]$ code with $d \geq 4$. Define a subcode of C as follows

$$C_0 = \{(x_1, x_2, \ldots, x_n) \in C \mid x_{n-1} = x_n\}.$$

Since $C^\perp$ has no codeword of weight 2, C_0 has dimension $n/2 - 1$. We may assume that there is a codeword $x = (x_1, \ldots, x_n)$ of weight d in C with $x_{n-1} = x_n$. Then, the following code

$$C_1 = \{(x_1, x_2, \ldots, x_{n-2}) \mid (x_1, x_2, \ldots, x_n) \in C_0\}$$

is a self-dual $[n-2, n/2-1, d-2]$ code. Thus, the subcode C_0 has generator matrix of the form

$$G_0 = \begin{pmatrix} & & & \vline & a_1 & a_1 \\ & G_1 & & \vline & \vdots & \vdots \\ & & & \vline & a_{n/2-1} & a_{n/2-1} \end{pmatrix}, \tag{1}$$

where G_1 is a generator matrix of C_1 and $a_i \in \mathbb{F}_4$ $(i = 1, \ldots, n/2 - 1)$. It follows that any Hermitian self-dual $[n, n/2, d]$ code is constructed as the code $\langle C_0, x \rangle$ for some code C_0 with generator matrix of the form (1) and some vector $x \in C_0^\perp \setminus C_0$, where $\langle C_0, x \rangle$ denotes the code generated by the codewords of C_0 and x.

In this way, all Hermitian self-dual $[n, n/2, d]$ codes, which must be checked further for equivalence, are constructed, by taking generator matrices of all inequivalent self-dual $[n - 2, n/2 - 1, d - 2]$ codes as matrices G_1, and by considering $a_1 \in \{0, 1\}$ and $a_i \in \mathbb{F}_4$ $(i = 2, \ldots, n/2 - 1)$ in (1). As described in [5], the number of possibilities for a_i $(i = 2, \ldots, n/2 - 1)$ is decreased by applying elements of $\mathrm{Aut}(C_1)$ to the first $n - 2$ coordinates of (1).

2.2   Mass formula for weight enumerators

Now we give a mass formula for weight enumerators of Hermitian self-dual codes.

Lemma 2. *Let n be an even positive integer. Let $W_C(y)$ denote the weight enumerator of a code C. Then*

$$\sum_C W_C(y) = \prod_{i=0}^{n/2-1} (2^{2i+1} + 1) + \sum_{j=1}^{n/2} \binom{n}{2j} 3^{2j} \prod_{i=0}^{n/2-2} (2^{2i+1} + 1) y^{2j}, \quad (2)$$

where C runs through the set of all Hermitian self-dual codes of length n.

Proof. Let $\mathrm{wt}(x)$ denote the weight of a vector $x \in \mathbb{F}_4^n$.

$$\sum_C W_C(y)$$

$$= \sum_{j=0}^{n/2} \sum_C \#\{x \in C \mid \mathrm{wt}(x) = 2j\} y^{2j}$$

$$= \sum_{j=0}^{n/2} \sum_{\substack{x \in \mathbb{F}_4^n \\ \mathrm{wt}(x)=2j}} \#\{C \mid x \in C\} y^{2j}$$

$$= \prod_{i=0}^{n/2-1} (2^{2i+1} + 1) + \sum_{j=1}^{n/2} \sum_{\mathrm{wt}(x)=2j} \#\{C \mid x \in C\} y^{2j} \qquad \text{(by [7, Theorem 19])}$$

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$$= \prod_{i=0}^{n/2-1} (2^{2i+1} + 1) + \sum_{j=1}^{n/2} \sum_{\mathrm{wt}(x)=2j} \prod_{i=0}^{n/2-2} (2^{2i+1} + 1)y^{2j} \qquad \text{(by [7, Theorem 23])}$$

$$= \prod_{i=0}^{n/2-1} (2^{2i+1} + 1) + \sum_{j=1}^{n/2} \binom{n}{2j} 3^{2j} \prod_{i=0}^{n/2-2} (2^{2i+1} + 1)y^{2j}.$$

