Optimal Self-Dual $\mathbb{Z}_4$-Codes and a Unimodular Lattice in Dimension 41

Masaaki Harada*

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Abstract

For lengths up to 47 except 37, we determine the largest minimum Euclidean weight among all Type I $\mathbb{Z}_4$-codes of that length. We also give the first example of an optimal odd unimodular lattice in dimension 41 explicitly, which is constructed from some Type I $\mathbb{Z}_4$-code of length 41.

1 Introduction

Let $\mathbb{Z}_4$ $(= \{0, 1, 2, 3\})$ denote the ring of integers modulo 4. A $\mathbb{Z}_4$-code C of length n is a $\mathbb{Z}_4$-submodule of $\mathbb{Z}_4^n$. The dual code $C^\perp$ of C is defined as $\{x \in \mathbb{Z}_4^n \mid x \cdot y = 0 \text{ for all } y \in C\}$ under the standard inner product $x \cdot y$. A code C is *self-dual* if $C = C^\perp$. The Euclidean weight of a codeword $x = (x_1, \ldots, x_n)$ is $n_1(x) + 4n_2(x) + n_3(x)$, where $n_\alpha(x)$ denotes the number of components i with $x_i = \alpha$ $(\alpha = 1, 2, 3)$. The minimum Euclidean weight $d_E(C)$ of C is the smallest Euclidean weight among all nonzero codewords of C. A self-dual code which has the property that all Euclidean weights are divisible by eight, is called *Type II* [2] (see also [15]). A self-dual code which is not Type II, is called *Type I*. A Type II $\mathbb{Z}_4$-code of length n exists if and only if $n \equiv 0 \pmod 8$ [2], while a Type I $\mathbb{Z}_4$-code exists for every length.

*Department of Mathematical Sciences, Yamagata University, Yamagata 990–8560, Japan, and PRESTO, Japan Science and Technology Agency, Kawaguchi, Saitama 332–0012, Japan. email: mharada@sci.kj.yamagata-u.ac.jp

It was shown in [2] that the minimum Euclidean weight $d_E(C)$ of a Type II code C of length n is bounded by $d_E(C) \leq 8\lfloor \frac{n}{24} \rfloor + 8$. A Type II code meeting this bound with equality is called *extremal*. It was also shown in [19] that the minimum Euclidean weight $d_E(C)$ of a Type I code C of length n is bounded by $d_E(C) \leq 8\lfloor \frac{n}{24} \rfloor + 8$ if $n \not\equiv 23 \pmod{24}$, and $d_E(C) \leq 8\lfloor \frac{n}{24} \rfloor + 12$ if $n \equiv 23 \pmod{24}$. It is a fundamental problem to determine the largest minimum Euclidean weight among self-dual codes of that length. We denote the largest minimum Euclidean weight among Type I codes of length n by $d^I_{max,E}(n)$. These values $d^I_{max,E}(n)$ have been determined in [8] and [18] for $n \leq 24$. We say that a Type I code of length n is *optimal* or *Euclidean-optimal* if it has minimum Euclidean weight $d^I_{max,E}(n)$. We pay attention to the Euclidean weight from the viewpoint of a connection with optimal odd unimodular lattices.

In this paper, we determine the largest minimum Euclidean weight $d^I_{max,E}(n)$ for lengths $n \leq 47$, $n \neq 37$. To do this, we slightly improve upper bounds on the minimum Euclidean weights for lengths $n = 25, 26, \ldots, 31, 33, 34, 35$ (see the bound (4)), and we construct Type I $\mathbb{Z}_4$-codes meeting the bound (4) with equality. The values $d^I_{max,E}(n)$ are listed in Table 1. For length 37, our extensive search failed to discover a Type I $\mathbb{Z}_4$-code with minimum Euclidean weight 16. However, we have found a Type I code with minimum Euclidean weight 12. We also give the first explicit example of an optimal odd unimodular lattice in dimension 41, which is constructed from an optimal code of length 41 by Construction A. All computer calculations in this paper were done using Magma [4].

