On the Classification of Weighing Matrices and Self-Orthogonal Codes

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Abstract

We provide a classification method of weighing matrices based on a classification of self-orthogonal codes. Using this method, we classify weighing matrices of orders up to 15 and order 17, by revising some known classification. In addition, we give a revised classification of weighing matrices of weight 5. A revised classification of ternary maximal self-orthogonal codes of lengths 18 and 19 is also presented.

1 Introduction

A weighing matrix W of order n and weight k is an $n \times n$ $(1, -1, 0)$-matrix W such that $WW^T = kI_n$, where I_n is the identity matrix of order n and W^T denotes the transpose of W. A weighing matrix of order n and weight n is also called a Hadamard matrix. We say that two weighing matrices W_1 and W_2 of order n and weight k are *equivalent* if there exist $(1, -1, 0)$-monomial matrices P and Q with $W_1 = PW_2Q$.

Chan, Rodger and Seberry [4] began a classification of weighing matrices and they classified all weighing matrices of weight $k \leq 5$ and all weighing matrices of orders $n \leq 11$. Ohmori [17] and [19] classified weighing matrices

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of orders 12 and 13, respectively. At order 14, weighing matrices of weights $k \leq 8$ and 13 were classified in [4] and [18], respectively. At order 17, all weighing matrices of weight 9 with intersection number 8 were classified in [20].

In this paper, we extend the classification of weighing matrices using the known classification of self-orthogonal codes. Let $\mathbb{Z}_m$ be the ring of integers modulo m, where m is an integer greater than 1. Let W be a weighing matrix of order n and weight k, and suppose that m is a divisor of k. If we regard the entries of W as elements of $\mathbb{Z}_m$, then the rows of W generate a self-orthogonal $\mathbb{Z}_m$-code. This means that W can be regarded as a subset of codewords in some maximal self-orthogonal code. For example, a classification of weighing matrices of order 16 and weight 6 can be derived from the known classification of ternary self-dual codes of length 16 given in [2].

The paper is organized as follows. In Section 2, we review the known classification of maximal self-orthogonal codes needed for our classification of weighing matrices. It turns out that there are errors in the classification of ternary maximal self-orthogonal codes of lengths 18 and 19 given in [22], and we correct them. In Section 3, we give a detailed description of our classification method of weighing matrices of order n and weight k based on the classification of self-orthogonal $\mathbb{Z}_m$-codes of length n, where m is a divisor of k. Our method, applied to the known classification of self-dual $\mathbb{F}_5$-codes of length 12, leads to a classification of weighing matrices of order 12 and weight 5. This reveals an omission in the classification given in [4, Theorem 5], and a revised classification of weighing matrices of weight 5 for all orders is given in Section 4, while a revised classification of weighing matrices of order 12 for all weights is given in Section 5. In Section 6, we classify weighing matrices of orders $14, 15$ and 17. Again, there is an error in the number of weighing matrices of order 14 and weight 8 given in [17, Theorem 3], and we correct it. This completes a classification of weighing matrices of orders $n \leq 17$ except $n = 16$. Weighing matrices of order n and k are also classified for

$$(n, k) = (16, 6), (16, 9), (16, 12), (18, 9)$$

in Section 7. All weighing matrices given in this paper can be obtained electronically from [11].

2 Maximal self-orthogonal codes

2.1 Codes

We shall exclusively deal with the case $\mathbb{Z}_p = \mathbb{F}_p$ and $\mathbb{Z}_4$, where $\mathbb{F}_p$ denotes the finite field of odd prime order p. A $\mathbb{Z}_m$-code C of length n (or a code C of length n over $\mathbb{Z}_m$) is a $\mathbb{Z}_m$-submodule of $\mathbb{Z}_m^n$. The dual code $C^\perp$ of C is defined as $C^\perp = \{x \in \mathbb{Z}_m^n \mid x \cdot y = 0 \text{ for all } y \in C\}$ under the standard inner product $x \cdot y$. A code C is *self-dual* if $C = C^\perp$, and C is *self-orthogonal* if $C \subset C^\perp$. A self-dual $\mathbb{F}_p$-code of length n exists if and only if n is even for $p \equiv 1 \pmod 4$, and $n \equiv 0 \pmod 4$ for $p \equiv 3 \pmod 4$. A self-dual $\mathbb{Z}_4$-code exists for every length.