$\square$

Lemma 3. *Let n and d be even positive integers. Let $\mathcal{C}$ be a family of pairwise inequivalent Hermitian self-dual codes of length n with minimum weight at most d. Then $\mathcal{C}$ is a complete set of representatives for equivalence classes of Hermitian self-dual codes of length n with minimum weight at most d, if and only if*

$$\sum_{C \in \mathcal{C}} \frac{3^n n!}{\# \operatorname{Aut}(C)} \#\{x \in C \mid \mathrm{wt}(x) = d\} = \binom{n}{d} 3^d \prod_{i=0}^{n/2-2} (2^{2i+1} + 1). \qquad (3)$$

Proof. Consider the coefficient of y^d in the formula (2) in Lemma 2. $\square$

3 Classification of self-dual codes of length 20

In this section, we give a classification of Hermitian self-dual codes of length 20.

3.1 Decomposable codes

We first consider decomposable Hermitian self-dual codes. By Theorem 28 in [7], any Hermitian self-dual code with minimum weight 2 is decomposable as $C_2 \oplus C_{18}$, where C_2 is the unique Hermitian self-dual code of length 2 and C_{18} is some Hermitian self-dual code of length 18. Hence, from Table 1, there are 245 inequivalent Hermitian self-dual codes with minimum weight 2. Every decomposable Hermitian self-dual code with minimum weight 4 is a direct sum of indecomposable codes of length at least 6, since there are no Hermitian self-dual code with minimum weight at least 4 for lengths less than 6. From Table 1, the numbers of indecomposable Hermitian self-dual codes of lengths $6, 8, 10, 12, 14$ are $1, 1, 2, 4, 10$, respectively. It follows that the numbers of Hermitian self-dual codes of the forms $C_6 \oplus C_6 \oplus C_8$, $C_6 \oplus C_{14}$,

$C_8 \oplus C_{12}$ or $C_{10} \oplus C_{10}$, where C_n denotes an indecomposable code of length n, are $1, 10, 4, 3$, respectively. Therefore, there are 18 inequivalent decomposable Hermitian self-dual codes with minimum weight 4. There is no decomposable Hermitian self-dual code with minimum weight $d \geq 6$.

3.2 Indecomposable codes

From the set of inequivalent Hermitian self-dual $[18, 9, 2]$ codes given in [3], the method given in Subsection 2.1 allows to enumerate the set $\mathcal{C}_{20,4}$ of the 2181 inequivalent Hermitian self-dual $[20, 10, 4]$ codes. Let $\mathcal{C}_{20,2}$ denote the set of the 245 inequivalent Hermitian self-dual codes of length 20 and minimum weight 2. Setting $\mathcal{C} = \mathcal{C}_{20,2} \cup \mathcal{C}_{20,4}$ in Lemma 3, one can verify that there is no other Hermitian self-dual $[20, 10, 4]$ code, by calculating the summand in the right-hand side of (3) for all codes of $\mathcal{C}$. Similarly, we found the set of 999 (resp. 2) inequivalent Hermitian self-dual codes with minimum weight 6 (resp. 8). In this way, we found the set $\mathcal{C}_{20}$ of 3182 inequivalent Hermitian self-dual codes with minimum weight $d \geq 4$ satisfying

$$\sum_{C \in \mathcal{C}_{20} \cup \mathcal{C}_{20,2}} \frac{3^{20} \cdot 20!}{\# \operatorname{Aut}(C)} = 2229034892015508532492061011707,$$

which is the constant term of (2). This constant term $\prod_{i=0}^{9}(2^{2i+1} + 1)$ gives the number of the distinct self-dual codes of length 20. The mass formula shows that there is no other Hermitian self-dual code of length 20 and the classification is complete. Therefore, we have Theorem 1.

This computation was performed in MAGMA [1]. In principle, such a computation can be done by classifying Hermitian self-dual codes by the MAGMA function `IsIsomorphic`, then their automorphism groups can be calculated by `AutomorphismGroup`. The orders of the automorphism groups of the 3427 codes are listed in Table 2, where $(\# \operatorname{Aut}, N(\# \operatorname{Aut}))$ lists the number $N(\# \operatorname{Aut})$ of the codes with an automorphism group of order $\# \operatorname{Aut}$.

