ON THE EXISTENCE OF EXTREMAL TYPE II $\mathbb{Z}_{2k}$-CODES

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In memory of Boris Venkov

Abstract. For lengths $8, 16$ and 24, it is known that there is an extremal Type II $\mathbb{Z}_{2k}$-code for every positive integer k. In this paper, we show that there is an extremal Type II $\mathbb{Z}_{2k}$-code of lengths $32, 40, 48, 56$ and 64 for every positive integer k. For length 72, it is also shown that there is an extremal Type II $\mathbb{Z}_{4k}$-code for every positive integer k with $k \geq 2$.

1. Introduction

As described in [29], self-dual codes are an important class of linear codes for both theoretical and practical reasons. It is a fundamental problem to classify self-dual codes of modest lengths and determine the largest minimum weight among self-dual codes of that length. For binary doubly even self-dual codes, much work has been done concerning this fundamental problem (see [7, 16, 29]). Binary doubly even self-dual codes are often called Type II codes. For general k, Type II $\mathbb{Z}_{2k}$-codes were defined in [2] as self-dual codes with the property that all Euclidean weights are divisible by $4k$, where $\mathbb{Z}_m$ is the ring of integers modulo m. By Construction A, Type II $\mathbb{Z}_{2k}$-codes give even unimodular lattices. It follows that a Type II $\mathbb{Z}_{2k}$-code of length n exists if and only if n is divisible by eight (see [2]).

Let C be a Type II $\mathbb{Z}_{2k}$-code of length $n \equiv 0 \pmod 8$. If $n \leq 136$ then we have the following bound on the minimum Euclidean weight $d_E(C)$ of C:

$$d_E(C) \leq 4k \left\lfloor \frac{n}{24} \right\rfloor + 4k, \tag{1.1}$$

for every positive integer k [2, 3, 13, 21]. We say that a Type II $\mathbb{Z}_{2k}$-code meeting the bound (1.1) with equality is *extremal* for length $n \leq 136$.

The existence of an extremal Type II $\mathbb{Z}_{2k}$-code of length $n \leq 64$ ($n \equiv 0 \pmod 8$) is known for $k = 1, 2, \dots, 6$ (see [13, Table 1]). The existence of a binary extremal Type II code of length 72 is a long-standing open question (see [16, 29]). Moreover, the existence of an extremal Type II $\mathbb{Z}_{2k}$-code of length 72 is not known for any positive integer k. It is well known that the binary extended Golay code is a binary extremal Type II code of length 24. For every positive integer $k \geq 2$, it was shown in [5, 10] that the Leech lattice contains a $2k$-frame[1]. Hence, there is an extremal Type II $\mathbb{Z}_{2k}$-code of length 24 for every positive integer k. This motivates our investigation of the existence of an extremal Type II $\mathbb{Z}_{2k}$-code of larger lengths. The main aim of this paper is to establish the existence of an extremal Type II $\mathbb{Z}_{2k}$-code of lengths $32, 40, 48, 56$ and 64 for every positive integer k. This result yields that if

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[1]Recently, the second author [22] has shown that the odd Leech lattice contains a k-frame for every positive integer k with $k \geq 3$ by the approach which is similar to that in [5, 10].

L is an extremal even unimodular lattice with theta series $\theta_L(q) = \sum_{m=0}^{\infty} A_{2m} q^{2m}$ for dimensions $n = 32, 40, 48, 56$ and 64 then $A_{2m} > 0$ for $m \geq \lfloor \frac{n}{24} \rfloor + 1$. For length 72, it is also shown that there is an extremal Type II $\mathbb{Z}_{4k}$-code for every positive integer k with $k \geq 2$. It is known that a unimodular lattice L contains a k-frame if and only if there exists a self-dual $\mathbb{Z}_k$-code C such that L is isomorphic to the lattice obtained from C by Construction A. The powerful tool in the study of this paper is to consider the existence of $2k$-frames in some extremal even unimodular lattices.

This paper is organized as follows. In Section 2, we give definitions and some basic properties of self-dual codes, weighing matrices and unimodular lattices used in this paper. In Section 3, we provide methods for constructing m-frames in unimodular lattices, which are constructed from some self-dual $\mathbb{Z}_k$-codes by Construction A. Using the theory of modular forms (see [23] for details), we also derive number theoretical results to give infinite families of m-frames based on the above methods (Theorems 3.8 and 3.9). In Section 4, by the approach, which is similar to that used in [5, 10], we show the existence of an extremal Type II $\mathbb{Z}_{2k}$-code of lengths 32 and 40 for every positive integer k (Theorem 4.3). This is done by finding $2k$-frames in some extremal even unimodular lattices constructed from extremal Type II $\mathbb{Z}_4$-codes, along with constructing extremal Type II $\mathbb{Z}_{22}$-codes. In Sections 5, 6 and 7, for every positive integer k, we show the existence of an extremal Type II $\mathbb{Z}_{2k}$-code of lengths $48, 56$ and 64, respectively (Theorems 5.5, 6.6 and 7.5). Our approach in Sections 5, 6 and 7 is similar to that in Section 4, but we have to find self-dual $\mathbb{Z}_k$-codes ($k = 3, 5$), instead of extremal Type II $\mathbb{Z}_4$-codes, since extremal even unimodular lattices have minimum norm 6. Hence, some number theoretical results established in Section 3 are required for these lengths. The first example of an extremal Type II $\mathbb{Z}_{2k}$-code of length n is also explicitly found for $(n, 2k) = (48, 14), (48, 46), (56, 14), (56, 34), (56, 46), (64, 14)$ and $(64, 46)$. Some of examples are used to complete the proofs of Theorems 5.5 and 6.6. In Section 8, it is shown that there is an extremal Type II $\mathbb{Z}_{4k}$-code of length 72 for every positive integer k with $k \geq 2$ (Theorem 8.1), by finding a $4k$-frame in the extremal even unimodular lattice in dimension 72, which has been recently found by Nebe [25]. Finally, in Section 9, we discuss the positivity of coefficients of theta series of extremal even unimodular lattices.

Most of computer calculations in this paper were done by Magma [4].

2. Preliminaries

In this section, we give definitions and some basic properties of self-dual codes, weighing matrices and unimodular lattices used in this paper.

2.1. **Self-dual codes.**

Let $\mathbb{Z}_k$ be the ring of integers modulo k, where k is a positive integer. In this paper, we always assume that $k \geq 2$ and we take the set $\mathbb{Z}_k$ to be $\{0, 1, \dots, k-1\}$. A $\mathbb{Z}_k$-code C of length n (or a code C of length n over $\mathbb{Z}_k$) is a $\mathbb{Z}_k$-submodule of $\mathbb{Z}_k^n$. A $\mathbb{Z}_2$-code and a $\mathbb{Z}_3$-code are called binary and ternary, respectively. A $\mathbb{Z}_k$-code C is *self-dual* if $C = C^\perp$, where the dual code $C^\perp$ of C is defined as $C^\perp = \{x \in \mathbb{Z}_k^n \mid x \cdot y = 0 \text{ for all } y \in C\}$ under the standard inner product $x \cdot y$. The Euclidean weight of a codeword $x = (x_1, \dots, x_n)$ of C is $\sum_{\alpha=1}^{\lfloor k/2 \rfloor} n_\alpha(x)\alpha^2$, where $n_\alpha(x)$ denotes the number of components i with $x_i \equiv \pm\alpha$

(mod k) ($\alpha = 1, 2, \dots, \lfloor k/2 \rfloor$). The minimum Euclidean weight $d_E(C)$ of C is the smallest Euclidean weight among all nonzero codewords of C.

Binary doubly even self-dual codes have been widely studied (see [7, 16, 29]). Binary doubly even self-dual codes are often called Type II codes. For general k, *Type II* $\mathbb{Z}_{2k}$-codes were defined in [2] as self-dual codes with the property that all Euclidean weights are divisible by $4k$ (see [3] and [15] for $\mathbb{Z}_4$-codes). Type II $\mathbb{Z}_{2k}$-codes are an important class of self-dual $\mathbb{Z}_{2k}$-codes, since Type II $\mathbb{Z}_{2k}$-codes give even unimodular lattices by Construction A. It is known that a Type II $\mathbb{Z}_{2k}$-code of length n exists if and only if n is divisible by eight [2].

Let C be a Type II $\mathbb{Z}_{2k}$-code of length $n \equiv 0 \pmod 8$. The bound (1.1) is established for $k = 1$ [21], for $k = 2$ [3], for $k = 3, 4, 5, 6$ [13]. For $k \geq 3$, the bound (1.1) is known under the assumption that $\lfloor n/24 \rfloor \leq k - 2$ [2]. Therefore, if $n \leq 136$ then we have (1.1) for every positive integer k. We say that a Type II $\mathbb{Z}_{2k}$-code of length $n \leq 136$ meeting the bound (1.1) with equality is *extremal*. The existence of an extremal Type II $\mathbb{Z}_{2k}$-code of length $n \leq 64$ ($n \equiv 0 \pmod 8$) is known for $k = 1, 2, \dots, 6$ (see [13, Table 1]). For lengths $8, 16$ and 24, it is known that there is an extremal Type II $\mathbb{Z}_{2k}$-code for every positive integer k.