A self-orthogonal code C is *maximal* if C is the only self-orthogonal code containing C. The dimension of a maximal self-orthogonal $\mathbb{F}_p$-code of length n is a constant depending only on n and p, and a self-dual code is automatically maximal. More precisely, for $p \equiv 1 \pmod 4$, a maximal self-orthogonal $\mathbb{F}_p$-code of length n has dimension $(n-1)/2$ if n is odd. For $p \equiv 3 \pmod 4$, a maximal self-orthogonal $\mathbb{F}_p$-code of length n has dimension $(n-1)/2$ if n is odd, $n/2-1$ if $n \equiv 2 \pmod 4$. It is easy to see that a maximal self-orthogonal $\mathbb{Z}_4$-code is necessarily self-dual for every length.

Two codes C and C' are *equivalent* if there exists a $(1,-1,0)$-monomial matrix P with $C' = CP = \{xP \mid x \in C\}$. The *automorphism group* $\mathrm{Aut}(C)$ of C is the group of all $(1,-1,0)$-monomial matrices P with $C = CP$. Our classification method of weighing matrices of order n and weight $k = mt$ requires a classification of maximal self-orthogonal $\mathbb{Z}_m$-codes of length n (see Section 3). In this paper, some classifications of maximal self-orthogonal $\mathbb{Z}_m$-codes are used for $m = 3, 4, 5, 7$ to classify weighing matrices. The current knowledge on the classifications of such codes is listed in Table 1.

2.2 Ternary maximal self-orthogonal codes

An $\mathbb{F}_3$-code is called ternary. All ternary maximal self-orthogonal codes of lengths $4m+1, 4m+2, 4m+3$ can be obtained from self-dual codes of length $4m+4$ by subtracting (see [2]). A classification of ternary maximal self-orthogonal codes of lengths $3, \dots, 12$, lengths $13, 14, 15, 16$ and lengths $17, 18, 19, 20$ was done in [15], [2] and [22], respectively.

In the course of reproducing a classification of ternary maximal self-orthogonal codes of lengths up to 20, we discovered errors in the classification

Table 1: Maximal self-orthogonal $\mathbb{Z}_m$-codes of length n

$\mathbb{Z}_m$	Lengths n	References
$\mathbb{F}_3$	$1, \dots, 12$	[15]
	$13, \dots, 16$	[2]
	$17, \dots, 20$	[22] (see also this section)
	24	[9]
$\mathbb{Z}_4$	$1, \dots, 9$	[3]
	$10, \dots, 15$	[5]
	16 (Type II)	[21]
	16 (Type I), $17, 18, 19$	[10]
$\mathbb{F}_5$	$1, \dots, 12$	[14]
	$13, \dots, 16$	[8]
$\mathbb{F}_7$	$1, \dots, 9$	[23]
	$10, \dots, 13$	[7]

for lengths 18 and 19. The numbers of ternary maximal self-orthogonal codes of lengths 18 and 19 are listed in [22, Table IV] as 154 and 54, respectively. However, we verified that the correct numbers are 160 and 56, respectively. Let $C_{20,i}$ denote the i-th self-dual code of length 20 given in [22, Tables II and III], and let $n_{18}(i)$ and $n_{19}(i)$ denote the numbers of inequivalent maximal self-orthogonal codes of lengths 18 and 19, respectively, obtained from $C_{20,i}$ by subtracting. Let $C_{20,i}^{(k)}$ denote the self-orthogonal code of length 19 obtained from $C_{20,i}$ by subtracting the k-th coordinate. Although the numbers $n_{19}(20)$ and $n_{19}(23)$ are listed as both 1 in [22, Table IV], we verified that the codes $C_{20,20}^{(i)}$ $(i = 1, \dots, 20)$ are equivalent to one of the two inequivalent codes $C_{20,20}^{(1)}, C_{20,20}^{(20)}$, and the codes $C_{20,23}^{(i)}$ $(i = 1, \dots, 20)$ are equivalent to one of the two inequivalent codes $C_{20,23}^{(1)}, C_{20,23}^{(20)}$. In fact, these four codes have different automorphism groups, of orders $32, 128, 576$ and 5184, respectively. Hence, we conclude that $n_{19}(20) = n_{19}(23) = 2$. Since [22, Table IV] also contains incorrect values for $n_{18}(i)$, we list their correct values in Table 2.