Alternatively, the set of 3182 inequivalent Hermitian self-dual codes of length 20 and minimum weight $d \geq 4$ can be found by a method similar to the one given in [3] as follows. Recall that two self-dual codes C and C' of length n are said to be *neighbors* if the dimension of $C \cap C'$ is $n/2-1$. Let D_{20} be the extremal Hermitian self-dual code of length 20 generated by the second generator matrix in [5, Fig. 4]. Let $N_0 = \{D_{20}\}$ and $N_{i+1} = \bigcup_{C \in N_i} \operatorname{Nei}(C)$

$(i = 0, 1, 2, 3)$, where $\mathrm{Nei}(C)$ denotes the set of inequivalent Hermitian self-dual neighbors with minimum weight $d \geq 4$ of C. Then the set

$$N_0 \cup N_1 \cup N_2 \cup N_3 \cup N_4,$$

contains 3182 inequivalent Hermitian self-dual codes of length 20 and minimum weight $d \geq 4$.

3.3 A weaker equivalence

In the above classification, we employ monomial matrices over $\mathbb{F}_4$ in the definition for equivalence of codes. To define a weaker equivalence, one could consider a conjugation γ of $\mathbb{F}_4$ sending each element to its square, in the definition of equivalence, that is, two codes C and C' are weakly equivalent if there is some monomial matrix M over $\mathbb{F}_4$ such that $C' = CM$ or $C' = CM\gamma$ (see [5]). In fact, the classification of extremal self-dual codes of length 20 in [5] was done under the weaker equivalence. Our classification shows that the equivalence classes of such codes are the same under both definitions.

We have verified that there are $15, 636$ and 323 pairs of self-dual codes with minimum weights $2, 4$ and 6, respectively, under the weaker equivalence. Hence, there are $3427 - (15 + 636 + 323) = 2453$ self-dual codes under the weaker equivalence for length 20.

4 Classification of extremal self-dual codes of length 22

From the set of inequivalent Hermitian self-dual $[20, 10, 6]$ codes classified in the previous section, the method given in Subsection 2.1 allows to enumerate extremal Hermitian self-dual codes of length 22.

For every self-dual $[20, 10, 6]$ code, we have verified that the subcode generated by codewords of weight 6 has dimension at least 4. Thus, we may assume that the first four rows of a generator matrix G_1 in (1) have weight 6. This yields that $a_1 = 1$ and $a_i \in \{1, \omega, \bar{\omega}\}$ $(i = 2, 3, 4)$. From the 999 self-dual $[20, 10, 6]$ codes, 723 inequivalent extremal self-dual codes of length 22 are obtained. Therefore, we have the following:

Proposition 4. *There are* 723 *inequivalent extremal Hermitian self-dual codes of length* 22.

