Residue codes of extremal Type II $\mathbb{Z}_4$-codes and the moonshine vertex operator algebra

Masaaki Harada[*], Ching Hung Lam[†] and Akihiro Munemasa[‡]

August 9, 2012

Dedicated to Professor Masahiko Miyamoto on the occasion of his sixtieth birthday

Abstract

In this paper, we study the residue codes of extremal Type II $\mathbb{Z}_4$-codes of length 24 and their relations to the famous moonshine vertex operator algebra. The main result is a complete classification of all residue codes of extremal Type II $\mathbb{Z}_4$-codes of length 24. Some corresponding results associated to the moonshine vertex operator algebra are also discussed.

Keywords: moonshine vertex operator algebra, framed vertex operator algebra, binary triply even code, binary doubly even code, extremal Type II $\mathbb{Z}_4$-code.

AMS Subject Classification: 17B69, 94B05, 20E32

1 Introduction

In this paper, we study the residue codes of extremal Type II $\mathbb{Z}_4$-codes of length 24 and their relationship to the famous moonshine vertex operator algebra (VOA). The main

[*]Department of Mathematical Sciences, Yamagata University, Yamagata 990–8560, Japan, and PRESTO, Japan Science and Technology Agency, Kawaguchi, Saitama 332–0012, Japan. email: mharada@sci.kj.yamagata-u.ac.jp, telephone number: +81-236-28-4533, fax number: +81-236-28-4538

[†]Institute of Mathematics, Academia Sinica, Taipei 106-17, Taiwan. email: chlam@math.sinica.edu.tw

[‡]Graduate School of Information Sciences, Tohoku University, Sendai 980–8579, Japan. email: munemasa@math.is.tohoku.ac.jp

result is a complete classification of all residue codes of extremal Type II $\mathbb{Z}_4$-codes of length 24. Some corresponding results about the structure codes of the moonshine vertex operator algebra are also discussed. Since the residue code of an extremal Type II $\mathbb{Z}_4$-code of length 24 is contained in some binary doubly even self-dual codes and binary doubly even self-dual codes of length 24 are classified in [PS75], we can list all binary doubly even codes B satisfying the condition that its dual code $B^{\perp}$ is even and $B^{\perp}$ has minimum weight ≥ 4. It turns out that there are 179 such codes up to equivalence (Table 1). Then, by using the algorithm given in [Ra99], we determine all binary doubly even codes that can be realized as the residue codes of some extremal Type II $\mathbb{Z}_4$-codes. There are 149 codes that are realizable (Table 1). We also prove that if $B' \supset B$ is a weight 4 augmentation of B (see Definition 3.1) and B is realized as the residue code of an extremal Type II $\mathbb{Z}_4$-code, then B' is also realized (Lemma 3.3). Not only does this result reduce the amount of computation, but it also helps us to express the main result in a nicer form (Theorem 3.8).

Our main motivation, on the other hand, is to determine the possible $\frac{1}{16}$-codes of the moonshine vertex operator algebra. We call a triply even code of length 48 a moonshine code if it is a $\frac{1}{16}$-code of the moonshine vertex operator algebra. Given an extremal Type II $\mathbb{Z}_4$-code $\mathcal{C}$ of length 24, one can obtain the Leech lattice by Construction A (Lemma 2.1),

$$A_4(\mathcal{C}) = \frac{1}{2}\left\{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid (x_1 \bmod 4, \ldots, x_n \bmod 4) \in \mathcal{C}\right\}.$$

The code $\mathcal{C}$ determines a 4-frame of the Leech lattice, which will define a Virasoro frame of the Leech lattice VOA [DGH98]. Since the Moonshine VOA $V^{\natural}$ is constructed as a $\mathbb{Z}_2$-orbifold from the Leech lattice VOA, the 4-frame of the Leech lattice also defines a Virasoro frame of $V^{\natural}$ and the corresponding moonshine code is the extended doubling (see Definition 4.4) of the residue code of $\mathcal{C}$ (see [DGH98] and Proposition 4.6). We show that the converse also holds and the extended doubling of a doubly even code B of length 24 is a moonshine code if and only if B is the residue code of some extremal Type II $\mathbb{Z}_4$-code (Theorem 4.8). Together with our main result, this means that we know all the moonshine codes which are extended doublings.

The organization of the paper is as follows. In Section 2, definitions and some basic results of codes, which are used in this paper, are given. In Section 3, we classify residue codes of extremal Type II $\mathbb{Z}_4$-codes of length 24. We also show that a binary doubly even code is the residue code of an extremal Type II $\mathbb{Z}_4$-code of length 24 if and only if it can be obtained by successive application of weight 4 augmentation to one of the codes listed in Table 2. In Section 4, we study the structure codes of the moonshine vertex operator algebra, which we call moonshine codes. In particular, we show that a binary triply even code is a moonshine code if and only if it can be obtained by successive application of weight 8 augmentation to a moonshine code of minimum weight 16. As a consequence, we also show that the direct sum of the extended doublings of its components are moonshine codes.

2 Binary codes and $\mathbb{Z}_4$-codes

In this paper, we deal with binary codes and $\mathbb{Z}_4$-codes, and all binary codes and $\mathbb{Z}_4$-codes are linear. In addition, codes mean binary codes unless otherwise specified. Let C be a code of length n. The *weight* $\mathrm{wt}(x)$ of a codeword $x \in C$ is the number of non-zero coordinates. A code C is called *even, doubly even* and *triply even* if the weights of all codewords of C are divisible by 2, 4 and 8, respectively. The dual code $C^{\perp}$ of C is defined as $\{x \in \mathbb{Z}_2^n \mid \langle x, y\rangle = 0 \text{ for all } y \in C\}$, where $\langle x, y\rangle$ denotes the standard inner product. A code C is *self-orthogonal* if $C \subset C^{\perp}$, and C is *self-dual* if $C = C^{\perp}$. Two codes are *equivalent* if one can be obtained from the other by a permutation of coordinates. Throughout this paper, we denote the all-one vector by $\mathbf{1}$ and the zero vector by $\mathbf{0}$. For a code C of length n and a vector $\delta \in \mathbb{Z}_2^n$, we denote by $\langle C, \delta\rangle_{\mathbb{Z}_2}$ the code generated by the codewords of C and δ.

For a $\mathbb{Z}_4$-code $\mathcal{C}$ of length n, define two codes:

$$\mathcal{C}_0 = \{\alpha \bmod 2 \mid \alpha \in \mathbb{Z}_4^n,\ 2\alpha \in \mathcal{C}\} \quad \text{and} \quad \mathcal{C}_1 = \{\alpha \bmod 2 \mid \alpha \in \mathcal{C}\}.$$

These codes $\mathcal{C}_0$ and $\mathcal{C}_1$ are called *torsion* and *residue* codes, respectively. It holds that $\mathcal{C}_1 \subset \mathcal{C}_0$. For a $\mathbb{Z}_4$-code $\mathcal{C}$, the dual code $\mathcal{C}^{\perp}$ is defined similarly to binary codes. Then self-orthogonal codes and self-dual codes are also defined similarly. If $\mathcal{C}$ is self-dual, then $\mathcal{C}_1$ is doubly even and $\mathcal{C}_0 = \mathcal{C}_1^{\perp}$. The *Euclidean weight* of a codeword $x = (x_1, \ldots, x_n)$ of $\mathcal{C}$ is $n_1(x) + 4n_2(x) + n_3(x)$, where $n_\alpha(x)$ denotes the number of components i with $x_i = \alpha$ $(\alpha = 1, 2, 3)$. A $\mathbb{Z}_4$-code $\mathcal{C}$ is called a *Type II* $\mathbb{Z}_4$-code if $\mathcal{C}$ is self-dual and the Euclidean weights of all codewords of $\mathcal{C}$ are divisible by 8. The *minimum Euclidean weight* d_E of $\mathcal{C}$ is the smallest Euclidean weight among all nonzero codewords of $\mathcal{C}$. A Type II $\mathbb{Z}_4$-code of length n and $d_E = 8\lfloor n/24\rfloor + 8$ is called *extremal*. Two $\mathbb{Z}_4$-codes are *equivalent* if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates.