In order to check that a classification is complete, in all of the classification results, we first verified by Magma that all codes are inequivalent. This was done by the Magma function `IsIsomorphic`, as well as by checking that all codes have different numbers $(B_0, B_1, \dots, B_n)$, where B_j is the number of distinct cosets of weight j. Then we checked the mass formula, that is, we computed the sum in

$$\sum_{C\in\mathcal{C}} \frac{2^n \cdot n!}{|\operatorname{Aut}(C)|}, \tag{1}$$

where $\mathcal{C}$ is the set of inequivalent maximal self-orthogonal codes of length n and we checked against the known formula for the number N_0 of distinct maximal self-orthogonal codes of length n, which is given in [15, p. 650]. The automorphism group $\operatorname{Aut}(C)$ of C is calculated by the Magma function `AutomorphismGroup`. Note that each summand in (1) expresses the cardinality of the equivalence class of a given a code C and the sum of all these cardinalities is equal to N_0. The numbers # of all inequivalent maximal self-orthogonal codes of lengths up to 20 are listed in Table 3, and generator matrices of those codes can be obtained electronically from [11].

Proposition 1. *Up to equivalence, there are* 160 *and* 56 *ternary maximal self-orthogonal codes of lengths* 18 *and* 19*, respectively.*

Table 2: Ternary maximal self-orthogonal codes of lengths 18 and 19

i	$n_{18}(i)$	$n_{19}(i)$	i	$n_{18}(i)$	$n_{19}(i)$	i	$n_{18}(i)$	$n_{19}(i)$
1	2	1	9	5	2	17	16	5
2	5	2	10	8	2	18	12	4
3	4	2	11	4	2	19	3	1
4	7	3	12	12	4	20	10	2
5	7	3	13	8	3	21	4	1
6	6	3	14	9	3	22	5	1
7	4	2	15	10	3	23	5	2
8	5	2	16	7	2	24	2	1
					Total		160	56

3 Classification method

When n is odd, the existence of a weighing matrix of order n and weight k implies that k is a square and $(n-k)^2 + (n-k) + 1 \geq n$. When $n \equiv 2 \pmod 4$, the existence of a weighing matrix of order n and weight k implies that k is the sum of two squares and $k \leq n-1$ [4].

Table 3: Ternary maximal self-orthogonal codes

Length	#	References	Length	#	References
3	1	[15]	12	3	[15]
4	1	[15]	13	7	[2]
5	1	[15]	14	22	[2]
6	2	[15]	15	12	[2]
7	1	[15]	16	7	[2]
8	1	[15]	17	23	[22]
9	2	[15]	18	160	Section 2.2
10	5	[15]	19	56	Section 2.2
11	3	[15]	20	24	[22]

For the remainder of this section, let $W = (w_{ij})$ be a weighing matrix of order n and weight k. The number $\sum_{s=1}^{n} w_{is}^2 w_{js}^2$ is called the intersection number of i-th row r_i and the j-th row r_j $(i \neq j)$. The maximum number among intersection numbers for rows of W and W^T is called the intersection number of W [20]. We say that r_j intersects r_i in 2ℓ places if the intersection number is 2ℓ [4]. For a fixed row r_i, let $x_{2\ell}$ be the numbers of rows r_j other than r_i such that the intersection number of r_i and r_j is 2ℓ. The sequence $(x_0, x_2, \dots, x_{2\lfloor n/2 \rfloor})$ is called the intersection pattern corresponding to r_i [4]. The number $\sum_{j=1}^{n} w_{sj}^2 w_{tj}^2 w_{uj}^2$ is called the generalized intersection number and the following set of generalized intersection numbers

$$N(i) = \left| \left\{ \{s,t,u\} \mid \sum_{j=1}^{n} w_{sj}^2 w_{tj}^2 w_{uj}^2 = i, 1 \leq s,t,u \leq n \ (s \neq t, s \neq u, t \neq u) \right\} \right|$$

is called the g-distribution (see [20]). Note that there are inequivalent weighing matrices with the same g-distribution.

Let $C_m(W)$ be the $\mathbb{Z}_m$-code generated by the rows of W, where the entries of W are regarded as elements of $\mathbb{Z}_m$. The following is trivial.

Lemma 2. *If k is divisible by m, then $C_m(W)$ is self-orthogonal.*

Proposition 3. *Let p be an odd prime. If k is divisible by p but k is not divisible by p^2, then $C_p(W)$ is a self-dual $\mathbb{F}_p$-code.*

Proof. Suppose that $k = pt$, where t is not divisible by p. Since

$$\det(W^2) = \det(WW^T) = \det(kI_n) = k^n,$$

we have $|\det(W)| = k^{n/2}$. Let $d_1|d_2|\cdots|d_n$ be the elementary divisors of W (see e.g. [16, II.17] for the definition of elementary divisors). Then

$$|\det(W)| = d_1 d_2 \cdots d_n = k^{\frac{n}{2}} = p^{\frac{n}{2}} t^{\frac{n}{2}}.$$

Since t is not divisible by p, n must be even, and at most $n/2$ d_i's are divisible by p. Hence, $\dim C_p(W) \geq n/2$. By Lemma 2, $\dim C_p(W) \leq n/2$. The result follows. $\square$