2.2. **Weighing matrices and negacirculant matrices.** A *weighing matrix* W of order n and weight k is an $n \times n$ $(1, -1, 0)$-matrix W such that $WW^T = kI$, where I is the identity matrix and W^T denotes the transpose of W (see [9] for details of weighing matrices). A weighing matrix W is called *skew-symmetric* if $W^T = -W$.

In this paper, to construct self-dual $\mathbb{Z}_k$-codes, we use weighing matrices M of order n and weight $\equiv -1 \pmod k$ and $n \times n$ $(0, \pm 1, \pm 2, \dots, \pm \lfloor k/2 \rfloor)$-matrices M with $MM^T = mI$ and $m \equiv -1 \pmod k$ as follows. Let $C_k(M)$ be the $\mathbb{Z}_k$-code of length $2n$ with generator matrix $\begin{pmatrix} I & M \end{pmatrix}$, where the entries of the matrix are regarded as elements of $\mathbb{Z}_k$. Then it is easy to see that $C_k(M)$ is self-dual.

An $n \times n$ matrix M is circulant and *negacirculant* if M has the following form:

$$\begin{pmatrix} r_0 & r_1 & \cdots & r_{n-1} \\ r_{n-1} & r_0 & \cdots & r_{n-2} \\ r_{n-2} & r_{n-1} & \cdots & r_{n-3} \\ \vdots & \vdots & & \vdots \\ r_1 & r_2 & \cdots & r_0 \end{pmatrix} \text{ and } \begin{pmatrix} r_0 & r_1 & \cdots & r_{n-1} \\ -r_{n-1} & r_0 & \cdots & r_{n-2} \\ -r_{n-2} & -r_{n-1} & \cdots & r_{n-3} \\ \vdots & \vdots & & \vdots \\ -r_1 & -r_2 & \cdots & r_0 \end{pmatrix},$$

respectively. Most of matrices including weighing matrices constructed in this paper are based on negacirculant matrices. For example, in order to construct self-dual $\mathbb{Z}_k$-codes of length $4n$, we often consider a generator matrix of the form:

$$\begin{pmatrix} & I & & A & B \\ & & & -B^T & A^T \end{pmatrix}, \tag{2.1}$$

where A and B are $n \times n$ negacirculant matrices. It is easy to see that the code is self-dual if $AA^T + BB^T = -I$.

2.3. **Unimodular lattices.** A (Euclidean) lattice $L \subset \mathbb{R}^n$ in dimension n is *unimodular* if $L = L^*$, where the dual lattice L^* of L is defined as $\{x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L\}$ under the standard inner product (x, y). Two lattices L and L' are *isomorphic*, denoted $L \cong L'$, if there exists an orthogonal matrix A with $L' = L \cdot A$. The norm of a vector x is defined as (x, x). The minimum norm $\min(L)$ of a unimodular lattice L is the smallest norm among all nonzero vectors of L. A unimodular lattice with even norms is said to be *even*, and that containing a vector of odd norm is said to be odd. An odd unimodular lattice exists for every dimension. Indeed, $\mathbb{Z}^n$ is an odd unimodular lattice in dimension n.

Let L be an even unimodular lattice in dimension n. Denote by $\theta_L(q) = \sum_{x\in L} q^{(x,x)} = 1 + \sum_{m=1}^{\infty} A_{2m}q^{2m}$ the theta series of L, that is, A_{2m} is the number of vectors of norm $2m$ in L. In this subsection, we simply consider the theta series as formal power series. Then there exist integers $a_0 = 1, a_1, \dots, a_{\lfloor \frac{n}{24} \rfloor}$ so that

$$\theta_L(q) = \sum_{r=0}^{\lfloor \frac{n}{24} \rfloor} a_r E_4(q)^{\frac{n}{8}-3r}\Delta(q)^r, \tag{2.2}$$

where $E_4(q) = 1+240\sum_{m=1}^{\infty} \sigma_3(m)q^{2m}$, $\Delta(q) = q^2\prod_{m=1}^{\infty}(1-q^{2m})^{24}$, and $\sigma_3(m) = \sum_{0<d|m} d^3$ [7, p. 193] (see also [27, Proposition 15]). As a consequence of (2.2), an even unimodular lattice in dimension n exists if and only if $n \equiv 0 \pmod 8$. It was shown in [30] that the coefficient $A_{2\lfloor \frac{n}{24} \rfloor+2}$ is always positive when $A_2 = A_4 = \cdots = A_{2\lfloor \frac{n}{24} \rfloor} = 0$ (see also [20]). Hence, the minimum norm $\min(L)$ of L is bounded by

$$\min(L) \le 2\left\lfloor \frac{n}{24} \right\rfloor + 2.$$

We say that an even unimodular lattice meeting the upper bound is *extremal*. It follows that the theta series of an extremal even unimodular lattice in each dimension is uniquely determined (see [7, p. 193]).

The existence of an extremal even unimodular lattice is known for dimension $n \le 80$ and $n \equiv 0 \pmod 8$ (see [26]). The existence of an extremal even unimodular lattice in dimension 72 was a long-standing open question. Recently, the first example of such a lattice has been found by Nebe [25]. On the other hand, Mallows, Odlyzko and Sloane [20] showed that there is no extremal even unimodular lattice in dimension n for all sufficiently large n by verifying that $A_{2\lfloor \frac{n}{24} \rfloor+4}$ is negative. Recently, it has been shown in [17] that there is no extremal even unimodular lattice in dimension n with $n > 163264$ by verifying that the largest n for which all A_i are non-negative is 163264.

Two lattices L and L' are *neighbors* if both lattices contain a sublattice of index 2 in common. Let L be an odd unimodular lattice and let L_0 denote its sublattice of vectors of even norms. Then L_0 is a sublattice of L of index 2. $L_0^* \setminus L$ is called the shadow S of L [6]. There are cosets L_1, L_2, L_3 of L_0 such that $L_0^* = L_0 \cup L_1 \cup L_2 \cup L_3$, where $L = L_0 \cup L_2$ and $S = L_1 \cup L_3$. If L is an odd unimodular lattice in dimension divisible by eight, then L has two even unimodular neighbors of L, namely, $L_0 \cup L_1$ and $L_0 \cup L_3$.

2.4. **Construction A and k-frames.** We give a method to construct unimodular lattices from self-dual $\mathbb{Z}_k$-codes, which is referred to as *Construction A* (see [2, 14]). Let ρ be a map from $\mathbb{Z}_k$ to $\mathbb{Z}$ sending $0, 1, \dots, k-1$ to $0, 1, \dots, k-1$, respectively. If C is a self-dual $\mathbb{Z}_k$-code of length n, then the lattice

$$A_k(C) = \frac{1}{\sqrt{k}}\{\rho(C) + k\mathbb{Z}^n\}$$

is a unimodular lattice in dimension n, where $\rho(C) = \{(\rho(c_1), \dots, \rho(c_n)) \mid (c_1, \dots, c_n) \in C\}$. The minimum norm of $A_k(C)$ is $\min\{k, d_E(C)/k\}$. Moreover, C is a Type II $\mathbb{Z}_{2k}$-code if and only if $A_{2k}(C)$ is an even unimodular lattice [2].

A set $\{f_1, \dots, f_n\}$ of n vectors $f_1, \dots, f_n$ of a unimodular lattice L in dimension n with $(f_i, f_j) = k\delta_{i,j}$ is called a *k-frame* of L, where $\delta_{i,j}$ is the Kronecker delta.

It is known that a unimodular lattice L contains a k-frame if and only if there exists a self-dual $\mathbb{Z}_k$-code C with $A_k(C) \cong L$. In addition, an even unimodular lattice L contains a $2k$-frame if and only if there exists a Type II $\mathbb{Z}_{2k}$-code C with $A_{2k}(C) \cong L$ (see [5, 14]). Hence, we directly have the following lemma.

Lemma 2.1. *Suppose that $n \le 136$ and $k \ge \lfloor n/24 \rfloor + 1$. There is an extremal even unimodular lattice in dimension n containing a $2k$-frame if and only if there is an extremal Type II $\mathbb{Z}_{2k}$-code of length n.*

3. Methods for constructing k-frames

By the following lemma, it is enough to consider a p-frame (resp. $2p$-frame) of an odd (resp. even) unimodular lattice for each prime p.