Let $\mathcal{C}$ be a self-orthogonal $\mathbb{Z}_4$-code of length n. Define

$$A_4(\mathcal{C}) = \frac{1}{2}\{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid (x_1 \bmod 4, \ldots, x_n \bmod 4) \in \mathcal{C}\}.$$

It is well-known that $A_4(\mathcal{C})$ is even unimodular if and only if $\mathcal{C}$ is Type II. The following result is also well-known (cf. [BSBM97]).

Lemma 2.1. *Let $\mathcal{C}$ be a Type II $\mathbb{Z}_4$-code of length n. Then, $\mathcal{C}$ has minimum Euclidean weight at least 16 if and only if $A_4(\mathcal{C})$ has minimum norm 4. In particular, for $n = 24$, $\mathcal{C}$ is extremal if and only if $A_4(\mathcal{C})$ is isomorphic to the Leech lattice Λ.*

Lemma 2.2. *Let C be the residue code of an extremal Type II $\mathbb{Z}_4$-code of length 24. Then C satisfies the following conditions:*

$$C \text{ is doubly even;} \tag{1}$$

$$C \ni \mathbf{1}; \tag{2}$$

$$C^{\perp} \text{ has minimum weight at least } 4. \tag{3}$$

In particular, $\dim C \geq 6$.

Proof. For the proofs of the assertions (1), (2) and (3), see [CS93], [HSG99] and [Ha10], respectively. It is known that a $[24, k, 4]$ code exists only if $k \leq 18$ [Br98]. This gives the last assertion. ∎

3 Classification of residue codes of extremal Type II $\mathbb{Z}_4$-codes

In this section, we classify all residue codes of extremal Type II $\mathbb{Z}_4$-codes of length 24.

3.1 Weight 4 augmentation

Definition 3.1. Let C be a subcode of a doubly even code C' and let k be a positive integer divisible by 4. We call C' a *weight k augmentation of C* if $C' \neq C$, and C' is generated by C and a vector of weight k.

Recall that two lattices L and L' are neighbors if both lattices contain a sublattice of index 2 in common.

Lemma 3.2. *Let Λ be an even unimodular lattice with minimum norm 4, and suppose that $\alpha \in \Lambda$ satisfies $\|\alpha\|^2 = 4$, where $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$. Define*

$$\Lambda_\alpha = \{\gamma \in \Lambda \mid \langle \alpha, \gamma \rangle \equiv 0 \pmod 2\}. \tag{4}$$

If $\beta \in \Lambda \setminus \Lambda_\alpha$, then

$$\Lambda'_{\alpha,\beta} = \Lambda_\alpha \cup (\frac{1}{2}\alpha + \beta + \Lambda_\alpha) \tag{5}$$

is an even unimodular lattice with minimum norm 4, which is a neighbor of Λ sharing Λ_α.

Proof. Since Λ is even, $\frac{1}{2}\alpha \notin \Lambda = \Lambda^*$. This implies that Λ_α is a sublattice of index 2 in Λ. Clearly, $\frac{1}{2}\alpha \in \Lambda_\alpha^*$, and $\beta \in \Lambda = \Lambda^* \subset \Lambda_\alpha^*$, hence $\frac{1}{2}\alpha + \beta \in \Lambda_\alpha^*$. Since $2(\frac{1}{2}\alpha + \beta) \in \Lambda_\alpha$, we conclude that $\Lambda'_{\alpha,\beta}$ is a unimodular lattice. Moreover, since $\langle \alpha, \beta \rangle \equiv 1 \pmod 2$, we have $\|\frac{1}{2}\alpha + \beta\|^2 \equiv 0 \pmod 2$. Thus $\Lambda'_{\alpha,\beta}$ is even.

It remains to show that $\Lambda'_{\alpha,\beta}$ has minimum norm 4. Since $\alpha \in \Lambda_\alpha \subset \Lambda$, it suffices to show that

$$\|\frac{1}{2}\alpha + \beta + \gamma\|^2 \geq 4$$

for all $\gamma \in \Lambda_\alpha$. This follows from the inequality

$$\|\frac{1}{2}\alpha + \beta + \gamma\|^2 = \frac{1}{2}\|\alpha + \beta + \gamma\|^2 + \frac{1}{2}\|\beta + \gamma\|^2 - 1 \geq 3,$$

noting that $\Lambda'_{\alpha,\beta}$ is even. ∎

The following lemma is very useful for our classification.

Lemma 3.3. *Let $\mathcal{C}$ be a Type II $\mathbb{Z}_4$-code of length n with minimum Euclidean weight* 16*. Let $a \in \mathbb{Z}_2^n \setminus \mathcal{C}_1$. Suppose $a + \mathcal{C}_1$ has a vector of weight* 4 *and the code $\langle \mathcal{C}_1, a \rangle_{\mathbb{Z}_2}$ is doubly even. Then there exists a Type II $\mathbb{Z}_4$-code $\mathcal{C}'$ such that the minimum Euclidean weight is* 16*, $\mathcal{C}'_1 = \langle \mathcal{C}_1, a \rangle_{\mathbb{Z}_2}$ and $A_4(\mathcal{C}')$ is a neighbor of $A_4(\mathcal{C})$.*

Proof. By the assumption, $\Lambda = A_4(\mathcal{C})$ is an even unimodular lattice with minimum norm 4. Without loss of generality, we may also assume a has weight 4. Let $e_1, \dots, e_n$ be the standard orthonormal basis of $\mathbb{R}^n$, and let

$$\alpha = \sum_{i \in \mathrm{supp}(a)} e_i,$$

where $\mathrm{supp}(a)$ denotes the support of a. Since $\langle \mathcal{C}_1, a \rangle_{\mathbb{Z}_2}$ is self-orthogonal, we have $a \in \mathcal{C}_1^\perp = \mathcal{C}_0$. Thus, $\alpha \in \Lambda$ and $\|\alpha\|^2 = 4$. Define Λ_α by (4). Since $a \notin \mathcal{C}_1 = \mathcal{C}_0^\perp$, there exists $b \in \mathcal{C}_0$ such that $\langle a, b \rangle = 1$. Let $\beta \in \mathbb{Z}^n$ be a vector satisfying $\beta \bmod 2 = b$. Then, $b \in \mathcal{C}_0$ implies $\beta \in \Lambda$. Moreover, $\langle a, b \rangle = 1$ implies $\langle \alpha, \beta \rangle \equiv 1 \pmod 2$. Thus Lemma 3.2 implies that the lattice $\Lambda'_{\alpha,\beta}$ defined by (5) is an even unimodular lattice with minimum norm 4, which is a neighbor of Λ. Since the standard 4-frame $\{2e_i\}_{i=1}^n$ is contained in $\Lambda_\alpha \subset \Lambda'_{\alpha,\beta}$, there exists a Type II $\mathbb{Z}_4$-code $\mathcal{C}'$ such that $A_4(\mathcal{C}') = \Lambda'_{\alpha,\beta}$.