2 Preliminaries

2.1 Self-dual $\mathbb{Z}_4$-codes

Every $\mathbb{Z}_4$-code C of length n has two binary codes $C^{(1)}$ and $C^{(2)}$ associated with C:

$$C^{(1)} = \{c \bmod 2 \mid c \in C\} \text{ and } C^{(2)} = \{c \bmod 2 \mid c \in \mathbb{Z}_4^n, 2c \in C\}.$$

The binary codes $C^{(1)}$ and $C^{(2)}$ are called the *residue* and *torsion* codes of C, respectively. If C is a self-dual $\mathbb{Z}_4$-code, then $C^{(1)}$ is a binary doubly even code with $C^{(2)} = {C^{(1)}}^{\perp}$ [5]. It is easy to see that

$$\min\{d(C^{(1)}), 4d(C^{(2)})\} \leq d_E(C) \leq 4d(C^{(2)}), \tag{1}$$

Table 1: Largest minimum Euclidean weights of Type I $\mathbb{Z}_4$-codes

Length n	$d^I_{max,E}(n)$	Reference	Length n	$d^I_{max,E}(n)$	Reference
25	8	sub(C_{26})	37	12, 16	
26	12	C_{26}	38	16	[9]
27	12	C_{27}	39	16	[11]
28	12	C_{28}	40	16	[12]
29	12	C_{29}	41	16	C_{41}
30	12	[9]	42	16	[9]
31	12	sub(C_{32})	43	16	C_{43}
32	16	C_{32}	44	16	C_{44}
33	12	C_{33}	45	16	C_{45}
34	12	C_{34}	46	16	[13]
35	12	sub(C_{36})	47	16	[13]
36	16	C_{36}			

where $d(C^{(i)})$ denotes the minimum weight of $C^{(i)}$ $(i = 1, 2)$.

Codes differing by only a permutation of coordinates are called permutation-equivalent. Any self-dual $\mathbb{Z}_4$-code of length n is permutation-equivalent to a code C with generator matrix of the standard form

$$\begin{pmatrix} I_{k_1} & A & B_1 + 2B_2 \\ O & 2I_{k_2} & 2D \end{pmatrix}, \tag{2}$$

where A, B_1, B_2 and D are $(1, 0)$-matrices, I_k denotes the identity matrix of order k, and O denotes the zero matrix [5]. The residue code $C^{(1)}$ of C is an $[n, k_1]$ code with generator matrix $\begin{pmatrix} I_{k_1} & A & B_1 \end{pmatrix}$, and the torsion code $C^{(2)}$ is an $[n, k_1 + k_2]$ code with generator matrix $\begin{pmatrix} I_{k_1} & A & B_1 \\ O & I_{k_2} & D \end{pmatrix}$.

2.2 Unimodular lattices and upper bounds

A (Euclidean) lattice L in dimension n is *unimodular* if $L = L^*$, where the dual lattice L^* of L is defined as $\{x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L\}$ under the standard inner product (x, y). The norm of a vector x is (x, x). Two lattices L and L' are *isomorphic*, denoted $L \cong L'$, if there exists an orthogonal matrix A with $L' = L \cdot A = \{xA \mid x \in L\}$. The minimum norm of L is the smallest norm among all nonzero vectors of L. The theta series $\theta_L(q)$ of L is the formal power series $\theta_L(q) = \sum_{x \in L} q^{(x,x)} = \sum_{m=0}^{\infty} N_m q^m$, where N_m is the number of vectors of norm m. The kissing number is the second nonzero coefficient of the theta series, that is, the number of vectors of L with minimum norm.

Let $\mu^O_{max}(n)$ denote the largest minimum norm among odd unimodular lattices in dimension n. We say that an odd unimodular lattice is *optimal* if it has the largest minimum norm $\mu^O_{max}(n)$. These values $\mu^O_{max}(n)$ have been determined for $n \le 47, n \ne 37, 41$ (see [6] and [16]). For $25 \le n \le 47$, the following values are known

$$(3) \qquad \mu^O_{max}(n) = \begin{cases} 2 & (n = 25), \\ 3 & (n = 26, 27, \dots, 31, 33, 34, 35), \\ 4 & (n = 32, 36, 38, 39, 40, 42, \dots, 47), \\ 3 \text{ or } 4 & (n = 37, 41). \end{cases}$$

In this paper, we give the first example of an odd unimodular lattice in dimension 41 having minimum norm 4. Hence, we have $\mu^O_{max}(41) = 4$.