Lemma 3.1 (Chapman [5, Lemma 5.1]). *If a lattice L in dimension $n \equiv 0 \pmod 4$ contains a k-frame, then L contains a km-frame for every positive integer m.*

As a consequence, we have the following:

Lemma 3.2. *If an odd unimodular lattice L in dimension $n \equiv 0 \pmod 8$ contains a k-frame, then both even unimodular neighbors $L_0 \cup L_1$ and $L_0 \cup L_3$ contain a $2k$-frame.*

Chapman [5] showed that the Leech lattice Λ_{24} contains a $2k$-frame for every positive integer k with $k \ge 2$ and $k \ne 11^\ell$, where ℓ is a positive integer. This was established from a construction of $2k$-frames of Λ_{24}, which is the case $(n, m) = (12, 11)$ of Proposition 3.3, using some extremal Type II $\mathbb{Z}_4$-code C with $A_4(C) \cong \Lambda_{24}$, along with Lemma 3.1 and Theorem 3.5.

Proposition 3.3. *Let W be a skew-symmetric weighing matrix of order $n \equiv 0 \pmod 4$ and weight $m \equiv 3 \pmod 8$. Let $\tilde{C}_4(W)$ be the Type II $\mathbb{Z}_4$-code of length $2n$ with generator matrix $\begin{pmatrix} I & W + 2I \end{pmatrix}$, where the entries of the matrix are regarded as elements of $\mathbb{Z}_4$. Let a, b, c and d be integers with $c \equiv 2a+b \pmod 4$ and $d \equiv a+2b \pmod 4$. Then the set of $2n$ rows of the following matrix*

$$\tilde{F}(W) = \frac{1}{2}\begin{pmatrix} aI + bW & cI + dW \\ -cI + dW & aI - bW \end{pmatrix}$$

forms a $\frac{1}{4}(a^2 + mb^2 + c^2 + md^2)$-frame of the even unimodular lattice $A_4(\tilde{C}_4(W))$.

Proof. By [11, Proposition 4], $\tilde{C}_4(W)$ is Type II. Then $A_4(\tilde{C}_4(W))$ is an even unimodular lattice. Since $\tilde{C}_4(W)$ is self-dual and W is skew-symmetric, the matrix $\begin{pmatrix} W + 2I & I \end{pmatrix}$, which is a parity-check matrix, is also a generator matrix of $\tilde{C}_4(W)$. Hence, it follows from $c \equiv 2a + b \pmod 4$ and $d \equiv a + 2b \pmod 4$ that the rows of $\tilde{F}(W)$ are vectors of $A_4(\tilde{C}_4(W))$. Since $\tilde{F}(W)\tilde{F}(W)^T = \frac{1}{4}(a^2+mb^2+c^2+md^2)I$, the set of the $2n$ rows of $\tilde{F}(W)$ forms a $\frac{1}{4}(a^2+mb^2+c^2+md^2)$-frame of $A_4(\tilde{C}_4(W))$. $\square$

Remark 3.4. It follows from the assumption that $a^2 + mb^2 + c^2 + md^2 \equiv 0 \pmod 8$.

Theorem 3.5 (Chapman [5, Theorem 5.2]). *There are integers a, b, c and d satisfying $c \equiv 2a + b \pmod 4$, $d \equiv a + 2b \pmod 4$ and $2p = \frac{1}{4}(a^2 + 11b^2 + c^2 + 11d^2)$ for each odd prime $p \ne 11$.*

For dimensions $2n = 32$ and 40, we are able to employ the approach which is similar to that in [5, 10], by considering skew-symmetric weighing matrices of order n and weight 11 (Section 4).

On the other hand, for dimensions $48, 56$, and 64, no $\mathbb{Z}_4$-code gives an extremal even unimodular lattice by Construction A. Hence, we consider extremal even unimodular lattices which are even unimodular neighbors of some odd unimodular lattices constructed from self-dual $\mathbb{Z}_k$-codes ($k = 3, 5$) (Sections 5, 6 and 7). To do this, we provide the following modification of Proposition 3.3.

Proposition 3.6. *Let k be a positive integer with $k \geq 2$. Let M be an $n \times n$ $(0, \pm 1, \pm 2, \ldots, \pm\lfloor k/2 \rfloor)$-matrix satisfying $M^T = -M$ and $MM^T = mI$, where $m \equiv -1 \pmod k$. Let $C_k(M)$ be the self-dual $\mathbb{Z}_k$-code of length $2n$ with generator matrix $\begin{pmatrix} I & M \end{pmatrix}$, where the entries of the matrix are regarded as elements of $\mathbb{Z}_k$. Let a, b, c and d be integers with $a \equiv d \pmod k$ and $b \equiv c \pmod k$. Then the set of $2n$ rows of the following matrix*

$$F(M) = \frac{1}{\sqrt{k}} \begin{pmatrix} aI + bM & cI + dM \\ -cI + dM & aI - bM \end{pmatrix}$$

forms a $\frac{1}{k}(a^2 + mb^2 + c^2 + md^2)$-frame of the unimodular lattice $A_k(C_k(M))$.

Proof. Since $MM^T = mI$ with $m \equiv -1 \pmod k$, $C_k(M)$ is a self-dual $\mathbb{Z}_k$-code of length $2n$. Thus, $A_k(C_k(M))$ is a unimodular lattice. Since $C_k(M)$ is self-dual and $M^T = -M$, the matrix $\begin{pmatrix} M & I \end{pmatrix}$ is also a generator matrix of $C_k(M)$. Hence, it follows from $a \equiv d \pmod k$ and $b \equiv c \pmod k$ that all rows of the matrix $F(M)$ are vectors of $A_k(C_k(M))$. Since $F(M)F(M)^T = \frac{1}{k}(a^2 + mb^2 + c^2 + md^2)I$, the set of the $2n$ rows of $F(M)$ forms a $\frac{1}{k}(a^2 + mb^2 + c^2 + md^2)$-frame of $A_k(C_k(M))$. $\square$

Remark 3.7. It follows from the assumption that $a^2 + mb^2 + c^2 + md^2 \equiv 0 \pmod k$.

For dimensions $48, 56$ and 64, we consider cases $(m, k) = (23, 3), (29, 5)$ in Proposition 3.6. Hence, we need the following modifications of Theorem 3.5. The proofs of Theorems 3.8 and 3.9 need some facts of modular forms for congruence subgroups. Our notation and terminology for modular forms follow from [23] (see [23] for undefined terms).

Theorem 3.8. *There are integers a, b, c and d satisfying $a \equiv d \pmod 3$, $b \equiv c \pmod 3$ and $p = \frac{1}{3}(a^2 + 23b^2 + c^2 + 23d^2)$ for each prime $p \neq 2, 5, 7, 23$.*

Proof. Consider the following lattice in dimension 4:

$$L_1 = \{(a, b, c, d) \in \mathbb{Z}^4 \mid a \equiv d \pmod 3 \text{ and } b \equiv c \pmod 3\}.$$

Here, we consider the inner product $\langle x, y \rangle_1$ induced by $(a^2 + 23b^2 + c^2 + 23d^2)/3$, instead of the standard inner product. This lattice is spanned by $(3, 0, 0, 0)$, $(0, 3, 0, 0)$, $(1, 0, 0, 1)$, and $(0, 1, 1, 0)$ with Gram matrix:

$$M_1 = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 69 & 0 & 23 \\ 1 & 0 & 8 & 0 \\ 0 & 23 & 0 & 8 \end{pmatrix}.$$

We have verified by Magma that the lattice L_1 has the following theta series:

$$\begin{aligned}\theta_{L_1}(q) &= \sum_{x\in L_1} q^{\langle x,x\rangle_1} = 1+4q^3+4q^6+4q^8+4q^9+8q^{11}+\cdots\\ &= \sum_{n=0}^{\infty} a_1(n)q^n \text{ (say).}\end{aligned}$$

Since $23M_1^{-1}$ has integer entries and $\det M_1 = 23^2$, $\theta_{L_1}(z)$ is a modular form (of weight 2) for $\Gamma_0(92)$ [23, Corollary 4.9.2], where $q = e^{2\pi i z}$, z is in the upper half plane, and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod N \right\}.$$

It is known that the dimension of the space of cusp forms of weight 2 for $\Gamma_0(23)$ is two and using Magma we have found some basis as follows:

$$\begin{aligned}f(z) &= q-q^3-q^4-2q^6+2q^7-q^8+2q^9+2q^{10}-4q^{11}+\cdots\\ &= \sum_{n=1}^{\infty} c_f(n)q^n \text{ (say),}\\ g(z) &= q^2-2q^3-q^4+2q^5+q^6+2q^7-2q^8-2q^{10}-2q^{11}+\cdots\\ &= \sum_{n=1}^{\infty} c_g(n)q^n \text{ (say).}\end{aligned}$$