Since

$$\begin{aligned}\mathcal{C}_1 &= \{\gamma \bmod 2 \mid \frac{1}{2}\gamma \in \Lambda\} \\ &= \{\gamma \bmod 2 \mid \frac{1}{2}\gamma \in \Lambda_\alpha\} \cup \{\gamma \bmod 2 \mid \frac{1}{2}\gamma \in \beta + \Lambda_\alpha\}\end{aligned}$$

$$= \{\gamma \bmod 2 \mid \gamma \in 2\Lambda_\alpha\}, \tag{6}$$

we have

$$\begin{aligned}
\mathcal{C}'_1 &= \{\gamma \bmod 2 \mid \frac{1}{2}\gamma \in \Lambda'_{\alpha,\beta}\} \\
&= \{\gamma \bmod 2 \mid \frac{1}{2}\gamma \in \Lambda_\alpha\} \cup \{\gamma \bmod 2 \mid \frac{1}{2}\gamma \in \frac{1}{2}\alpha + \beta + \Lambda_\alpha\} && \text{(by (5))} \\
&= \{\gamma \bmod 2 \mid \gamma \in 2\Lambda_\alpha\} \cup \{\gamma \bmod 2 \mid \gamma \in \alpha + 2\Lambda_\alpha\} \\
&= \mathcal{C}_1 \cup (a + \mathcal{C}_1) && \text{(by (6))} \\
&= \langle \mathcal{C}_1, a\rangle_{\mathbb{Z}_2}.
\end{aligned}$$

Since $A_4(\mathcal{C}')$ has minimum norm 4, $\mathcal{C}'$ has minimum Euclidean weight at least 16 by Lemma 2.1. Since $\mathcal{C}'_0 \supset \mathcal{C}'_1 \ni a$ and $\mathrm{wt}(a) = 4$, there is a codeword of Euclidean weight 16 in $\mathcal{C}'$. Hence, the minimum Euclidean weight of $\mathcal{C}'$ is exactly 16. ∎

A partial converse of the above lemma also holds.

Lemma 3.4. *Let $\mathcal{C}$ be a Type II $\mathbb{Z}_4$-code of length n with minimum Euclidean weight* 16*. Suppose $a \in \mathcal{C}_1$ and* $\mathrm{wt}(a) = 4$*. Then there exists a Type II $\mathbb{Z}_4$-code $\mathcal{C}'$ of length n such that the minimum Euclidean weight is* 16*, $\mathcal{C}'_1 \subsetneqq \langle \mathcal{C}'_1, a\rangle = \mathcal{C}_1$ and $A_4(\mathcal{C}')$ is a neighbor of $A_4(\mathcal{C})$.*

Proof. By the assumption, $\Lambda = A_4(\mathcal{C})$ is an even unimodular lattice with minimum norm 4. We may assume without loss of generality $a_1 = 1$ in $a = (a_1, \dots, a_n)$. Since $a \in \mathcal{C}_1$, there exists $\alpha' = (\alpha'_1, \dots, \alpha'_n) \in \mathbb{Z}^n$ such that $\alpha' \bmod 2 = a$ and

$$\frac{1}{2}\alpha' \in \Lambda. \tag{7}$$

We may assume without loss of generality

$$\alpha'_i = \pm 1 \quad (i \in \mathrm{supp}(a)).$$

Define $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ by

$$\alpha_i = \begin{cases} -\alpha'_1 & \text{if } i = 1, \\ \alpha'_i & \text{if } i \in \mathrm{supp}(a) \setminus \{1\}, \\ 0 & \text{otherwise,} \end{cases}$$

and set $c = \frac{1}{2}(\alpha - \alpha') \bmod 2 \in \mathbb{Z}_2^n$. Then $\alpha \in \Lambda$, $\|\alpha\|^2 = 4$, and

$$\alpha \bmod 2 = \alpha' \bmod 2 = a, \tag{8}$$

$$\langle \alpha, \alpha' \rangle = 2, \tag{9}$$

$$\langle a, c \rangle = 1. \tag{10}$$

Define Λ_α by (4), and set

$$\beta = -\frac{1}{2}\alpha' \in \Lambda.$$

Then by (9), we have $\beta \in \Lambda \setminus \Lambda_\alpha$. Thus Lemma 3.2 implies that the lattice $\Lambda'_{\alpha,\beta}$ defined by (5) is an even unimodular lattice with minimum norm 4, which is a neighbor of Λ. Since $\alpha \in \mathbb{Z}^n$, the standard 4-frame of Λ is contained in Λ_α. This implies that there exists a Type II $\mathbb{Z}_4$-code $\mathcal{C}'$ of length n such that $\Lambda'_{\alpha,\beta} = A_4(\mathcal{C}')$.

Since

$$\begin{aligned}
\Lambda_\alpha &= \{\frac{1}{2}\gamma \in \Lambda \mid \langle \alpha, \frac{1}{2}\gamma \rangle \equiv 0 \pmod 2\} && \text{(by (4))} \\
&= \{\frac{1}{2}\gamma \in \Lambda \mid \langle \alpha - \alpha', \frac{1}{2}\gamma \rangle \equiv 0 \pmod 2\} && \text{(by (7))} \\
&= \{\frac{1}{2}\gamma \in \Lambda \mid \langle c, \gamma \bmod 2 \rangle = 0\},
\end{aligned} \tag{11}$$

we have

$$\begin{aligned}
\mathcal{C}'_1 &= \{\gamma \bmod 2 \mid \frac{1}{2}\gamma \in \Lambda'_{\alpha,\beta}\} \\
&= \{\gamma \bmod 2 \mid \frac{1}{2}\gamma \in \Lambda_\alpha\} \cup \{\gamma \bmod 2 \mid \frac{1}{2}\gamma \in \frac{1}{2}\alpha + \beta + \Lambda_\alpha\} && \text{(by (5))} \\
&= \{\gamma \bmod 2 \mid \frac{1}{2}\gamma \in \Lambda_\alpha\} \cup \{\gamma \bmod 2 \mid \gamma \in \alpha - \alpha' + 2\Lambda_\alpha\} \\
&= \{\gamma \bmod 2 \mid \frac{1}{2}\gamma \in \Lambda_\alpha\} && \text{(by (8))} \\
&= \{\gamma \bmod 2 \mid \frac{1}{2}\gamma \in \Lambda, \langle c, \gamma \bmod 2 \rangle = 0\} && \text{(by (11))} \\
&= \{b \in \mathcal{C}_1 \mid \langle b, c \rangle = 0\}.
\end{aligned}$$

It follows from (10) that $a \notin \mathcal{C}'_1$, and hence $\mathcal{C}'_1 \subsetneqq \langle \mathcal{C}'_1, a \rangle = \mathcal{C}_1$.

Since $A_4(\mathcal{C}')$ has minimum norm 4, $\mathcal{C}'$ has minimum Euclidean weight at least 16 by Lemma 2.1. Since $a \in \mathcal{C}_1 \subset \mathcal{C}_0 \subset \mathcal{C}'_0$ and $\mathrm{wt}(a) = 4$, there is a codeword of Euclidean weight 16 in $\mathcal{C}'$. Hence, the minimum Euclidean weight of $\mathcal{C}'$ is exactly 16. ∎

In Section 4, we shall give analogues of Lemmas 3.3 and 3.4 for moonshine codes.

Definition 3.5. We say that a code C of length 24 satisfying (1)–(3) is *realizable* if C can be realized as the residue code of some extremal Type II $\mathbb{Z}_4$-code.

By Lemmas 3.3 and 3.4, we have the following corollary.

Corollary 3.6. *A doubly even code B of length* 24 *is realizable if and only if B can be obtained by successive weight* 4 *augmentations from a realizable code with minimum weight* 8.