Let C be a Type II (resp. Type I) $\mathbb{Z}_4$-code of length n and minimum Euclidean weight d_E. Then the following lattice

$$A_4(C) = \frac{1}{2}\{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid (x_1 \bmod 4, \dots, x_n \bmod 4) \in C\}$$

is an even (resp. odd) unimodular lattice with minimum norm $\min\{4, d_E/4\}$ [2]. Let C be a Type I code of length n and minimum Euclidean weight d_E. Since $A_4(C)$ is an odd unimodular lattice with minimum norm $\min\{4, d_E/4\}$, $\mu^O_{max}(n)$ in (3) implies upper bounds, which improve the upper bounds described in Section 1, for $25 \le n \le 35, n \ne 32$. Hence, we have the following bounds

$$(4) \qquad d^I_{max,E}(n) \le \begin{cases} 8 & (n = 25), \\ 12 & (n = 26, 27, \dots, 31, 33, 34, 35), \\ 16 & (n = 32, 36, 37, \dots, 47). \end{cases}$$

3 Optimal Type I $\mathbb{Z}_4$-codes

In this section, we determine the largest minimum Euclidean weights $d^I_{max,E}(n)$. This is done by constructing Type I codes meeting the bound (4) with equality for lengths $n = 25, \dots, 29, 31, \dots, 36, 41, 43, 44, 45$.

- Lengths $26, \dots, 29, 33, 34, 36, 41, 43, 44, 45$:

  We have found a binary doubly even code B_n of length n with $d(B_n) = 12$ and $d(B_n^\perp) \geq 3$ for $n = 26, \dots, 29, 33, 34$, and $d(B_n) = 16$ and $d(B_n^\perp) \geq 4$ for $n = 36, 41, 43, 44, 45$. Note that there is a self-dual $\mathbb{Z}_4$-code C with $C^{(1)} = B$ for any given binary doubly even code B (see [17]). It follows from (1) that there is a Type I code meeting the bound (4) with equality for these lengths.

  The method of construction of a self-dual $\mathbb{Z}_4$-code C with $C^{(1)} = B$ was given in [17, Section 3]. Using this method, we explicitly have found an optimal Type I code C_n with $C_n^{(1)} = B_n$ for these lengths. In order to save space, instead of listing generator matrices, we only list in Figure 1 the $k_1 \times (n - k_1)$ matrix

  $$M_n = \begin{pmatrix} A & B_1 + 2B_2 \end{pmatrix},$$

  in standard form (2), since the lower part of (2) can be obtained from M_n for each code C_n. The minimum Euclidean, Lee, Hamming weights d_E, d_L, d_H of C_n (see e.g. [2] for the definition of d_L) are listed in Table 2. The parameters $[n, k, d]$ and the orders $\#\,\mathrm{Aut}$ of the automorphism groups of their residue codes $C_n^{(1)}$ are also listed in Table 2. Note that the minimum Hamming weight of a self-dual $\mathbb{Z}_4$-code C is the same as $d(C^{(2)})$ [18].

- Length 32:

  Let L be a unimodular lattice in dimension n and let k be a positive integer. A set $\{f_1, \dots, f_n\}$ of n vectors $f_1, \dots, f_n$ in L with $(f_i, f_j) = k\delta_{ij}$ is called a *k-frame* of L, where δ_{ij} is the Kronecker delta. It is known that an even (resp. odd) unimodular lattice L contains a 4-frame if and only if there is a Type II (resp. Type I) $\mathbb{Z}_4$-code C with $A_4(C) \cong L$.