Let $\eta(z) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$ be the Dedekind η-function. Then

$$\frac{\eta(4z)^8}{\eta(2z)^4} = \sum_{n=1}^{\infty} \sigma_1(2n-1)q^{2n-1}$$

is a modular form for $\Gamma_0(4)$, where $\sigma_1(n) = \sum_{p|n} p$ (see [19, p. 145, Problem 10]). We define a modular form $h_{92}(z)$ for $\Gamma_0(92)$ as follows:

$$\begin{aligned}h_{92}(z) = \frac{4}{11}\Bigg(&\frac{\eta(4z)^8}{\eta(2z)^4} - 23\frac{\eta(92z)^8}{\eta(46z)^4} - f(z) + 2f(2z) - 4f(4z)\\ &- 3g(z) - 5g(2z) - 12g(4z)\Bigg) = \sum_{n=0}^{\infty} b_1(n)q^n \text{ (say).}\end{aligned}$$

Let

$$\chi_2(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 2\\ 1 & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}(\theta_{L_1}(z))_{\chi_2} &= \sum_{n=0}^{\infty} \chi_2(n)a_1(n)q^n = 4q^3+4q^9+8q^{11}+\cdots \text{ and}\\ (h_{92}(z))_{\chi_2} &= \sum_{n=0}^{\infty} \chi_2(n)b_1(n)q^n = 4q^3+4q^9+8q^{11}+\cdots\end{aligned}$$

are modular forms with character χ_2 for $\Gamma_0(368)$ [19, p. 127, Proposition 17]. Using Theorem 7 in [24] and the fact that the genus of $\Gamma_0(368)$ is 43, the verification by

MAGMA that $\chi_2(n)a_1(n) = \chi_2(n)b_1(n)$ for $n \leq 86$ shows

$$(\theta_{L_1}(z))_{\chi_2} = (h_{92}(z))_{\chi_2}.$$

Hence, for each odd prime p with $p \neq 23$, we have

$$a_1(p) = \frac{4}{11}(p+1-(c_f(p)+3c_g(p))). \tag{3.1}$$

Set $h_1(z)$ and $h_2(z)$ as follows:

$$h_1(z) = f(z) + \frac{-1+\sqrt{5}}{2}g(z) = \sum_{n=1}^{\infty} c_{h_1}(n)q^n \text{ (say) and}$$

$$h_2(z) = f(z) + \frac{-1-\sqrt{5}}{2}g(z) = \sum_{n=1}^{\infty} c_{h_2}(n)q^n \text{ (say).}$$

Let $T(n)$ be the Hecke operator considered on the space of modular forms for $\Gamma_0(23)$ (see [19, p. 161, Proposition 37]). Then, by [18, Proposition 9.15] we have

$$T(2)f(z) = g(z) \text{ and } T(2)g(z) = f(z) - g(z).$$

Namely, $h_1(z)$ and $h_2(z)$ are eigen forms for $T(2)$. Since the algebra of Hecke operators is commutative [23, Theorem 4.5.3], $h_1(z)$ and $h_2(z)$ are normalized Hecke eigen forms. In addition, for each prime p and $i = 1, 2$,

$$|c_{h_i}(p)| \leq 2\sqrt{p}$$

(see [19, p. 164]). Since $f(z) = \frac{5+\sqrt{5}}{10}h_1(z) + \frac{5-\sqrt{5}}{10}h_2(z)$ and $g(z) = \frac{\sqrt{5}}{5}h_1(z) + \frac{-\sqrt{5}}{5}h_2(z)$, we have

$$f(z) + 3g(z) = \frac{5+7\sqrt{5}}{10}h_1(z) + \frac{5-7\sqrt{5}}{10}h_2(z).$$

Hence, for each prime p, we have

$$|c_f(p) + 3c_g(p)| \leq \left(\frac{5+7\sqrt{5}}{10} + \frac{-5+7\sqrt{5}}{10}\right)2\sqrt{p}.$$

Using (3.1), $a_1(p)$ is bounded below by

$$\frac{4}{11}\left(p+1-\frac{14\sqrt{p}}{\sqrt{5}}\right). \tag{3.2}$$

Hence, (3.2) is positive for $p > 37$, namely, $a_1(p) > 0$ for $p > 37$. We have verified by MAGMA that $a_1(p) > 0$ for each prime p with $p \leq 37$ and $p \neq 2, 5, 7, 23$, where $a_1(p)$ is listed in Table 1 for a prime $p \leq 37$. □

TABLE 1. Coefficients $a_1(p)$ for primes $p \leq 37$

p	$a_1(p)$	p	$a_1(p)$	p	$a_1(p)$	p	$a_1(p)$
2	0	7	0	17	8	29	12
3	4	11	8	19	8	31	4
5	0	13	4	23	0	37	16

Theorem 3.9. *There are integers a, b, c and d satisfying $a \equiv d \pmod 5$, $b \equiv c \pmod 5$ and $p = \frac{1}{5}(a^2 + 29b^2 + c^2 + 29d^2)$ for each prime $p \neq 2, 3, 7, 17, 23$.*

Proof. We follow the same line as in the previous proof. Consider the following lattice in dimension 4:

$$L_2 = \{(a,b,c,d) \in \mathbb{Z}^4 \mid a \equiv d \pmod 5 \text{ and } b \equiv c \pmod 5\}.$$

Here, we consider the inner product $\langle x, y\rangle_2$ induced by $(a^2+29b^2+c^2+29d^2)/5$. This lattice is spanned by $(5,0,0,0)$, $(0,5,0,0)$, $(1,0,0,1)$, and $(0,1,1,0)$ with Gram matrix:

$$M_2 = \begin{pmatrix} 5 & 0 & 1 & 0 \\ 0 & 145 & 0 & 29 \\ 1 & 0 & 6 & 0 \\ 0 & 29 & 0 & 6 \end{pmatrix}.$$

We have verified by MAGMA that L_2 has the following theta series:

$$\begin{aligned}\theta_{L_2}(q) &= \sum_{x\in L_2} q^{\langle x,x\rangle_2} = 1 + 4q^5 + 4q^6 + 4q^9 + 4q^{10} + 8q^{11} + \cdots \\ &= \sum_{n=0}^{\infty} a_2(n) q^n \text{ (say)}.\end{aligned}$$

Since $29M_2^{-1}$ has integer entries and $\det M_2 = 29^2$, $\theta_{L_2}(z)$ is a modular form for $\Gamma_0(116)$ [23, p. 192].

It is known that the dimension of the space of cusp forms of weight 2 for $\Gamma_0(116)$ is thirteen and using MAGMA we have found some basis $f_1(z), \ldots, f_{13}(z)$ such that

$$f_i(z) = q^i + c_{i,12}q^{12} + c_{i,13}q^{13} + \cdots,$$

for $i = 1, 2, \ldots, 11$, and $f_i(z) = c_i q^i + \cdots$, where $c_i \neq 0$ for $i = 12, 13$. In particular, we use $f_i(z)$ $(i = 1, 3, 5, 7, 9, 11, 13)$, which are explicitly written as:

$$\begin{aligned}
f_1(z) &= q + 5q^{13} + 3q^{15} - 5q^{17} - 6q^{19} + 3q^{21} - 4q^{23} + 3q^{25} - 6q^{27} - \cdots, \\
f_3(z) &= q^3 + 10q^{13} + 7q^{15} - 10q^{17} - 17q^{19} + 10q^{21} - 4q^{23} + 10q^{25} - 12q^{27} - \cdots, \\
f_5(z) &= q^5 + 8q^{13} + 5q^{15} - 7q^{17} - 14q^{19} + 7q^{21} - 4q^{23} + 9q^{25} - 10q^{27} - \cdots, \\
f_7(z) &= q^7 + 5q^{13} + 3q^{15} - 5q^{17} - 8q^{19} + 5q^{21} - 3q^{23} + 5q^{25} - 6q^{27} - \cdots, \\
f_9(z) &= q^9 + 9q^{13} + 7q^{15} - 9q^{17} - 16q^{19} + 9q^{21} - 4q^{23} + 10q^{25} - 12q^{27} - \cdots, \\
f_{11}(z) &= q^{11} + 6q^{13} + 5q^{15} - 7q^{17} - 9q^{19} + 5q^{21} - 4q^{23} + 6q^{25} - 7q^{27} - \cdots, \\
f_{13}(z) &= 14q^{13} + 11q^{15} - 13q^{17} - 24q^{19} + 15q^{21} - 6q^{23} + 14q^{25} - 16q^{27} - \cdots.
\end{aligned}$$

For $i = 1, 3, 5, 7, 9, 11, 13$, we denote by $c_{f_i}(n)$ the coefficient of $f_i(z)$ as follows:

$$f_i(z) = \sum_{n=1}^{\infty} c_{f_i}(n) q^n.$$

We define a modular form $h_{116}(z)$ for $\Gamma_0(116)$ as follows:

$$\begin{aligned} h_{116}(z) = \frac{4}{15}\Bigg(&\frac{\eta(4z)^8}{\eta(2z)^4} + 29\frac{\eta(116z)^8}{\eta(58z)^4} - f_1(z) - 4f_3(z) + 9f_5(z) \\ &- 8f_7(z) + 2f_9(z) + 18f_{11}(z) - 8f_{13}(z)\Bigg) = \sum_{n=0}^{\infty} b_2(n) q^n \text{ (say)}.\end{aligned}$$