3.2 Complete classification

There exist nine inequivalent doubly even self-dual codes of length 24 [PS75]. The extended Golay code g_{24} is the unique doubly even self-dual $[24, 12, 8]$ code, and the other codes have minimum weight 4 and these codes are described by specifying the subcodes spanned by the codewords of weight 4, namely, $d_{12}^2, d_{10}e_7^2, d_8^3, d_6^4, d_{24}, d_4^6, e_8^3, d_{16}e_8$. Since any code C of length 24 satisfying (1)–(3) is contained in a doubly even self-dual code of length 24, the classification of codes C satisfying (1)–(3) can be done by taking successively subcodes of codimension 1 starting from doubly even self-dual codes. This method allows us to classify all codes satisfying (1)–(3). We list in the second column of Table 1 the numbers of inequivalent $[24, k]$ codes satisfying (1)–(3). We remark that this classification is of independent interest, as it forms a basis for a possible classification of extremal Type II $\mathbb{Z}_4$-codes of length 24.

Table 1: Numbers of inequivalent codes of length 24 satisfying (1)–(3)

Dimensions k	Total	$R_{k,8}$	$R_{k,4}$	$N_{k,8}$	$N_{k,4}$
12	9	1	8	0	0
11	21	1	20	0	0
10	49	3	44	0	2
9	60	6	40	4	10
8	32	4	16	8	4
7	7	3	2	2	0
6	1	1	0	0	0

$R_{k,d}$ = the number of inequivalent realizable $[24, k, d]$ codes.
$N_{k,d}$ = the number of inequivalent non-realizable $[24, k, d]$ codes.

We use the algorithm of Rains [Ra99] to determine if a given $[24, k]$ code C satisfying (1)–(3) is realizable or not. The algorithm is described in the form of the proof of [Ra99, Theorem 3] for classifying self-dual $\mathbb{Z}_4$-codes, and its modification to Type II $\mathbb{Z}_4$-codes is straightforward. Here, we describe the algorithm briefly. We first construct the action of the automorphism group $\mathrm{Aut}(C)$ of C on the quotient $Q(C)$ of the $1+\frac{k(k-1)}{2}$ dimensional space of all Type II $\mathbb{Z}_4$-codes $\mathcal{C}$ with $\mathcal{C}_1 = C$, by column negations. This defines a homomorphism from $\mathrm{Aut}(C)$ to $AGL(m,2)$, where $m = \dim Q(C)$, and the orbits are in one-to-one correspondence with equivalence classes of Type II $\mathbb{Z}_4$-codes $\mathcal{C}$ with $\mathcal{C}_1 = C$. By enumerating orbit representatives, we obtain all Type II $\mathbb{Z}_4$-codes $\mathcal{C}$ with $\mathcal{C}_1 = C$ up to equivalence. If none of the codes $\mathcal{C}$ with $\mathcal{C}_1 = C$ is extremal, we conclude that C is non-realizable. This algorithm can be executed when $\dim C \leq 10$, since the maximum value of m turns out to be 26. Note that Rains [Ra99, p. 220] in 1999 commented that direct orbit finding of a 26-dimensional matrix group is somewhat tricky. However, with 10GB of memory, such a computation can be done without problem nowadays. In particular, we obtain the following result.

Proposition 3.7. *Up to equivalence, there is a unique extremal Type II $\mathbb{Z}_4$-code of length* 24 *whose residue code has dimension* 6*.*

We denote this code by $\mathcal{C}^\natural$ and its generator matrix is given in Figure 1.

$$\begin{bmatrix}
1111 & 1111 & 1111 & 1111 & 0000 & 0000\\
0200 & 0000 & 1111 & 3111 & 0002 & 0000\\
1111 & 1111 & 0000 & 0000 & 1111 & 1111\\
0100 & 1011 & 1013 & 0102 & 1011 & 0100\\
1110 & 0001 & 1130 & 0201 & 0001 & 1110\\
0111 & 1000 & 3020 & 0111 & 0111 & 1000\\
0000 & 0000 & 2022 & 0200 & 0000 & 0000\\
0000 & 0000 & 2220 & 0002 & 0000 & 0000\\
0000 & 0000 & 2000 & 0222 & 0000 & 0000\\
0200 & 2022 & 0000 & 0000 & 0000 & 0000\\
2220 & 0002 & 0000 & 0000 & 0000 & 0000\\
0222 & 2000 & 0000 & 0000 & 0000 & 0000\\
0200 & 0002 & 2020 & 0000 & 0000 & 0000\\
0200 & 2000 & 2000 & 0200 & 0000 & 0000\\
0220 & 0000 & 2000 & 0002 & 0000 & 0000\\
0200 & 0002 & 0000 & 0000 & 0002 & 0200\\
0200 & 2000 & 0000 & 0000 & 0022 & 0000\\
0220 & 0000 & 0000 & 0000 & 0002 & 2000
\end{bmatrix}$$

Figure 1: A generator matrix of the extremal Type II $\mathbb{Z}_4$-code $\mathcal{C}^\natural$

When $\dim C = 11$ or 12, the value of m ranges from 33 to 46, and a direct method will fail. However, we randomly found an extremal Type II $\mathbb{Z}_4$-code $\mathcal{C}$ with $\mathcal{C}_1 = C$ without finding all inequivalent Type II $\mathbb{Z}_4$-codes. The two doubly even self-dual codes with labels g_{24} and d_{24} can be realized as the residue codes of some extremal Type II $\mathbb{Z}_4$-codes of length 24 [CS97]. Moreover, Young and Sloane claim to have found an extremal Type II $\mathbb{Z}_4$-code $\mathcal{C}$ with $C = \mathcal{C}_1$ for each C of the remaining seven codes (cf. [CS97, Postscript]), although no explicit information about such codes has been published since then. In Appendix A.1, we give such an extremal Type II $\mathbb{Z}_4$-code for each of the seven doubly even self-dual codes. In particular, our result for the case $\dim C = 12$ confirms the claim in [CS97, Postscript].

In Table 1, we list the number $R_{k,d}$ of inequivalent realizable $[24, k, d]$ codes. All such codes with $d = 8$ are listed in Table 2, where $C_6 = \mathcal{C}^{\natural}{}_1$ and $C_{7,1}, C_{7,2}$ are defined in Appendix A.2, and the codes other than $C_6, C_{7,1}, C_{7,2}$ are generated by the code C and the vectors v listed in Table 3. Note that $C_6, C_{7,1}, C_{7,2}$ are minimal subject to (1)–(3), with respect to the subspace relation. Also, in Table 1, we list the number $N_{k,d}$ of inequivalent non-realizable $[24, k, d]$ codes. Maximal codes (with respect to the subspace relation) among these codes are listed in Table 4, where the codes are generated by C_6 and the vectors v listed in Table 5. All other non-realizable codes can be obtained from a maximal one; see Theorem 3.8 (iii) below. Also, in Table 4, we give the dimension m of the quotient space $Q(C)$ and the number N of the equivalence classes of Type II $\mathbb{Z}_4$-codes $\mathcal{C}$ with $\mathcal{C}_1 = C$ for the maximal non-realizable codes C. Generator matrices of all codes in Table 1 can be obtained electronically from

`http://www.math.is.tohoku.ac.jp/~munemasa/de24extremalresidue.htm`

The following is the main theorem of the paper.

Theorem 3.8. *Let C be a doubly even code of length 24 containing $\mathbf{1}$ such that $C^{\perp}$ has minimum weight at least 4. Then the following are equivalent.*

(i) the code C is the residue code of some extremal Type II $\mathbb{Z}_4$-code;

(ii) successive applications of weight 4 augmentation to one of the codes in Table 2 gives a code equivalent to C;

(iii) none of the codes in Table 4 can be obtained by successive applications of weight 4 augmentation to a code equivalent to C.