  Conway and Sloane [6] showed that there are exactly five odd unimodular lattices in dimension 32 having minimum norm 4, up to isomorphism. In addition, such a lattice Λ contains vectors of the form

  $$\frac{1}{\sqrt{8}}(\pm 4, \pm 4, 0, \dots, 0), \dots, \frac{1}{\sqrt{8}}(0, \dots, 0, \pm 4, \pm 4).$$

  Hence, Λ contains a 4-frame. This means that there is a Type I $\mathbb{Z}_4$-code C_{32} of length 32 with $A_4(C_{32}) \cong \Lambda$. Since Λ has minimum norm 4, C_{32} has minimum Euclidean weight 16, that is, C_{32} is optimal.

$$M_{26} = \begin{pmatrix} 01111011110001230212 \\ 11001011111000101120 \\ 11011110000100311232 \\ 00100110101110333302 \\ 11000110111101210021 \\ 01110001111110003003 \end{pmatrix} \qquad M_{27} = \begin{pmatrix} 10011100010113320130 \\ 01011010100013333030 \\ 01010101010002131131 \\ 00010010111112303103 \\ 11110010010101330201 \\ 11111111110002220230 \\ 01111100101111000003 \end{pmatrix}$$

$$M_{28} = \begin{pmatrix} 001101001010102330111 \\ 000110100101011211211 \\ 100011010010101123101 \\ 010001101001011132130 \\ 101000110100100331031 \\ 110100001010011211101 \\ 011010010101001121332 \end{pmatrix} \qquad M_{29} = \begin{pmatrix} 111000100000111103030 \\ 110110011010100200130 3 \\ 110000100110111203010 3 \\ 110111001001000211021 3 \\ 011011100011101001021 2 \\ 101010011001101120201 1 \\ 000111011111100323220 0 \end{pmatrix}$$

$$M_{33} = \begin{pmatrix} 100101011010001020213213 \\ 110010100011100203101010 \\ 011111010001001010333131 \\ 110111000110111200023000 \\ 001010111111011113223001 \\ 101011111111100312212322 \\ 001001111100011302030032 \\ 111110101111110230220033 \\ 010110111101001020201010 \end{pmatrix} \qquad M_{34} = \begin{pmatrix} 010110100101010311033313 \\ 001001110101000113202213 \\ 011010010101003100021323 \\ 000111011101110221331112 \\ 101000101001001031101030 \\ 011111010001100322300012 \\ 110010100111111103122323 \\ 111010010111112213011100 \\ 110000111011110000232003 \\ 111111111100002202022023 \end{pmatrix}$$

$$M_{36} = \begin{pmatrix} 1110000011001101100100 3323121 \\ 01110110011000100010011231213 \\ 11010011111010100111111102121 \\ 01000011111101001010110130212 \\ 00011010110110010100013213121 \\ 00011100011010111101111022010 \\ 11111111111111000000000002023 \end{pmatrix} \qquad M_{41} = \begin{pmatrix} 1001101011010100100110113101331 \\ 1011111100100111100012210033331 \\ 1100110001000111100003301002331 \\ 0001000111100111010012301003121 \\ 0011111001111011101002203222032 \\ 1000011011111010001111023333310 \\ 1011111111100101011011201310220 \\ 0110100100000101010013122113213 \\ 1000010101101100100100130112332 \\ 1110110011011110111101132113031 \end{pmatrix}$$

Figure 1: Generator matrices

$$M_{43} = \begin{pmatrix} 1010101010011010100010011032 10230 \\ 11001100111011001100000122113 31132 \\ 00100101111101110110001313122 0013 \\ 10110111101001001100100303311 0211 \\ 01010111000000110011000330033 0330 \\ 11111110101100100111110110220 2120 \\ 10110111100110111100111223001 2100 \\ 01001111001111001101111102021 1320 \\ 00010011100011111011111302222 1133 \\ 11111111111111100000000000202 32222 \end{pmatrix}$$

$$M_{44} = \begin{pmatrix} 011110001001001001011010122113202 \\ 001111000100100100101103210031100 \\ 000111100011010010010120101021330 \\ 100011110001101001001002010320313 \\ 010001111000110100100130203010013 \\ 001000111101011010010011022123003 \\ 000100011110101101001031320012300 \\ 100010001110010110100103312003012 \\ 110001000111001011010002331022301 \\ 111000100010100101101032013122012 \\ 111100010000010010110121023132003 \end{pmatrix}$$