Then

$$(\theta_{L_2}(z))_{\chi_2} = \sum_{n=0}^{\infty} \chi_2(n)a_2(n)q^n = 4q^5 + 4q^9 + 8q^{11} + \cdots \text{ and}$$
$$(h_{116}(z))_{\chi_2} = \sum_{n=0}^{\infty} \chi_2(n)b_2(n)q^n = 4q^5 + 4q^9 + 8q^{11} + \cdots$$

are modular forms with character χ_2 for $\Gamma_0(464)$ [19, p. 127, Proposition 17]. Using Theorem 7 in [24] and the fact that the genus of $\Gamma_0(464)$ is 55, the verification by MAGMA that $\chi_2(n)a_2(n) = \chi_2(n)b_2(n)$ for $n \leq 110$ shows

$$(\theta_{L_2}(z))_{\chi_2} = (h_{116}(z))_{\chi_2}.$$

Hence, for each odd prime p with $p \neq 29$, we have

$$a_2(p) = \frac{4}{15}(p + 1 - (c_{f_1}(p) + 4c_{f_3}(p) - 9c_{f_5}(p) + 8c_{f_7}(p) - 2c_{f_9}(p) - 18c_{f_{11}}(p) + 8c_{f_{13}}(p))). \tag{3.3}$$

Set $\hat{h}_i(z)$ $(i = 1, 3, 5, 7, 9, 11, 13)$ as follows:

$$\begin{pmatrix} \hat{h}_1(z) \\ \hat{h}_3(z) \\ \hat{h}_5(z) \\ \hat{h}_7(z) \\ \hat{h}_9(z) \\ \hat{h}_{11}(z) \\ \hat{h}_{13}(z) \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 & 4 & 6 & -1 & -5 \\ 1 & 2 & -2 & 4 & 1 & -6 & 0 \\ 1 & 1 & 3 & -4 & -2 & 3 & -1 \\ 1 & -3 & -3 & -2 & 6 & -1 & 1 \\ 1 & -1 & 1 & -2 & -2 & -3 & 3 \\ 1 & 1+\sqrt{2} & -1 & -2\sqrt{2} & 2\sqrt{2} & 1-\sqrt{2} & -1-\sqrt{2} \\ 1 & 1-\sqrt{2} & -1 & 2\sqrt{2} & -2\sqrt{2} & 1+\sqrt{2} & -1+\sqrt{2} \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_3(z) \\ f_5(z) \\ f_7(z) \\ f_9(z) \\ f_{11}(z) \\ f_{13}(z) \end{pmatrix}. \tag{3.4}$$

For $i = 1, 3, 5, 7, 9, 11, 13$, we denote by $c_{\hat{h}_i}(n)$ the coefficient of $\hat{h}_i(z)$ as follows:

$$\hat{h}_i(z) = \sum_{n=1}^{\infty} c_{\hat{h}_i}(n)q^n.$$

Let $T(n)$ be the Hecke operator considered on the space of modular forms for $\Gamma_0(116)$ (see [19, p. 161, Proposition 37]). Then, by [18, Proposition 9.15] we have

$$\begin{aligned}
T(3)f_1(z) &= 3f_3(z) + 3f_5(z) + 3f_7(z) - 6f_9(z) - 2f_{11}(z), \\
T(3)f_3(z) &= f_1(z) + 7f_5(z) + 10f_7(z) - 9f_9(z) - 6f_{11}(z) + f_{13}(z), \\
T(3)f_5(z) &= 5f_5(z) + 7f_7(z) - 10f_9(z) - 3f_{11}(z) + 3f_{13}(z), \\
T(3)f_7(z) &= 3f_5(z) + 5f_7(z) - 6f_9(z) - 3f_{11}(z) + 2f_{13}(z), \\
T(3)f_9(z) &= f_3(z) + 7f_5(z) + 9f_7(z) - 12f_9(z) - 4f_{11}(z) + 2f_{13}(z), \\
T(3)f_{11}(z) &= 5f_5(z) + 5f_7(z) - 7f_9(z) - 2f_{11}(z) + f_{13}(z), \\
T(3)f_{13}(z) &= 11f_5(z) + 15f_7(z) - 16f_9(z) - 6f_{11}(z) + 2f_{13}(z).
\end{aligned}$$

Namely, $\hat{h}_i(z)$ $(i = 1, 3, 5, 7, 9, 11, 13)$ are eigen forms for $T(3)$. Since the algebra of Hecke operators is commutative [23, Theorem 4.5.3], $\hat{h}_i(z)$ $(i = 1, 3, 5, 7, 9, 11, 13)$ are normalized Hecke eigen forms. In addition, for each prime p and $i = 1, 3, 5, 7, 9, 11, 13$,

$$|c_{\hat{h}_i}(p)| \leq 2\sqrt{p}.$$

By (3.4), we have

$$\begin{pmatrix} f_1(z) \\ f_3(z) \\ f_5(z) \\ f_7(z) \\ f_9(z) \\ f_{11}(z) \\ f_{13}(z) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{12} & -\frac{\sqrt{2}}{8} & \frac{\sqrt{2}}{8} \\ -\frac{5}{24} & \frac{2}{3} & \frac{7}{8} & \frac{1}{3} & -\frac{2}{3} & \frac{-\sqrt{2}-4}{8} & \frac{\sqrt{2}-4}{8} \\ -\frac{1}{20} & \frac{7}{15} & \frac{3}{4} & \frac{1}{4} & -\frac{5}{12} & \frac{-\sqrt{2}-4}{8} & \frac{\sqrt{2}-4}{8} \\ -\frac{1}{24} & \frac{1}{3} & \frac{3}{8} & \frac{1}{6} & -\frac{1}{3} & \frac{-\sqrt{2}-2}{8} & \frac{\sqrt{2}-2}{8} \\ -\frac{7}{60} & \frac{8}{15} & \frac{3}{4} & \frac{5}{12} & -\frac{7}{12} & \frac{-\sqrt{2}-4}{8} & \frac{\sqrt{2}-4}{8} \\ -\frac{1}{10} & \frac{4}{15} & \frac{1}{2} & \frac{1}{4} & -\frac{5}{12} & \frac{-\sqrt{2}-2}{8} & \frac{\sqrt{2}-2}{8} \\ -\frac{31}{120} & \frac{4}{5} & \frac{9}{8} & \frac{7}{12} & -\frac{3}{4} & \frac{-\sqrt{2}-3}{4} & \frac{\sqrt{2}-3}{4} \end{pmatrix} \begin{pmatrix} \hat{h}_1(z) \\ \hat{h}_3(z) \\ \hat{h}_5(z) \\ \hat{h}_7(z) \\ \hat{h}_9(z) \\ \hat{h}_{11}(z) \\ \hat{h}_{13}(z) \end{pmatrix}.$$

Hence, we have

$$\begin{aligned} &f_1(z) + 4f_3(z) - 9f_5(z) + 8f_7(z) - 2f_9(z) - 18f_{11}(z) + 8f_{13}(z) \\ &= -\frac{3}{4}\hat{h}_1(z) + 2\hat{h}_3(z) - \frac{5}{4}\hat{h}_5(z) + \hat{h}_9(z). \end{aligned}$$

For each prime p, $|c_{f_1}(p)+4c_{f_3}(p)-9c_{f_5}(p)+8c_{f_7}(p)-2c_{f_9}(p)-18c_{f_{11}}(p)+8c_{f_{13}}(p)|$ is bounded above by

$$\left(\frac{3}{4} + 2 + \frac{5}{4} + 1\right) 2\sqrt{p} = 10\sqrt{p}.$$

Using (3.3), $a_2(p)$ is bounded below by

$$\frac{4}{15}\left(p + 1 - 10\sqrt{p}\right). \tag{3.5}$$

Hence, (3.5) is positive for $p > 97$, namely, $a_2(p) > 0$ for $p > 97$. We have verified by Magma that $a_2(p) > 0$ for each prime p with $p \leq 97$ and $p \neq 2, 3, 7, 17, 23$, where $a_2(p)$ is listed in Table 2 for a prime $p \leq 97$. □

Table 2. Coefficients $a_2(p)$ for primes $p \leq 97$

p	$a_2(p)$	p	$a_2(p)$	p	$a_2(p)$	p	$a_2(p)$	p	$a_2(p)$
2	0	13	4	31	16	53	12	73	8
3	0	17	0	37	8	59	16	79	24
5	4	19	8	41	8	61	16	83	16
7	0	23	0	43	8	67	32	89	24
11	8	29	16	47	8	71	16	97	24

4. Lengths 32 and 40

In this section, we show the existence of an extremal Type II $\mathbb{Z}_{2k}$-code of lengths 32 and 40 for every positive integer k. Our approach is similar to that in [5, 10].