Proof. That (i) is equivalent to (ii) follows from Corollary 3.6. The implication (i) $\Longrightarrow$ (iii) follows from Lemma 3.3. The implication (iii) $\Longrightarrow$ (ii) can be verified by classifying all the pairs (S, C) such that C is a weight 4 augmentation of a subcode S of C of codimension 1. ∎

Table 2: Realizable codes with minimum weight 8

Codes	C	v	Codes	C	v
C_6			$C_{9,3}$	$C_{7,3}$	v_{931}, v_{932}
$C_{7,1}$			$C_{9,4}$	$C_{8,3}$	v_{94}
$C_{7,2}$			$C_{9,5}$	$C_{8,1}$	v_{95}
$C_{7,3}$	C_6	v_7	$C_{9,6}$	$C_{8,3}$	v_{96}
$C_{8,1}$	$C_{7,3}$	v_{81}	$C_{10,1}$	$C_{9,4}$	v_{101}
$C_{8,2}$	$C_{7,3}$	v_{82}	$C_{10,2}$	$C_{9,4}$	v_{102}
$C_{8,3}$	$C_{7,3}$	v_{83}	$C_{10,3}$	$C_{9,4}$	v_{103}
$C_{8,4}$	C_6	v_{841}, v_{842}	C_{11}	$C_{10,1}$	v_{11}
$C_{9,1}$	$C_{8,3}$	v_{91}	C_{12}	C_{11}	v_{12}
$C_{9,2}$	$C_{8,4}$	v_{92}			

4 Moonshine vertex operator algebra and its structure codes

In this section, we study the relationship between the moonshine vertex operator algebra and extremal Type II $\mathbb{Z}_4$-codes. Our notations for vertex operator algebras (VOA) and framed VOAs are standard. We shall refer to [FLM88, DGH98, LY08] for details.

4.1 Moonshine codes and extremal Type II $\mathbb{Z}_4$-codes

Recall that the moonshine VOA $V^\natural$ is constructed by [FLM88] as a $\mathbb{Z}_2$-orbifold of the Leech lattice VOA V_Λ. Namely,

$$V^\natural = \tilde{V}_\Lambda = (V_\Lambda)^\theta \oplus (V_\Lambda^T)^\theta, \tag{12}$$

where θ is an automorphism of V_Λ lifted by the (-1)-isometry of the Leech lattice Λ, $V_\Lambda^T = V_\Lambda^T(\theta)$ is the unique irreducible θ-twisted module for V_Λ and $(V_\Lambda^T)^\theta$ is the submodule fixed by θ (see [FLM88]). It was shown in [DMZ94] that $V^\natural$ is a framed VOA, i.e., $V^\natural$ contains a subVOA T, called a frame, which is isomorphic to the tensor product of 48 copies of the simple Virasoro VOA $L(\frac{1}{2}, 0)$.

Table 3: Vectors for realizable codes

Vectors	
v_7	$(1,0,0,0,0,1,1,1,0,0,1,1,0,0,1,1,0,1,1,0,0,1,1,0)$
v_{81}	$(0,1,0,0,0,0,0,1,0,0,0,0,1,0,1,0,0,0,1,1,0,0,1,1)$
v_{82}	$(1,0,0,0,0,1,1,1,0,0,1,0,1,1,0,1,0,0,1,1,0,0,1,1)$
v_{83}	$(1,0,0,0,0,1,1,1,0,0,1,0,1,1,0,1,0,0,0,0,0,0,0,0)$
v_{841}	$(0,0,0,0,1,1,0,0,0,0,0,1,1,1,0,1,0,1,0,0,0,1,0,0)$
v_{842}	$(0,0,0,0,1,1,1,1,0,0,0,0,1,1,0,0,0,0,1,1,0,0,0,0)$
v_{91}	$(0,1,0,0,0,0,0,1,0,0,0,0,1,1,1,1,0,0,1,0,1,0,0,0)$
v_{92}	$(1,0,0,0,0,1,0,0,0,0,1,0,0,0,1,0,0,1,0,1,1,0,0,1)$
v_{931}	$(0,0,0,0,1,0,0,1,0,0,1,0,0,0,0,1,0,0,0,1,1,0,1,1)$
v_{932}	$(0,0,0,0,1,1,0,0,0,0,0,0,1,1,1,1,0,1,1,1,1,0,1,1)$
v_{94}	$(0,1,0,0,0,0,1,0,0,0,0,0,0,0,1,1,0,1,1,1,0,0,1,0)$
v_{95}	$(1,0,0,0,0,1,1,1,0,0,1,0,1,1,0,1,0,0,1,1,0,0,1,1)$
v_{96}	$(1,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,1,1,0,0,1,1)$
v_{101}	$(1,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,1,1,0,0,1,1)$
v_{102}	$(1,0,0,0,0,0,1,0,0,0,1,0,0,0,0,1,0,0,0,1,0,1,1,1)$
v_{103}	$(1,0,0,0,0,0,0,1,0,0,1,0,1,0,0,0,0,1,1,0,0,1,0,1)$
v_{11}	$(1,0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,0,1,0,0,0,1,1,1)$
v_{12}	$(1,0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,0,0,0,1,1,1,0,1)$

Table 4: Maximal non-realizable codes

Codes	C	v	m	N	Codes	C	v	m	N
$N_{9,1}$	C_6	w_7, w_{81}, w_{91}	14	159	$N_{9,6}$	C_6	w_7, w_{82}, w_{96}	14	254
$N_{9,2}$	C_6	w_7, w_{81}, w_{92}	14	372	$N_{9,7}$	C_6	w_7, w_{82}, w_{97}	14	287
$N_{9,3}$	C_6	w_7, w_{81}, w_{93}	14	170	$N_{9,8}$	C_6	w_7, w_{82}, w_{98}	14	488
$N_{9,4}$	C_6	w_7, w_{82}, w_{94}	14	388	$N_{10,1}$	C_6	$w_7, w_{81}, w_9, w_{101}$	23	299
$N_{9,5}$	C_6	w_7, w_{82}, w_{95}	14	228	$N_{10,2}$	C_6	$w_7, w_{81}, w_9, w_{102}$	23	378

Table 5: Vectors for non-realizable codes

Vectors	
w_7	$(0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1)$
w_{81}	$(0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 1, 1)$
w_{82}	$(0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1)$
w_{91}	$(0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0)$
w_{92}	$(0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1)$
w_{93}	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1)$
w_{94}	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1)$
w_{95}	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0)$
w_{96}	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0)$
w_{97}	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1)$
w_{98}	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0)$
w_9	$(0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 0)$
w_{101}	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1)$
w_{102}	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)$

Given a holomorphic framed VOA V and a frame T, one can associate a triply even code D, called the structure code or the $\frac{1}{16}$-code of V (cf. [DGH98, LY08]). Let $C = D^{\perp}$. Then V can be decomposed as $V = \oplus_{\beta \in D} V^{\beta}$ such that $V^{\beta}, \beta \in D$, are irreducible V^0-modules and V^0 is isomorphic to the code VOA M_C as constructed in [Mi98]. However, the structure code D depends on the choice of T and there are many possible choices for the frame T in general. The main purpose of this section is to study the structure codes of the moonshine VOA $V^{\natural}$.

Definition 4.1. We call a triply even code of length 48 a *moonshine code* if it can be realized as a $\frac{1}{16}$-code of $V^{\natural}$ with respect to some Virasoro frame.

Remark 4.2. A holomorphic framed VOA V of central charge 24 is said to be extremal if $V_1 = 0$. In [LY07] (see also [LY08]), it was shown that any extremal holomorphic framed VOA V of central charge 24 is isomorphic to the moonshine VOA. Therefore, the notion of moonshine codes can be regarded as a VOA-analogue of the residue codes of extremal Type II $\mathbb{Z}_4$-codes.