Table 2: Orders of the automorphism groups for length 20

d	$(\#\,\mathrm{Aut},\,N(\#\,\mathrm{Aut}))$
8	(4320, 1), (5760, 1)
6	(3, 419), (6, 328), (9, 13), (12, 103), (15, 1), (18, 26), (24, 15), (27, 1), (30, 5), (36, 33), (48, 9), (54, 6), (60, 3), (72, 11), (96, 2), (108, 4), (144, 3), (192, 3), (216, 2), (288, 4), (360, 2), (384, 1), (576, 1), (1440, 1), (2160, 1), (3456, 1), (5760, 1)
4	(12, 377), (24, 258), (36, 22), (48, 363), (72, 38), (96, 134), (144, 41), (192, 176), (216, 4), (240, 3), (288, 60), (384, 82), (432, 13), (576, 31), (720, 1), (768, 44), (864, 9), (1008, 1), (1080, 2), (1152, 92), (1296, 3), (1440, 3), (1536, 30), (1728, 9), (2160, 3), (2304, 42), (2592, 1), (2880, 3), (3072, 6), (3456, 9), (3600, 1), (4320, 2), (4608, 31), (5184, 3), (5760, 4), (6144, 1), (6480, 1), (6912, 42), (7560, 2), (8640, 1), (9216, 22), (10368, 3), (10800, 1), (11520, 3), (12288, 1), (13824, 18), (15360, 1), (16128, 1), (17280, 9), (18432, 3), (20736, 1), (23040, 4), (24576, 1), (25920, 1), (27648, 9), (34560, 1), (36288, 1), (36864, 2), (39312, 1), (41472, 8), (48384, 2), (51840, 1), (55296, 21), (69120, 8), (82944, 7), (92160, 1), (96768, 1), (103680, 3), (110592, 3), (124416, 1), (138240, 3), (145152, 1), (165888, 4), (184320, 1), (193536, 1), (207360, 4), (221184, 4), (248832, 2), (290304, 1), (331776, 3), (345600, 1), (368640, 2), (387072, 1), (414720, 2), (518400, 1), (552960, 3), (580608, 2), (622080, 2), (663552, 5), (725760, 1), (777600, 1), (829440, 5), (884736, 2), (1105920, 2), (1161216, 1), (1244160, 1), (1382400, 1), (1658880, 1), (1866240, 1), (2073600, 2), (2488320, 1), (3110400, 1), (3538080, 1), (3732480, 1), (4064256, 1), (4147200, 2), (4354560, 1), (5806080, 1), (6635520, 1), (7464960, 1), (10886400, 1), (11612160, 1), (13271040, 1), (13934592, 1), (14929920, 1), (15552000, 1), (22118400, 1), (23224320, 1), (37324800, 1), (41472000, 1), (41803776, 1), (58060800, 1), (66355200, 1), (121927680, 1), (124416000, 1), (182891520, 1), (278691840, 1), (279936000, 1), (933120000, 1), (1045094400, 1), (5573836800, 1), (9405849600, 1)
2	(36, 1), (72, 5), (108, 1), (144, 9), (162, 1), (216, 6), (288, 4), (324, 2), (432, 7), (576, 8), (864, 5), (1080, 2), (1152, 9), (1296, 2), (1728, 10), (2304, 6), (2592, 1), (3024, 1), (3456, 3), (3888, 1), (4608, 2), (5184, 2), (6480, 1), (6912, 13), (7776, 1), (9216, 1), (10368, 3), (13824, 6), (17496, 1), (18432, 1), (20736, 3), (25920, 1), (27648, 4), (31104, 1), (34560, 1), (36864, 1), (41472, 7), (55296, 1), (62208, 2), (72576, 1), (77760, 1), (82944, 8), (84240, 1), (103680, 2), (110592, 3), (124416, 2), (139968, 1), (146880, 1), (165888, 4), (207360, 2), (248832, 1), (290304, 1), (311040, 1), (331776, 4), (373248, 1), (414720, 1), (497664, 3), (622080, 1), (746496, 1), (829440, 2), (870912, 1), (995328, 3), (1244160, 3), (1327104, 3), (1492992, 4), (2239488, 1), (2488320, 1), (3110400, 1), (3317760, 1), (3732480, 1), (3981312, 1), (4245696, 1), (4354560, 1), (4478976, 1), (5806080, 1), (6635520, 1), (6967296, 1), (8957952, 1), (9953280, 2), (11943936, 1), (12192768, 1), (15925248, 1), (17915904, 1), (18662400, 1), (22394880, 1), (24883200, 3), (26127360, 1), (37324800, 1), (44789760, 1), (67184640, 1), (69984000, 1), (107495424, 1), (139345920, 1), (195955200, 1), (219469824, 1), (322486272, 1), (335923200, 1), (447897600, 2), (522547200, 1), (940584960, 1), (1114767360, 1), (1254113280, 1), (1672151040, 1), (1679616000, 1), (2149908480, 1), (2341011456, 1), (5374771200, 1), (5643509760, 1), (20155392000, 1), (45349632000, 1), (72559411200, 1), (135444234240, 1), (1523747635200, 1), (219419659468800, 1)

The orders of the automorphism groups of the 723 codes are listed in Table 3, where $(\#\operatorname{Aut}, N(\#\operatorname{Aut}))$ lists the number $N(\#\operatorname{Aut})$ of the codes with an automorphism group of order $\#\operatorname{Aut}$. The code with an automorphism group of order 1330560 is equivalent to the code $C_{22,P1}$ in [8, Table 2], which is one of the three inequivalent pure double circulant extremal self-dual codes of length 22. We have verified that the automorphism group is isomorphic to the direct product of the Mathieu group M_{22} of degree 22 and the cyclic group of order 3.