Some weighing matrix of order 16 and weight 11 is given in [11, p. 280] and it is denoted by $W_{16,11}$. Let $W_{20,11}$ be the 20×20 $(0, \pm 1)$-matrix $\begin{pmatrix} A & B \\ -B^T & A^T \end{pmatrix}$, where A and B are negacirculant matrices with first rows r_A and r_B:

$$r_A = (0, 1, 0, 1, 0, 1, 0, 1, 0, 1) \text{ and } r_B = (1, 1, -1, -1, 1, 0, 0, 1, 0, 0),$$

respectively. Then $W_{n,11}$ $(n = 16, 20)$ is a skew-symmetric weighing matrix of order n and weight 11. By Proposition 3.3, $\tilde{C}_4(W_{n,11})$ $(n = 16, 20)$ is a Type II $\mathbb{Z}_4$-code. Moreover, $\tilde{C}_4(W_{16,11})$ is extremal [11], and we have verified by Magma that $\tilde{C}_4(W_{20,11})$ is extremal. Hence, $A_4(\tilde{C}_4(W_{n,11}))$ is an extremal even unimodular lattice in dimension $2n$ for $n = 16, 20$.

If a, b, c and d are integers with $c \equiv 2a + b \pmod 4$ and $d \equiv a + 2b \pmod 4$, then the extremal even unimodular lattice $A_4(\tilde{C}_4(W_{n,11}))$ $(n = 16, 20)$ contains a $\frac{1}{4}(a^2 + 11b^2 + c^2 + 11d^2)$-frame by Proposition 3.3. Moreover, by Lemma 3.1 and Theorem 3.5, we have the following:

Lemma 4.1. $A_4(\tilde{C}_4(W_{n,11}))$ $(n = 16, 20)$ *contains a* $2k$*-frame for every positive integer* k *with* $k \geq 2$ *and* $k \neq 11^m$*, where* m *is a positive integer.*

TABLE 3. Extremal Type II $\mathbb{Z}_{22}$-codes of lengths $32, 40$

Code	r_A	r_B
$C_{22,32}$	$(0,0,0,0,1,10,21,6)$	$(11,2,20,20,9,3,21,11)$
$C_{22,40}$	$(0,0,0,0,1,19,2,10,1,6)$	$(13,1,10,14,10,16,13,6,9,4)$

Let $C_{22,2n}$ $(n = 16, 20)$ be the $\mathbb{Z}_{22}$-code of length $2n$ with generator matrix of the form (2.1), where the first rows r_A and r_B of negacirculant matrices A and B are listed in Table 3. These codes were found by considering pairs of binary Type II codes and self-dual $\mathbb{Z}_{11}$-codes with generator matrices of the form (2.1) (see [8, Theorem 2.3] for the construction method). Since $AA^T + BB^T = -I$ and the Euclidean weights of all rows of the generator matrix are divisible by 44, it follows from [2, Lemma 2.2] that $C_{22,2n}$ is a Type II $\mathbb{Z}_{22}$-code of length $2n$ $(n = 16, 20)$. Moreover, we have verified by MAGMA that $C_{22,2n}$ is extremal. Hence, we have the following:

Lemma 4.2. $A_{22}(C_{22,2n})$ $(n = 16, 20)$ *contains a* $22k$*-frame for every positive integer* k*.*

Since binary extremal Type II codes of length 40 are known, by Lemmas 2.1, 4.1 and 4.2, we have the following:

Theorem 4.3. *There is an extremal Type II* $\mathbb{Z}_{2k}$*-code of lengths* $32, 40$ *for every positive integer* k*.*

5. LENGTH 48

In this section and the next two sections, we show the existence of an extremal Type II $\mathbb{Z}_{2k}$-code of lengths $48, 56$ and 64 for every positive integer k. Main tools of these sections are Proposition 3.6 and Theorems 3.8 and 3.9, whereas the main tools in the previous section were Proposition 3.3 and Theorem 3.5.

Let $W_{24,23}$ be the 24×24 $(0, \pm 1)$-matrix of the form:

$$\begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & & & \\ \vdots & & A & \\ -1 & & & \end{pmatrix},$$

where A is the circulant matrix with first row

$$(0, 1, 1, 1, 1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, -1, 1, -1, -1, -1, -1).$$

Note that 1's are at the nonzero squares modulo 23. It is well known that $W_{24,23}$ is a skew-symmetric weighing matrix of order 24 and weight 23.

If a, b, c and d are integers with $a \equiv d \pmod 3$ and $b \equiv c \pmod 3$, then the odd unimodular lattice $A_3(C_3(W_{24,23}))$ contains a $\frac{1}{3}(a^2 + 23b^2 + c^2 + 23d^2)$-frame by Proposition 3.6. Moreover, by Lemma 3.1 and Theorem 3.8, we have the following:

Lemma 5.1. *$A_3(C_3(W_{24,23}))$ contains a k-frame for every positive integer k with $k \geq 3$, $k \neq 2^{m_1}5^{m_2}7^{m_3}23^{m_4}$, where m_i is a non-negative integer ($i = 1, 2, 3, 4$).*

Remark 5.2. $C_3(W_{24,23})$ is a ternary extremal self-dual code of length 48 [28] and it is called the Pless symmetry code.

TABLE 4. Extremal Type II $\mathbb{Z}_{2k}$-codes of length 48 ($2k = 14, 46$)

Code	r_A	r_B
$C_{14,48}$	$(1, 11, 6, 6, 9, 9, 11, 1, 7, 0, 7, 7)$	$(5, 3, 11, 12, 4, 5, 1, 0, 4, 6, 11, 8)$
$C_{46,48}$	$(0, 0, 1, 26, 29, 3, 13, 13, 45, 21, 0, 23)$	$(30, 9, 23, 37, 33, 37, 35, 40, 6, 8, 33, 28)$

Let $C_{2k,48}$ ($2k = 14, 46$) be the $\mathbb{Z}_{2k}$-code of length 48 with generator matrix of the form (2.1), where the first rows r_A and r_B of A and B are listed in Table 4. Similarly to Table 3, these codes were found by considering pairs of binary Type II codes and self-dual $\mathbb{Z}_k$-codes. Since $AA^T + BB^T = -I$ and the Euclidean weights of all rows of the generator matrix are divisible by $4k$, $C_{2k,48}$ is a Type II $\mathbb{Z}_{2k}$-code of length 48 and $A_{2k}(C_{2k,48})$ is an even unimodular lattice ($2k = 14, 46$). In addition, we have verified by MAGMA that $A_{2k}(C_{2k,48})$ ($2k = 14, 46$) is extremal. Hence, we have the following:

Lemma 5.3. *For $2k = 14, 46$, $C_{2k,48}$ is an extremal Type II $\mathbb{Z}_{2k}$-code of length 48.*

The existence of an extremal Type II $\mathbb{Z}_{2k}$-code of length 48 is known for $k = 1, 2, \ldots, 6$ (see [13, Table 1]). Denote by $C_{2k,48}$ an existing extremal Type II $\mathbb{Z}_{2k}$-code of length 48 for $2k = 8$ and 10. It is known that one of the two even unimodular neighbors $L_0 \cup L_1$ and $L_0 \cup L_3$, which are given in Section 2.3, of $L = A_3(C_3(W_{24,23}))$ is extremal, and this is denoted by P_{48p} in [7] (see also [12]). By Lemma 3.2, we have the following:

Lemma 5.4.
(1) *P_{48p} contains a $2k$-frame for every positive integer k with $k \geq 3$, $k \neq 2^{m_1}5^{m_2}7^{m_3}23^{m_4}$, where m_i is a non-negative integer ($i = 1, 2, 3, 4$).*
(2) *$A_8(C_{8,48})$ contains an $8k$-frame for every positive integer k.*
(3) *$A_{10}(C_{10,48})$ contains a $10k$-frame for every positive integer k.*
(4) *$A_{14}(C_{14,48})$ contains a $14k$-frame for every positive integer k.*
(5) *$A_{46}(C_{46,48})$ contains a $46k$-frame for every positive integer k.*

Since P_{48p} and $A_{2k}(C_{2k,48})$ ($2k = 8, 10, 14, 46$) are extremal even unimodular lattices, by Lemma 2.1 we have the following:

Theorem 5.5. *There is an extremal Type II $\mathbb{Z}_{2k}$-code of length 48 for every positive integer k.*

Remark 5.6. Three non-isomorphic extremal even unimodular lattices are known for dimension 48 (see [7]). It is worthwhile to determine whether $A_{14}(C_{14,48})$ and $A_{46}(C_{46,48})$ are new or not.