Lemma 4.3 ([DGH98, Mi04]). *Let D be a moonshine code. Then D satisfies the following* *conditions:*

$$D \text{ is triply even,} \tag{13}$$

$$D \ni \mathbf{1}, \tag{14}$$

$$D^{\perp} \text{ has minimum weight at least } 4. \tag{15}$$

Moreover, $\dim D \geq 7$*.*

Now, let us define two linear maps $d, \ell : \mathbb{Z}_2^n \to \mathbb{Z}_2^{2n}$ such that

$$d(a_1, a_2, \ldots, a_n) = (a_1, a_1, a_2, a_2, \ldots, a_n, a_n),$$
$$\ell(a_1, a_2, \ldots, a_n) = (a_1, 0, a_2, 0, \ldots, a_n, 0),$$

for any $(a_1, a_2, \ldots, a_n) \in \mathbb{Z}_2^n$.

Definition 4.4. Let C be a code of length n. We define

$$\mathcal{D}(C) = \langle d(C), \ell(\mathbf{1}) \rangle_{\mathbb{Z}_2}$$

to be the code generated by $d(C) = \{d(x) \mid x \in C\}$ and $\ell(\mathbf{1})$. We call the code $\mathcal{D}(C)$ the *extended doubling* of C.

Lemma 4.5. *If C is a doubly even $[8n, k]$ code, then the extended doubling $\mathcal{D}(C)$ is a triply even $[16n, k+1]$ code. Moreover, if $C^{\perp}$ is even and has minimum weight ≥ 4, then $\mathcal{D}(C)^{\perp}$ is even and has minimum weight ≥ 4.*

Proof. Straightforward. ∎

The following result is essentially proved in [DGH98] (see also [LY08]).

Proposition 4.6. *Let $\mathcal{C}$ be an extremal Type II $\mathbb{Z}_4$-code of length 24. Then the extended doubling $\mathcal{D}(\mathcal{C}_1)$ can be realized as a $\frac{1}{16}$-code of the moonshine VOA $V^{\natural}$.*

The converse also holds. First let us recall a $\mathbb{Z}_2$-orbifold construction for holomorphic framed VOAs (see Theorem 8 of [LY08]).

Lemma 4.7 ([LY08])**.** *Let $V = \bigoplus_{\beta \in D} V^{\beta}$ be a framed VOA and $C = D^{\perp}$. Let $\delta \in \mathbb{Z}_2^n \setminus C$ be a vector of even weight and denote $D^0 = \{\beta \in D \mid \langle \beta, \delta \rangle = 0\}$. Then*

$$\tilde{V}(\delta) = \bigoplus_{\beta \in D^0} \Big(V^{\beta} \oplus (M_{\delta + C} \times_{M_C} V^{\beta}) \Big)$$

is also a holomorphic framed VOA and D^0 is the $\frac{1}{16}$-code, where $\times_{M_C}$ denotes the fusion product with respect to the VOA M_C.

The following is the main theorem of this section.

Theorem 4.8. *Let B be a doubly even code of length 24. Then, the extended doubling $\mathcal{D}(B)$ is a moonshine code if and only if there exists an extremal Type II $\mathbb{Z}_4$-code $\mathcal{C}$ of length 24 with $B = \mathcal{C}_1$.*

Proof. The "if" part follows from Proposition 4.6. To prove the converse, suppose that $D = \mathcal{D}(B)$ is a moonshine code.

Let $\delta = (1, 1, 0, 0, \ldots, 0) \in \mathbb{Z}_2^{48}$ and $C = D^\perp$. Since the minimum weight of C is at least 4 by Lemma 4.3, we have $\delta \notin C$. By Lemma 4.7, the $\frac{1}{16}$-code of the $\mathbb{Z}_2$-orbifold VOA $\tilde{V}^\natural(\delta)$ is $D^0 = \{\beta \in \mathcal{D}(B) \mid \langle \beta, \delta \rangle = 0\} = d(B)$. Since $D^\perp = \langle \ell(B^\perp), d(\langle \mathbf{1} \rangle_{\mathbb{Z}_2}^\perp)\rangle_{\mathbb{Z}_2}$, we have $(D^0)^\perp \supset \langle d(\langle \mathbf{1}\rangle_{\mathbb{Z}_2}^\perp), \delta\rangle_{\mathbb{Z}_2} = d(\mathbb{Z}_2^{24})$.

Thus, $\tilde{V}^\natural(\delta)$ contains a subalgebra $M_{d(\mathbb{Z}_2^{24})}$, which is isomorphic to the lattice VOA $V_{\sqrt{2}A_1}^{\otimes 24}$ (cf. [LY08]). Hence, $\tilde{V}^\natural(\delta)$ must be isomorphic to a lattice VOA V_L for some even unimodular lattice L (cf. [Do93]), and $L/(\sqrt{2}A_1^{\oplus 24})$ defines a Type II $\mathbb{Z}_4$-code $\mathcal{C}$ with $\mathcal{C}_1 = B$. By [LY07, Proposition 4.2] (see also [DM04]), $\tilde{V}^\natural(\delta)$ is isomorphic to the Leech lattice VOA V_Λ. Therefore, L is isomorphic to Λ and $\mathcal{C}$ is an extremal Type II $\mathbb{Z}_4$-code by Lemma 2.1. ∎

Together with Theorem 3.8, we can determine all the moonshine codes which are extended doublings.

4.2 Weight 8 augmentation and other moonshine codes

In this subsection, we shall give analogues of Lemmas 3.3 and 3.4 for moonshine codes.

Recall that the full automorphism group $\mathrm{Aut}(V^\natural)$ of $V^\natural$ is the Monster simple group $\mathbb{M}$ and it has two conjugacy classes of involutions denoted by $2A$ and $2B$ in [ATLAS]. Their $\mathbb{Z}_2$-twisted modules $V^T(2A)$ and $V^T(2B)$ were constructed in [La00] and [Hu96], respectively. Their minimal weights are also determined.

Lemma 4.9. *(i) The minimal weight of the $2A$-twisted module $V^T(2A)$ is $\frac{1}{2}$, and* $\dim(V^T(2A))_{1/2} = 1$.

(ii) The minimal weight of the $2B$-twisted module $V^T(2B)$ is 1, and $\dim(V^T(2B))_1 = 24$.

The structure of the corresponding $\mathbb{Z}_2$-orbifold VOA is also determined.

Lemma 4.10 ([Hu96, La00, LY08, Ya05])**.** *Let g be an involution of* $\mathrm{Aut}(V^\natural) = \mathbb{M}$. *Then the $\mathbb{Z}_2$-orbifold VOA $\tilde{V}^\natural(g)$ is isomorphic to $V^\natural$ if g belongs to $2A$, V_Λ if g belongs to $2B$.*

The next theorem is an analogue of Lemma 3.3 for moonshine codes.

Theorem 4.11. *Suppose that D is a moonshine code. Let $\xi \in \mathbb{Z}_2^{48} \setminus D$ be such that $D' = \langle D, \xi\rangle_{\mathbb{Z}_2}$ is triply even. If $\xi + D$ has minimum weight 8, then D' is also a moonshine code.*

Proof. Let D be the $\frac{1}{16}$-code of $V^\natural$ with respect to a Virasoro frame T and $C = D^\perp$. Let $V^\natural = \bigoplus_{\beta\in D} V^\beta$ and $C^0 = \{\alpha \in C \mid \langle \alpha, \xi\rangle = 0\}$. Suppose that $\xi + D$ has minimum weight 8. Without loss of generality, we may assume that $\xi = (\xi_1, \dots, \xi_{48})$ has weight 8.