$$M_{45} = \begin{pmatrix} 000110111110100101100000111131021032 \\ 100011011111010010110000011213322301 \\ 110001101011101001111000001121332032 \\ 111000110101110100111100000232133223 \\ 011100011010111010011110000323013320 \\ 101110001001011101001111000232103112 \\ 110111000100101110000111100001230333 \\ 011011100010010111000011110120101213 \\ 001101110101001011000001111130230121 \end{pmatrix}$$

Figure 1: Generator matrices (continued)

- Lengths $25, 31, 35$:

  Let C be a self-dual code of length n ($n \geq 2$). Then the following code

  $$\text{sub}(C) = \{(x_2, \ldots, x_n) \mid (x_1, x_2, \ldots, x_n) \in C, x_1 \in \{0, 2\}\}$$

  is a self-dual code of length $n - 1$. The codes $\text{sub}(C_{26})$, $\text{sub}(C_{32})$ and $\text{sub}(C_{36})$ are self-dual codes of lengths $25, 31$ and 35, respectively. Moreover, from (4), the codes $\text{sub}(C_{26})$, $\text{sub}(C_{32})$ and $\text{sub}(C_{36})$ have minimum Euclidean weights $8, 12$ and 12, respectively, since C_{26}, C_{32} and C_{36} have minimum Euclidean weight $12, 16$ and 16, respectively.

  Since there are self-dual codes of lengths $8, 17$ and minimum Euclidean weight 8, the direct sum of the codes is also a self-dual code of length 25 and minimum Euclidean weight 8.

By constructing Type I codes meeting the bound (4) with equality for lengths $n = 25, \ldots, 29, 31, \ldots, 36,\ 41, 43, 44, 45,$ we determine the largest minimum Euclidean weight $d^I_{max,E}(n)$ for $n \leq 47,\ n \neq 37$, as follows.

Proposition 1. *Let $d^I_{max,E}(n)$ denote the largest minimum Euclidean weight among Type I $\mathbb{Z}_4$-codes of length n. Then $d^I_{max,E}(25) = 8$, $d^I_{max,E}(n) = 12$ if $n = 26, \ldots, 31, 33, 34, 35$, and $d^I_{max,E}(n) = 16$ if $n = 32, 36, 38, \ldots, 47$.*

Table 2: Optimal Type I $\mathbb{Z}_4$-codes

	C_n			$C_n^{(1)}$	
Code	d_E	d_L	d_H	$[n,k,d]$	# Aut
C_{26}	12	6	3	$[26,6,12]$	120
C_{27}	12	6	3	$[27,7,12]$	240
C_{28}	12	8	4	$[28,7,12]$	10752
C_{29}	12	6	3	$[29,7,12]$	1
C_{33}	12	6	3	$[33,9,12]$	1
C_{34}	12	6	3	$[34,10,12]$	1
C_{36}	16	8	4	$[36,7,16]$	1451520
C_{41}	16	8	4	$[41,10,16]$	1
C_{43}	16	8	4	$[43,10,16]$	1
C_{44}	16	8	4	$[44,11,16]$	11
C_{45}	16	8	4	$[45,9,16]$	9

For length 37, our extensive search failed to discover a Type I $\mathbb{Z}_4$-code with minimum Euclidean weight 16. However, we have found a Type I code with minimum Euclidean weight 12. Hence, $d^I_{max,E}(37) = 12$ or 16 (see Table 1).

4 Optimal odd unimodular lattices

4.1 Dimension 41

By Construction A, optimal Type I $\mathbb{Z}_4$-codes C_n constructed in the previous section give optimal odd unimodular lattices $A_4(C_n)$. In particular, the first explicit example of an optimal odd unimodular lattice $A_4(C_{41})$ in dimension 41 can be constructed from C_{41}.

Proposition 2. *There is an odd unimodular lattice in dimension* 41 *having minimum norm* 4.