Table 3: Orders of the automorphism groups for length 22

$(\#\operatorname{Aut}, N(\#\operatorname{Aut}))$
(3, 308), (6, 229), (9, 8), (12, 73), (18, 12), (24, 39), (30, 2), (36, 19), (48, 1), (60, 1), (66, 2), (72, 5), (108, 1), (120, 1), (180, 3), (192, 1), (240, 1), (288, 4), (324, 1), (360, 1), (384, 2), (504, 2), (864, 2), (1728, 2), (17280, 2), (1330560, 1)

We have verified that there are 301 pairs of extremal self-dual codes of length 22, under the weaker equivalence given in Subsection 3.3. Hence, there are $723 - 301 = 422$ extremal self-dual codes of length 22, under the weaker equivalence. We have also verified that the 422 extremal self-dual codes have different numbers $(B_0, B_1, \ldots, B_{22})$, where B_j denotes the number of distinct cosets of weight j. This shows that the 422 extremal self-dual codes are certainly inequivalent.

The smallest possible automorphism group is of order 3. From the constant term of (2), the number of inequivalent self-dual codes of length 22 is at least

$$\frac{\prod_{i=0}^{10}(2^{2i+1}+1)}{22! \cdot 3^{21}} > 397588.$$

5   Trivial automorphism groups

It has been a question to determine if there is a Hermitian self-dual code with a trivial automorphism group (see [7, Open Problem (4)]). There is no Hermitian self-dual code with a trivial automorphism group for lengths up to 18 (see [2, 3, 7]). From our classification given in Section 3, we see that such a code exists at length 20 (see Table 2). Hence, we have an answer to the above problem.

Corollary 5. *The smallest length for which there is a quaternary Hermitian self-dual code with a trivial automorphism group is* 20.

We describe here an example of a code satisfying the conditions of Corollary 5. Let B_{20} be the pure double circulant code with generator matrix $(\,I\,,\,R\,)$, where R is the circulant matrix with first row $(\bar{\omega}, 0, \omega, 0, \omega, 0, \bar{\omega}, 1, \omega, 1)$. The following code

$$C_{20} = \langle B_{20} \cap \langle v \rangle^{\perp}, v \rangle$$

where $v = (\omega, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, \bar{\omega}, 0, 0, 1, 0, \omega, \bar{\omega}, 1, \bar{\omega})$, is a Hermitian self-dual code of length 20 with a trivial automorphism group.

In addition, there is no extremal Hermitian self-dual code with a trivial automorphism group for length 20, but there are 308 extremal Hermitian self-dual codes with trivial automorphism groups for length 22 (see Tables 2 and 3). Hence, the smallest length for which there is a quaternary extremal Hermitian self-dual code with a trivial automorphism group is 22. For example, the code with generator matrix $(\,I\,,\,M\,)$ is an extremal self-dual $[22, 11, 8]$ code with a trivial automorphism group, where M is given in Figure 1.

$$M = \begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & \omega & \bar{\omega} & \omega & \bar{\omega} & 0 \\
\omega & \omega & 0 & \omega & 1 & 0 & \omega & \omega & \bar{\omega} & \omega & 1 \\
\bar{\omega} & \omega & \bar{\omega} & 0 & 0 & \bar{\omega} & \omega & \bar{\omega} & \bar{\omega} & 1 & \omega \\
\bar{\omega} & 0 & 1 & 1 & 0 & \omega & \bar{\omega} & 1 & \omega & \omega & \omega \\
\omega & \omega & 0 & \omega & \bar{\omega} & 1 & \bar{\omega} & \bar{\omega} & \bar{\omega} & 0 & \bar{\omega} \\
1 & \bar{\omega} & 1 & 0 & 0 & \omega & 0 & \bar{\omega} & 0 & \bar{\omega} & 1 \\
0 & 1 & \bar{\omega} & 0 & 0 & \omega & 1 & 0 & 1 & \omega & \bar{\omega} \\
\omega & 0 & \bar{\omega} & \bar{\omega} & \bar{\omega} & 1 & \bar{\omega} & 0 & \omega & \bar{\omega} & \bar{\omega} \\
1 & \omega & \omega & \omega & 1 & 1 & 0 & \bar{\omega} & 1 & 0 & \omega \\
1 & \omega & \bar{\omega} & \omega & \bar{\omega} & 1 & 1 & \bar{\omega} & 0 & \omega & 0 \\
0 & 0 & \omega & \bar{\omega} & \bar{\omega} & \omega & 1 & 1 & 1 & \omega & \omega
\end{pmatrix}$$

Figure 1: An extremal self-dual $[22, 11, 8]$ code with a trivial automorphism group

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