Set

$$h_i^\xi = \begin{cases} 0 & \text{if } \xi_i = 0, \\ \frac{1}{16} & \text{if } \xi_i = 1, \end{cases}$$

and let U be an irreducible M_{C^0}-module which contains $\bigotimes_{i=1}^{48} L(\frac{1}{2}, h_i^\xi)$ as a T-submodule. Then the minimal weight of U is $\frac{1}{16}\mathrm{wt}(\xi) = \frac{1}{2}$. By [LY08, Theorem 1], there exists an automorphism $g \in \mathrm{Aut}(V)$ of order 2 such that $g(V^\beta) = V^\beta$ for all $\beta \in D$. Let $V^{\beta,\pm}$ be the ± 1-eigenspaces of g on V^β. Then $V^\beta \times_{M_{C^0}} U = (V^{\beta,+} \times_{M_{C^0}} U) \oplus (V^{\beta,-} \times_{M_{C^0}} U)$ is a sum of two irreducible M_{C^0}-modules. One has weights in $\mathbb{Z}$ and the other has weights in $\frac{1}{2} + \mathbb{Z}$ (cf. [La00, LY08]). Moreover, the M_{C^0}-module

$$V^T(g) = \bigoplus_{\beta\in D} V^\beta \times_{M_{C^0}} U \tag{16}$$

forms an irreducible g-twisted module of V. For each $\beta \in D$, let $U^\beta = V^{\beta,+}$ and let $U^{\xi+\beta}$ be the integral part of $V^\beta \times_{M_{C^0}} U$. Then by Theorem 8 of [LY08],

$$\tilde{V}^\natural(g) = \bigoplus_{\beta\in D} \left(U^\beta \oplus U^{\xi+\beta}\right)$$

is a holomorphic framed VOA whose $\frac{1}{16}$-code is $D' = \langle D, \xi\rangle_{\mathbb{Z}_2}$.

Since U is isomorphic to $M_{C^0} \times_{M_{C^0}} U \subset V^0 \times_{M_{C^0}} U$, the minimal weight of $(V^\natural)^T(g)$ is $\leq \frac{1}{2}$. Thus, the minimal weight of $(V^\natural)^T(g)$ is $\frac{1}{2}$ and $(V^\natural)^T(g)$ is a $2A$-twisted module by Lemma 4.9. Therefore, the $\mathbb{Z}_2$-orbifold VOA $\tilde{V}^\natural(g)$ is isomorphic to $V^\natural$ by Lemma 4.10 and we have the desired conclusion. ∎

By Lemma 4.5, the extended doublings $\mathcal{D}(e_8)$, $\mathcal{D}(d_{16}^+)$ and $\mathcal{D}(e_8 \oplus e_8)$ are triply even, where e_8 denotes the extended Hamming code of length 8 and d_{16}^+ denotes the unique indecomposable doubly even self-dual code of length 16. Thus, $\mathcal{D}(e_8) \oplus \mathcal{D}(e_8) \oplus \mathcal{D}(e_8)$, $\mathcal{D}(e_8 \oplus e_8) \oplus \mathcal{D}(e_8)$ and $\mathcal{D}(d_{16}^+) \oplus \mathcal{D}(e_8)$ are also triply even codes of length 48. By Theorem 4.11, we also have the following result.

Proposition 4.12. *The triply even codes $\mathcal{D}(e_8)\oplus\mathcal{D}(e_8)\oplus\mathcal{D}(e_8)$, $\mathcal{D}(e_8\oplus e_8)\oplus\mathcal{D}(e_8)$ and $\mathcal{D}(d_{16}^+)\oplus\mathcal{D}(e_8)$ are moonshine codes.*

Proof. We note that

$$\begin{aligned}
\mathcal{D}(e_8\oplus e_8)\oplus\mathcal{D}(e_8) &= \langle \mathcal{D}(e_8\oplus e_8\oplus e_8), ((0,0)^{16},(1,0)^8)\rangle_{\mathbb{Z}_2},\\
\mathcal{D}(e_8)\oplus\mathcal{D}(e_8)\oplus\mathcal{D}(e_8) &= \langle \mathcal{D}(e_8\oplus e_8)\oplus\mathcal{D}(e_8), ((1,0)^8,(0,0)^{16})\rangle_{\mathbb{Z}_2},\\
\mathcal{D}(d_{16}^+)\oplus\mathcal{D}(e_8) &= \langle \mathcal{D}(d_{16}^+\oplus e_8), ((0,0)^{16},(1,0)^8)\rangle_{\mathbb{Z}_2}.
\end{aligned}$$

The extended doublings $\mathcal{D}(e_8\oplus e_8\oplus e_8)$ and $\mathcal{D}(d_{16}^+\oplus e_8)$ of the two decomposable doubly even self-dual codes of length 24 are moonshine codes (see Table 1). Hence, we have the desired result by Theorem 4.11. ∎

Remark 4.13. The three codes above have dimensions greater than 13, while the extended doubling of any doubly even self-dual code has dimension 13 by Lemma 4.5. Hence, none of the three codes is equivalent to any extended doubling of a doubly even self-dual code.

The next theorem is a partial converse of Theorem 4.11, which can also be viewed as an analogue of Lemma 3.4 for moonshine codes.

Theorem 4.14. *Let D be a moonshine code. Suppose $\eta\in D$ and $\mathrm{wt}(\eta)=8$. Then there exists a moonshine code D' such that $D'\subsetneq\langle D',\eta\rangle = D$.*

Proof. Let D be the $\frac{1}{16}$-code of $V^\natural$ with respect to a frame T and $C=D^\perp$. Then $V^\natural=\bigoplus_{\beta\in D}V^\beta$ and $V^\beta,\beta\in D$, are irreducible modules for the corresponding code VOA $V^0=M_C$.

Set

$$h_i^\eta=\begin{cases}0 & \text{if } \eta_i=0,\\ \frac{1}{16} & \text{if } \eta_i=1.\end{cases}$$

Let U be an irreducible M_C-module which contains $\bigotimes_{i=1}^{48}L(\frac12,h_i^\xi)$ as a T-submodule such that the fusion product $V^\eta\times_{M_C}U$ has integral weights. Note that $V^\eta\times_{M_C}U$ is isomorphic to $M_{\alpha+C}$ for some $\alpha\in\mathbb{Z}_2^{48}$. Since the minimal weight of U is $\frac{1}{16}\mathrm{wt}(\eta)=\frac12$, we have $\langle\alpha,\eta\rangle\neq0$ and hence $\alpha\notin C$. In this case, we have a $\mathbb{Z}_2$-twisted module

$$(V^\natural)^T=\bigoplus_{\beta\in D}V^\beta\times_{M_C}U=\bigoplus_{\beta\in D}V^\beta\times_{M_C}M_{\alpha+C}.$$

By Lemma 4.9, $(V^\natural)^T$ is a $2A$-twisted module of $V^\natural$.

Now set $D' = \{\beta \in C \mid \langle \alpha, \beta \rangle = 0\}$. Then $D' \subsetneqq \langle D', \eta \rangle = D$. By Lemmas 4.7 and 4.10, the $\mathbb{Z}_2$-orbifold VOA

$$\tilde{V}(\alpha) = \bigoplus_{\beta \in D'} \left(V^\beta + V^\beta \times_{M_C} M_{\alpha + C} \right)$$

is isomorphic to the moonshine VOA $V^\natural$. Therefore, D' is a moonshine code. ∎

Corollary 4.15. *A triply even code D of length 48 is a moonshine code if and only if D can be obtained by successive weight 8 augmentations from a moonshine code with minimum weight 16.*

Remark 4.16. After this work has been completed, triply even codes of length 48 were classified by Betsumiya and Munemasa [BM10]. Their work is, in some sense, complementary to ours. In particular, the classification of moonshine codes can be reduced to checking the realizability of few triply even codes with minimum weight 16 which are not extended doublings based on their work and Theorems 4.11 and 4.14 in this work.