We consider the theta series of optimal odd unimodular lattices in dimension 41. Conway and Sloane [6] show that if the theta series of an odd unimodular lattice L in dimension n is written as

$$\theta_L(q) = \sum_{j=0}^{\lfloor n/8 \rfloor} a_j \theta_3(q)^{n-8j} \Delta_8(q)^j, \tag{5}$$

then the theta series of the shadow S (see [6] for the definition) is written as

$$\theta_S(q) = \sum_{j=0}^{\lfloor n/8 \rfloor} \frac{(-1)^j}{16^j} a_j \theta_2(q)^{n-8j} \theta_4(q^2)^{8j} = \sum_i B_i q^i \text{ (say)}, \tag{6}$$

where $\Delta_8(q) = q \prod_{m=1}^{\infty}(1-q^{2m-1})^8(1-q^{4m})^8$ and $\theta_2(q), \theta_3(q)$ and $\theta_4(q)$ are the Jacobi theta series [7]. As the additional conditions, it holds that there is at most one nonzero B_r for $r < (\mu+2)/2$; $B_r = 0$ for $r < \mu/4$; and $B_r \leq 2$ for $r < \mu/2$, where μ is the minimum norm of L.

In the case $n = 41$, since minimum norm μ is 4, $a_0, \dots, a_3$ in (5) and (6) are determined as follows: $a_0 = 1$, $a_1 = -82$, $a_2 = 1476$ and $a_3 = -3280$. Since the coefficients in the shadow must be non-negative integers, a_4 is divisible by 2^7 and a_5 is divisible by 2^{19}. Thus, we put $a_4 = 2^7\alpha$ and $a_5 = -2^{19}\beta$. Then we have the possible theta series θ_L and θ_S of an optimal odd unimodular lattice L in dimension 41 and its shadow S:

$$\begin{aligned}\theta_L = 1 &+ (15170 + 128\alpha)q^4 + (1226720 - 1792\alpha - 524288\beta)q^5 \\ &+ (42928640 + 8192\alpha + 19922944\beta)q^6 + \cdots \text{ and} \\ \theta_S = \beta q^{1/4} &+ (\alpha - 79\beta)q^{9/4} + (104960 - 55\alpha + 3040\beta)q^{17/4} + \cdots,\end{aligned}$$

respectively, where $\beta = 0$ or $\alpha = 79\beta$ by the above additional conditions. By calculating the kissing number of $A_4(C_{41})$ and the minimum norm of its shadow, we determine the theta series of the lattice $A_4(C_{41})$ as follows:

$$\begin{aligned}1 &+ 15426q^4 + 1223136q^5 + 42945024q^6 + 867179520q^7 \\ &+ 11719744560q^8 + 116521216256q^9 + 909236984832q^{10} + \cdots .\end{aligned}$$

4.2 Minimum norms and kissing numbers

In Table 3, we list the minimum norms $\mu(L)$ and the kissing numbers $N(L)$ of optimal odd unimodular lattices $L = A_4(C_n)$ constructed from C_n given in Table 2.

Table 3: Minimum norms and kissing numbers

L	$\mu(L)$	$N(L)$	L	$\mu(L)$	$N(L)$
$A_4(C_{26})$	3	3120	$A_4(C_{36})$	4	51032
$A_4(C_{27})$	3	2664	$A_4(C_{41})$	4	15426
$A_4(C_{28})$	3	1728	$A_4(C_{43})$	4	9286
$A_4(C_{29})$	3	1856	$A_4(C_{44})$	4	8392
$A_4(C_{33})$	3	752	$A_4(C_{45})$	4	7866
$A_4(C_{34})$	3	528			