6. Length 56

Let D_{28} be the 28×28 $(0, \pm 1, \pm 2)$-matrix $\begin{pmatrix} A & B \\ -B^T & A^T \end{pmatrix}$, where A and B are negacirculant matrices with first rows r_A and r_B:

$$r_A = (0, 1, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 1) \text{ and}$$
$$r_B = (-1, 0, 1, -1, 0, 2, -1, 2, 0, 1, 1, 0, -1, 2),$$

respectively. The matrix D_{28} satisfies that $D_{28}D_{28}^T = 29I$ and $D_{28}^T = -D_{28}$.

If a, b, c and d are integers with $a \equiv d \pmod 5$ and $b \equiv c \pmod 5$, then the odd unimodular lattice $A_5(C_5(D_{28}))$ contains a $\frac{1}{5}(a^2 + 29b^2 + c^2 + 29d^2)$-frame by Proposition 3.6. Moreover, by Lemma 3.1 and Theorem 3.9, we have the following:

Lemma 6.1. *$A_5(C_5(D_{28}))$ contains a k-frame for every positive integer k with $k \geq 5$, $k \neq 2^{m_1}3^{m_2}7^{m_3}17^{m_4}23^{m_5}$, where m_i is a non-negative integer $(i = 1, 2, 3, 4, 5)$.*

Remark 6.2. The weight enumerator of a code is given by $\sum_i A_i y^i$, where A_i denotes the number of codewords of weight i. We have verified by Magma that $C_5(D_{28})$ has weight enumerator $1+168y^{12}+224y^{14}+448y^{15}+9464y^{16}+\cdots$ and $A_5(C_5(D_{28}))$ has minimum norm 5.

Remark 6.3. We have verified by Magma that every ternary self-dual code $C_3(W)$ has minimum weight less than 15 if W is a skew-symmetric weighing matrix of order 28 and weight $k \equiv 2 \pmod 3$, which has the form $\begin{pmatrix} A & B \\ -B^T & A^T \end{pmatrix}$, where A and B are negacirculant matrices. This is a reason to find the above matrix D_{28}.

Table 5. Extremal Type II $\mathbb{Z}_{2k}$-codes of length 56 $(2k = 14, 34, 46)$

$2k$	r_A	r_B
14	$(0, 0, 0, 0, 1, 9, 0, 12, 8, 10, 12, 9, 12, 7)$	$(8, 1, 13, 4, 9, 7, 9, 0, 10, 7, 0, 0, 8, 1)$
34	$(0, 0, 0, 0, 1, 13, 6, 1, 4, 0, 6, 4, 22, 8)$	$(17, 32, 4, 30, 1, 1, 6, 32, 31, 23, 23, 14, 9, 27)$
46	$(0, 0, 0, 0, 1, 23, 6, 33, 43, 19, 30, 18, 29, 11)$	$(17, 13, 32, 23, 42, 16, 38, 31, 29, 1, 30, 25, 41, 22)$

Let $C_{2k,56}$ $(2k = 14, 34, 46)$ be the $\mathbb{Z}_{2k}$-code of length 56 with generator matrix of the form (2.1), where the first rows r_A and r_B of A and B are listed in Table 5. These codes were found by considering pairs of binary Type II codes and self-dual $\mathbb{Z}_k$-codes. Since $AA^T + BB^T = -I$ and the Euclidean weights of all rows of the generator matrix are divisible by $4k$, $C_{2k,56}$ is a Type II $\mathbb{Z}_{2k}$-code of length 56 and $A_{2k}(C_{2k,56})$ is an even unimodular lattice $(2k = 14, 34, 46)$. Moreover, we have verified by Magma that $A_{2k}(C_{2k,56})$ is extremal. Hence, we have the following:

Lemma 6.4. *For $2k = 14, 34, 46$, $C_{2k,56}$ is an extremal Type II $\mathbb{Z}_{2k}$-code of length 56.*

We have verified by Magma that one of the even unimodular neighbors $L_0 \cup L_1$ and $L_0 \cup L_3$ of $L = A_5(C_5(D_{28}))$ is extremal. Denote by L_{56} the extremal even unimodular neighbor of $A_5(C_5(D_{28}))$. The existence of an extremal Type II $\mathbb{Z}_{2k}$-code of length 56 is known for $k = 1, 2, \ldots, 6$ (see [13, Table 1]). Denote by $C_{2k,56}$ an existing extremal Type II $\mathbb{Z}_{2k}$-code of length 56 for $2k = 6$ and 8. By Lemma 3.2, we have the following:

Lemma 6.5. (1) L_{56} *contains a* $2k$*-frame for every positive integer* k *with* $k \geq 5$, $k \neq 2^{m_1}3^{m_2}7^{m_3}17^{m_4}23^{m_5}$, *where* m_i *is a non-negative integer* $(i = 1, 2, 3, 4, 5)$.
(2) $A_6(C_{6,56})$ *contains a* $6k$*-frame for every positive integer* k.
(3) $A_8(C_{8,56})$ *contains an* $8k$*-frame for every positive integer* k.
(4) $A_{14}(C_{14,56})$ *contains a* $14k$*-frame for every positive integer* k.
(5) $A_{34}(C_{34,56})$ *contains a* $34k$*-frame for every positive integer* k.
(6) $A_{46}(C_{46,56})$ *contains a* $46k$*-frame for every positive integer* k.

Since L_{56} and $A_{2k}(C_{2k,56})$ $(2k = 6, 8, 14, 34, 46)$ are extremal even unimodular lattices, by Lemma 2.1 we have the following:

Theorem 6.6. *There is an extremal Type II* $\mathbb{Z}_{2k}$*-code of length* 56 *for every positive integer* k.

7. Length 64

Let $W_{32,23}$ and $W_{32,17}$ be the 32×32 $(0, \pm 1)$-matrices $\begin{pmatrix} A & B \\ -B^T & A^T \end{pmatrix}$, where A, B are negacirculant matrices with the following first rows r_A, r_B:

$$r_A = (0, 1, 1, 0, -1, 1, -1, 0, 0, 0, -1, 1, -1, 0, 1, 1),$$
$$r_B = (0, 1, 0, 1, 1, 1, 1, -1, 0, -1, -1, 1, -1, -1, -1, 1)$$

and

$$r_A = (0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0),$$
$$r_B = (0, 1, -1, -1, 0, -1, 1, 0, 1, -1, 1, 1, 1, 0, -1, 1),$$

respectively. The matrix $W_{32,23}$ (resp. $W_{32,17}$) is a skew-symmetric weighing matrix of order 32 and weight 23 (resp. 17).

If a, b, c and d are integers with $a \equiv d \pmod 3$ and $b \equiv c \pmod 3$, then the odd unimodular lattice $A_3(C_3(W_{32,23}))$ contains a $\frac{1}{3}(a^2 + 23b^2 + c^2 + 23d^2)$-frame by Proposition 3.6. Moreover, by Lemma 3.1 and Theorem 3.8, we have the following:

Lemma 7.1. $A_3(C_3(W_{32,23}))$ *contains a* k*-frame for every positive integer* k *with* $k \geq 3$, $k \neq 2^{m_1}5^{m_2}7^{m_3}23^{m_4}$, *where* m_i *is a non-negative integer* $(i = 1, 2, 3, 4)$.

Lemma 7.2. $A_3(C_3(W_{32,17}))$ *contains a* $7k$*-frame and a* $23k$*-frame for every positive integer* k.

Proof. Take $(a, b, c, d) = (0, 1, -2, 0)$ and $(0, 2, -1, 0)$. By Proposition 3.6, the odd unimodular lattice $A_3(C_3(W_{32,17}))$ contains a 7-frame and a 23-frame. The result follows by Lemma 3.1. $\square$

Denote by $L_{64,t}$ any of two even unimodular neighbors $L_0 \cup L_1$ and $L_0 \cup L_3$ of $L = A_3(C_3(W_{32,t}))$ $(t = 23, 17)$. We have verified by MAGMA that $C_3(W_{32,t})$ $(t = 23, 17)$ have minimum weight 15. Hence, $L_{64,t}$ are extremal (see [12, Theorem 6]). The existence of an extremal Type II $\mathbb{Z}_{2k}$-code of length 64 is known for $k = 1, 2, \dots, 6$ (see [13, Table 1]). Denote by $C_{2k,64}$ an existing extremal Type II $\mathbb{Z}_{2k}$-code of length 64 for $2k = 8$ and 10. By Lemma 3.2, we have the following:

Lemma 7.3. (1) $L_{64,23}$ *contains a* $2k$*-frame for every positive integer* k *with* $k \geq 3$, $k \neq 2^{m_1}5^{m_2}7^{m_3}23^{m_4}$, *where* m_i *is a non-negative integer* $(i = 1, 2, 3, 4)$.

(2) $L_{64,17}$ *contains a* $14k$*-frame and a* $46k$*-frame for every positive integer* k.
(3) $A_8(C_{8,64})$ *contains an* $8k$*-frame for every positive integer* k.
(4) $A_{10}(C_{10,64})$ *contains a* $10k$*-frame for every positive integer* k.