A Appendix

A.1 Extremal Type II $\mathbb{Z}_4$-codes of length 24 whose residue codes are doubly even self-dual codes

Here, we give explicitly extremal Type II $\mathbb{Z}_4$-codes $\mathcal{C}$ of length 24 with $C = \mathcal{C}_1$ for each C of the seven doubly even self-dual codes with labels d_{12}^2, $d_{10}e_7^2$, d_8^3, d_6^4, d_4^6, e_8^3 and $d_{16}e_8$. This is done by listing their generator matrices $\begin{bmatrix} I_{12} & M \end{bmatrix}$ where M are as follows:

$$\begin{bmatrix} 311222000022 \\ 112302000002 \\ 310010020022 \\ 130221220020 \\ 330202300020 \\ 121311130200 \\ 323111323220 \\ 103111322100 \\ 101133102210 \\ 101331322003 \\ 132202213331 \\ 213131331333 \end{bmatrix}, \begin{bmatrix} 333220002022 \\ 132120002220 \\ 132012002022 \\ 332023220200 \\ 002220333202 \\ 213113033002 \\ 213113101200 \\ 233113312000 \\ 220002220131 \\ 121311022231 \\ 321111002301 \\ 123133202312 \end{bmatrix}, \begin{bmatrix} 131000222202 \\ 132122020002 \\ 110232020022 \\ 220001310020 \\ 202001321022 \\ 013310313220 \\ 231133213000 \\ 121331120100 \\ 101131102032 \\ 301111300221 \\ 130221300133 \\ 231330002113 \end{bmatrix}, \begin{bmatrix} 311222200002 \\ 130102222200 \\ 020213122002 \\ 220013010200 \\ 033301131202 \\ 213321132320 \\ 312230333300 \\ 121311003120 \\ 121130132010 \\ 303110112021 \\ 112203132233 \\ 031131002233 \end{bmatrix},$$

$$\begin{bmatrix} 311002220220 \\ 200333022220 \\ 220022311222 \\ 011031123102 \\ 121330332122 \\ 332121211120 \\ 301123121212 \\ 031110033232 \\ 312231130030 \\ 101231031201 \\ 310310303221 \\ 033123130021 \end{bmatrix}, \begin{bmatrix} 213120000200 \\ 103102202202 \\ 132100002020 \\ 131022002202 \\ 222001310220 \\ 002012130200 \\ 220011032000 \\ 022211320222 \\ 202002022131 \\ 000020203233 \\ 222200023303 \\ 022202201310 \end{bmatrix} \text{ and } \begin{bmatrix} 211302200000 \\ 103102022022 \\ 332320022222 \\ 333020220220 \\ 022001331331 \\ 200230113111 \\ 220013033113 \\ 220231321111 \\ 002231110131 \\ 022013313031 \\ 220231133321 \\ 002233113310 \end{bmatrix},$$

respectively.

A.2 Two $[24, 7]$ codes $C_{7,1}$ and $C_{7,2}$

Up to equivalence, there exist two $[24, 7]$ codes which are minimal subject to (1)–(3) (see Subsection 3.2). Here, we give generator matrices of the two $[24, 7]$ codes. Since these two codes along with C_6 are used to define other codes in Tables 2 and 4, we define the two codes by fixing the coordinates.

The first one $C_{7,1}$ has generator matrix

$$\begin{bmatrix} M_6 & M_6 & M_6 & M_6 \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix},$$

where M_6 denotes a generator matrix of the parity check $[6, 5, 2]$ code E_6. In order to construct the second one, we first construct a 6×12 bordered double circulant matrix

$$M_{12} = \begin{bmatrix} 1 & \mathbf{0} & 0 & \mathbf{1} \\ \mathbf{0}^T & I_5 & \mathbf{1}^T & C_5 \end{bmatrix} = \begin{bmatrix} 100000011111 \\ 010000101001 \\ 001000110100 \\ 000100101010 \\ 000010100101 \\ 000001110010 \end{bmatrix},$$

where C_5 denotes the adjacency matrix of a 5-cycle. The second one $C_{7,2}$ has generator matrix

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ M_{12} & M_{12} \end{bmatrix}.$$

Acknowledgments. The research for this paper was partially carried out while the first and the third authors were visiting Institute of Mathematics, Academia Sinica, Taiwan. These authors would like to thank Academia Sinica for the hospitality during this visit.

References

[BM10] K. Betsumiya and A. Munemasa, On triply even binary codes, *J. London Math. Soc.* **86** (2012), 1–16.

[BSBM97] A. Bonnecaze, P. Solé, C. Bachoc and B. Mourrain, Type II codes over $\mathbb{Z}_4$, *IEEE Trans. Inform. Theory* **43** (1997), 969–976.

[Br98] A.E. Brouwer, "Bounds on the size of linear codes," in Handbook of Coding Theory, V.S. Pless and W.C. Huffman (Editors), Elsevier, Amsterdam, 1998, pp. 295–461.

[CS97] A.R. Calderbank and N.J.A. Sloane, Double circulant codes over $\mathbb{Z}_4$ and even unimodular lattices, *J. Algebraic Combin.* **6** (1997), 119–131.

[ATLAS] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, ATLAS of Finite Groups, Clarendon Press, Oxford, 1985.

[CS93] J.H. Conway and N.J.A. Sloane, Self-dual codes over the integers modulo 4, *J. Combin. Theory Ser. A* **62** (1993), 30–45.

[Do93] C. Dong, Vertex algebras associated with even lattices, *J. Algebra* **161** (1993), 245–265.

[DGH98] C. Dong, R.L. Griess, Jr. and G. Höhn, Framed vertex operator algebras, codes and the moonshine module, *Commun. Math. Phys.* **193** (1998), 407–448.

[DM04] C. Dong and G. Mason, Holomorphic vertex operator algebras of small central charge, *Pacific J. Math.* **213** (2004), 253–266.

[DMZ94] C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module, *Proc. Symp. Pure. Math.* **56, Part 2** (1994), 295–316.

[FLM88] I.B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Academic Press, New York, 1988.

[Ha10] M. Harada, Extremal Type II $\mathbb{Z}_4$-codes of lengths 56 and 64, *J. Combin. Theory Ser. A* **117** (2010), 1285–1288.

[HSG99] M. Harada, P. Solé and P. Gaborit, Self-dual codes over $\mathbb{Z}_4$ and unimodular lattices: a survey, Algebras and combinatorics (Hong Kong, 1997), 255–275, Springer, Singapore, 1999.

[Hu96] Y.-Z. Huang, Virasoro vertex operator algebras, (non-meromorphic) operator product expansion and the tensor product theory, *J. Algebra* **182** (1996), 201–234.

[La00] C.H. Lam, Some twisted module for framed vertex operator algebras, *J. Algebra* **231** (2000), 331–341.

[LY07] C.H. Lam and H. Yamauchi, A characterization of the moonshine vertex operator algebra by means of Virasoro frames, *Int. Math. Res. Not. IMRN* **2007**, Art. ID rnm003, 10 pp.

[LY08] C.H. Lam and H. Yamauchi, On the structure of framed vertex operator algebras and their pointwise frame stabilizers, *Comm. Math. Phys.* **277** (2008), 237–285.

[Mi98] M. Miyamoto, Representation theory of code vertex operator algebras, *J. Algebra* **201** (1998), 115–150.

[Mi04] M. Miyamoto, A new construction of the moonshine vertex operator algebra over the real number field, *Ann. of Math.* **159** (2004), 535–596.

[PS75] V. Pless and N.J.A. Sloane, On the classification and enumeration of self-dual codes, *J. Combin. Theory Ser. A* **18** (1975), 313–335.

[Ra99] E. Rains, Optimal self-dual codes over $\mathbb{Z}_4$, *Discrete Math.* **203** (1999), 215–228.

[Ya05] H. Yamauchi, 2A-orbifold construction and the baby-monster vertex operator superalgebra, *J. Algebra* **284** (2005), 645–668.