For dimensions up to 28, optimal odd unimodular lattices have been classified (see [7, p. xliii–xliv]). Borcherds [3] showed that there is a unique optimal odd unimodular lattice S_{26} in dimension 26 (see [7, p. xliii]). Hence, the lattice $A_4(C_{26})$ gives an alternative construction of S_{26}. We list the symmetrized weight enumerator swe_{26} (see [5] for the definition) of C_{26} at the end of this section. Bacher and Venkov [1] showed that there are three (resp. 38) non-isomorphic optimal odd unimodular lattices in dimension 27 (resp. 28) (see [7, p. xliv]). By comparing the kissing numbers and the automorphism groups, we have that $A_4(C_{27}) \cong \mathbf{R_{27,1}}(\emptyset)$ in [1, Table 4], and $A_4(C_{28}) \cong \mathbf{R_{28,38e}}(\emptyset)$ in [1, Table 5]. Hence, these lattices S_{26}, $\mathbf{R_{27,1}}(\emptyset)$ and $\mathbf{R_{28,38e}}(\emptyset)$ contain a 4-frame. We remark that the lattices have no 3-frame.

For other dimensions, since the lattices $A_4(C_n)$ $(n = 33, 36, 44)$ have different kissing numbers than those of the known lattices in [16] and no example was given in [16] for dimension 30, the lattices $A_4(C_n)$ $(n = 30, 33, 36, 44)$ provide other examples of optimal odd unimodular lattices.

Since an odd unimodular lattice in dimension 41 having minimum norm 4 has been constructed, the largest minimum norm $\mu^O_{max}(41)$ is 4. The smallest dimension n for which the largest minimum norm $\mu^O_{max}(n)$ has not been determined is 37. Hence, it is worthwhile to determine if there is a Type I $\mathbb{Z}_4$-code of length 37 and minimum Euclidean weight 16. At dimension 48, the largest minimum norm $\mu^O_{max}(48)$ is exactly 5 ([10], [14] and [19]). No Type I $\mathbb{Z}_4$-code constructs an odd unimodular lattice with minimum norm 5 by Construction A. It seems that the connection between self-dual $\mathbb{Z}_4$-codes and unimodular lattices is no longer useful at this point.

$$
\begin{aligned}
swe_{26} =& a^{26} + 30a^{23}c^3 + 255a^{22}c^4 + 1100a^{21}c^5 + 3571a^{20}c^6 + 9990a^{19}c^7 + 24330a^{18}c^8 \\
&+ 49680a^{17}c^9 + 83237a^{16}c^{10} + 119004a^{15}c^{11} + 2880a^{14}b^{12} + 150750a^{14}c^{12} \\
&+ 40320a^{13}b^{12}c + 164680a^{13}c^{13} + 262080a^{12}b^{12}c^2 + 150750a^{12}c^{14} + 1048320a^{11}b^{12}c^3 \\
&+ 119004a^{11}c^{15} + 17408a^{10}b^{16} + 2882880a^{10}b^{12}c^4 + 83237a^{10}c^{16} + 174080a^9b^{16}c \\
&+ 5765760a^9b^{12}c^5 + 49680a^9c^{17} + 783360a^8b^{16}c^2 + 8648640a^8b^{12}c^6 + 24330a^8c^{18} \\
&+ 2088960a^7b^{16}c^3 + 9884160a^7b^{12}c^7 + 9990a^7c^{19} + 16384a^6b^{20} + 3655680a^6b^{16}c^4 \\
&+ 8648640a^6b^{12}c^8 + 3571a^6c^{20} + 98304a^5b^{20}c + 4386816a^5b^{16}c^5 + 5765760a^5b^{12}c^9 \\
&+ 1100a^5c^{21} + 245760a^4b^{20}c^2 + 3655680a^4b^{16}c^6 + 2882880a^4b^{12}c^{10} + 255a^4c^{22} \\
&+ 327680a^3b^{20}c^3 + 2088960a^3b^{16}c^7 + 1048320a^3b^{12}c^{11} + 30a^3c^{23} + 245760a^2b^{20}c^4 \\
&+ 783360a^2b^{16}c^8 + 262080a^2b^{12}c^{12} + 98304ab^{20}c^5 + 174080ab^{16}c^9 + 40320ab^{12}c^{13} \\
&+ 16384b^{20}c^6 + 17408b^{16}c^{10} + 2880b^{12}c^{14} + c^{26}
\end{aligned}
$$

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