Remark 7.4. We have found an extremal Type II $\mathbb{Z}_{2k}$-code $C_{2k,64}$ $(2k = 14, 46)$ of length 64 explicitly. These codes have generator matrices of the form (2.1), where the first rows r_A and r_B of negacirculant matrices A and B are as follows:

$$r_A = (0, 0, 0, 0, 1, 9, 1, 2, 7, 9, 13, 0, 10, 3, 10, 1) \text{ and}$$
$$r_B = (0, 5, 12, 13, 5, 6, 8, 8, 1, 10, 8, 1, 3, 0, 8, 3)$$

and

$$r_A = (0, 0, 0, 0, 1, 10, 24, 27, 35, 22, 7, 20, 22, 7, 36, 18) \text{ and}$$
$$r_B = (5, 15, 19, 43, 19, 18, 35, 5, 5, 42, 34, 27, 23, 36, 4, 32)$$

respectively. $A_{2k}(C_{2k,64})$ $(2k = 14, 46)$ contains a $2km$-frame for every positive integer m.

Since $L_{64,t}$ $(t = 17, 23)$ and $A_{2k}(C_{2k,64})$ $(2k = 8, 10)$ are extremal even unimodular lattices, by Lemma 2.1 we have the following:

Theorem 7.5. *There is an extremal Type II* $\mathbb{Z}_{2k}$*-code of length* 64 *for every positive integer* k.

8. Length 72

Although the approach is somewhat different to those in previous sections, it is shown in this section that there is an extremal Type II $\mathbb{Z}_{4k}$-code of length 72 for every positive integer k with $k \geq 2$.

The existence of an extremal even unimodular lattice in dimension 72 was a long-standing open question. Recently, the first example of such a lattice has been found by Nebe [25]. Roughly speaking, using the Leech lattice Λ_{24}, Nebe [25] found a pair (M, N) such that

$$\frac{1}{\sqrt{2}}\{(x_1 + y, x_2 + y, x_3 + y) \mid x_1, x_2, x_3 \in M, y \in N, x_1 + x_2 + x_3 \in M \cap N\}$$

is an extremal even unimodular lattice in dimension 72, where $M \cong N \cong \sqrt{2}\Lambda_{24}$, $\Lambda_{24} = M + N$ and $2\Lambda_{24} = M \cap N$. Hence, the extremal even unimodular lattice N_{72} in dimension 72, which has been found by Nebe [25], contains a sublattice

$$\frac{1}{\sqrt{2}}\{(x, \mathbf{0}, \mathbf{0}), (\mathbf{0}, y, \mathbf{0}), (\mathbf{0}, \mathbf{0}, z) \mid x, y, z \in 2\Lambda_{24}\},$$

where $\mathbf{0}$ denotes the zero vector of length 24. As described in Section 1, it was shown in [5, 10] that Λ_{24} contains a $2k$-frame for every positive integer k with $k \geq 2$. Let $\{f_1, \dots, f_{24}\}$ be a $2k$-frame of Λ_{24} $(k \geq 2)$. Then

$$\{(\sqrt{2}f_i, \mathbf{0}, \mathbf{0}), (\mathbf{0}, \sqrt{2}f_i, \mathbf{0}), (\mathbf{0}, \mathbf{0}, \sqrt{2}f_i) \mid i = 1, 2, \dots, 24\}$$

is a $4k$-frame of N_{72} for $k \geq 2$. Therefore, we have the following:

Theorem 8.1. *There is an extremal Type II* $\mathbb{Z}_{4k}$*-code of length* 72 *for every positive integer* k *with* $k \geq 2$.

Remark 8.2. It is worthwhile to determine whether there is an extremal Type II $\mathbb{Z}_{2m}$-code of length 72 or not for the cases that $m = 2$ and m is an odd positive integer. Of course, the case $m = 1$ is a long-standing open question (see [16, 29]).

As an example, an explicit generator matrix $\begin{pmatrix} I & M_{8,72} \end{pmatrix}$ of some extremal Type II $\mathbb{Z}_8$-code $C_{8,72}$ of length 72 with $A_8(C_{8,72}) \cong N_{72}$ is given by listing $M_{8,72}$ in Figure 1.

$$\begin{pmatrix}
0004044444444276260502007044004400004 \\
444400400444035522611141064440040444 \\
004040004004725430437502014040004004 \\
044444404444437162115704404040440400 \\
444440004044402144472442074400444044 \\
004404444040142273042604020004400400 \\
004400044040064160263204410444404444 \\
044404004044707461671100414000040000 \\
400404044000207033255140074004000440 \\
040040004044445560422004010000444044 \\
000000044040266504204702414044404444 \\
444444404440401366111547444040440404 \\
400404444440404004044424042414230661 \\
4000440040004004004044404604064336223 \\
444440040004444444044424744660227162 \\
400000000004400000400000404030136026 \\
040444040440040404440054340031762541 \\
440404444404444004440454347236540305 \\
044040404400044440404040705211520543 \\
0000000440400000004044060304300622360 \\
444000040044440044404050205221242642 \\
4044044004044404040400444046756741311 \\
444444440400440444444020544732763677 \\
0044000000000000040000020142220444752 \\
2724205422247004004444404044004400004 \\
430216002627000400004404400400400040 \\
723141607130004440004404004040004004 \\
034454706026400000400040000000000004 \\
002357524571004000400404400440004440 \\
143224276042004044044004040004400400 \\
464551763307440404444040004404044040 \\
706143413552044040444000444000040000 \\
203707035152404004040040044004000440 \\
044516363535404040400444444040004440 \\
666517121453440444444040040004044040 \\
402433523733440440400040444040440404
\end{pmatrix}$$

FIGURE 1. A generator matrix of $C_{8,72}$

9. POSITIVITY OF COEFFICIENTS OF THETA SERIES

In this section, we discuss the positivity of coefficients of theta series of extremal even unimodular lattices. As we have already mentioned in Section 2.3, it is important to study the positivity and non-negativity of coefficients of the theta series of extremal even unimodular lattices.

The positivity is also useful to construct spherical t-designs (see [1] for a recent survey on this subject). For a lattice L and a positive integer m, the *shell* of norm m

of L is defined by $\{x \in L \mid (x,x) = m\}$. Then shells of a lattice often give examples of spherical t-designs for some t. For example, any nonempty shell of an extremal even unimodular lattice in dimension n forms a spherical t-design, where $t = 11$ if $n \equiv 0 \pmod{24}$, $t = 7$ if $n \equiv 8 \pmod{24}$, $t = 3$ if $n \equiv 16 \pmod{24}$ [31] (see also [1, Theorem 3.8]). Note that 11 is the largest t among known spherical t-designs constructed as the shells of some lattices. The positivity of some coefficients of the theta series of extremal even unimodular lattices means that the corresponding shells of those lattices are not empty sets.

Let A_{2m} denote the number of vectors of norm $2m$ in an extremal even unimodular lattice L in dimensions n, where $n = 8, 16$ and 24. For the cases $n = 8$ and 16, it follows from (2.2) that $A_{2m} > 0$ for $m \geq 0$. For the case that $n = 24$, that is, L is the Leech lattice, it is well known that $A_{2m} > 0$ for $m \geq 2$ (see [7, p. 51]).

As a consequence of Theorems 4.3, 5.5, 6.6, 7.5 and 8.1, and the Fisher type bound (see [1, Theorem 2.12]), we give the following observation on the positivity of coefficients of the theta series of extremal even unimodular lattices.

Corollary 9.1. *Let L be an extremal even unimodular lattice in dimension n. Let A_{2m} denote the number of vectors of norm $2m$ in L, that is, $|L_{2m}| = A_{2m}$. Then*

$$A_{2m} \geq \begin{cases} 11968 & \text{if} \quad n = 32 \\ 80 & \text{if} \quad n = 40 \\ 5197920 & \text{if} \quad n = 48 \\ 61712 & \text{if} \quad n = 56 \\ 128 & \text{if} \quad n = 64 \end{cases}$$

for every positive integer m with $m \geq \lfloor \frac{n}{24} \rfloor + 1$. If $n = 72$, then

$$A_{4m} \geq 36949680$$

for every positive integer m with $m \geq 2$.

Proof. Suppose that $n = 32, 40, 48, 56, 64$. By Theorems 4.3, 5.5, 6.6 and 7.5, L_{2m} is a nonempty shell for every positive integer m with $m \geq \lfloor \frac{n}{24} \rfloor + 1$. Hence, L_{2m} is a spherical t-design, where $t = 7, 3, 11, 7, 3$, respectively. If L_{2m} is a spherical $(2e+1)$-design then $|L_{2m}| \geq 2\binom{n+e-1}{e}$ (see [1, Theorem 2.12]). The case $n = 72$ is similar. The result follows. □

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