From now on, suppose that $\mathbb{Z}_m$ is either $\mathbb{F}_p$ or $\mathbb{Z}_4$. Let $n_i(x)$ denote the number of components i of $x \in \mathbb{Z}_m^n$ ($i \in \mathbb{Z}_m$). Any row of W is a codeword x of $C_m(W)$ such that $n_0(x) = n - k$ and $n_1(x) + n_{-1}(x) = k$. By Lemma 2, $C_m(W)$ is self-orthogonal. It follows that the rows of W are composed of n codewords x with $n_0(x) = n - k$ and $n_1(x) + n_{-1}(x) = k$ in some maximal self-orthogonal $\mathbb{Z}_m$-code of length n.

We now describe how all weighing matrices of order n and weight $k = mt$ can be constructed from maximal self-orthogonal $\mathbb{Z}_m$-codes of length n. Let C be a maximal self-orthogonal $\mathbb{Z}_m$-code of length n, and let V be the set of pairs $\{x, -x\}$ satisfying the condition that $n_0(x) = n-k$, $n_1(x)+n_{-1}(x) = k$, $x \in C$. We define the simple undirected graph Γ, whose set of vertices is the set V and two vertices $\{x, -x\}, \{y, -y\} \in V$ are adjacent if $\overline{x}\,\overline{y}^T = 0$, where $\overline{x} = (\overline{x_1}, \ldots, \overline{x_n}) \in \{0, 1, -1\}^n \subset \mathbb{Z}^n$ is the vector with $\overline{x} \bmod m = x$.

It follows that the n-cliques in the graph Γ are precisely the set of weighing matrices which generate subcodes of C. It is clear that the group $\mathrm{Aut}(C)$ acts on the graph Γ as an automorphism group, and therefore, the classification of such weighing matrices reduces to finding a set of representatives of n-cliques of Γ up to the action of $\mathrm{Aut}(C)$. This computation was performed in Magma [1], the results were then converted to weighing matrices. In this way, by considering all inequivalent maximal self-orthogonal $\mathbb{Z}_m$-codes of length n, we obtain a set of weighing matrices which contain a representative of every equivalence class of weighing matrices of order n and weight $k = mt$.

Since a weighing matrix does not, in general generate a maximal self-orthogonal code, two equivalent weighing matrices may be contained in two inequivalent maximal self-orthogonal codes. One could consider not only maximal but also all self-orthogonal codes, and then list only those weighing matrices which generate the given code. This will avoid duplication of equivalent weighing matrices in the classification. However, we took a different approach for efficiency. Once we have a set of weighing matrices which could possibly contain equivalent pairs of weighing matrices, we perform equivalent testing by considering the associated incidence structures. This construction of incidence structures is given by [12, Theorem 6.8], and in our case, it is as follows. Given a weighing matrix W of order n, replacing $0, 1, -1$ in each entry by the matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively, we obtain a $(0, 1)$-matrix of order $2n$. This matrix defines a square incidence structure $\mathcal{D}(W)$ with $2n$ points and $2n$ blocks. We may take the set of points of $\mathcal{D}(W)$ to be $\mathcal{P} = \{\pm 1, \pm 2, \ldots, \pm n\}$, so that the permutation $\tau = (1, -1)(2, -2) \cdots (n, -n)$ is a fixed-point-free involutive automorphism of $\mathcal{D}(W)$. More precisely, the set of blocks $\mathcal{B}(W)$ of $\mathcal{D}(W)$ is

$$\mathcal{B}(W) = \{B_i^\varepsilon \mid 1 \leq i \leq n,\ \varepsilon = \pm 1\},$$

where

$$B_i^\varepsilon = \{\varepsilon w_{ij} j \mid 1 \leq j \leq n,\ w_{ij} \neq 0\}.$$

Here an automorphism of $\mathcal{D}(W)$ is a permutation of $\mathcal{P}$ which maps $\mathcal{B}(W)$ to $\mathcal{B}(W)$. The set of all automorphisms is called the automorphism group and is denoted by $\mathrm{Aut}(\mathcal{D}(W))$. If we denote the orbits on $\mathcal{P}$ under τ by $P_1, \ldots, P_n$, then the following conditions hold.

(i) $|B \cap P_i| \leq 1$ for any i $(1 \leq i \leq n)$ and any block $B \in \mathcal{B}(W)$,

(ii) for any two blocks $B, B' \in \mathcal{B}(W)$ such that $B' \neq B, B^\tau$,

$$|\{i \mid B \cap P_i = B' \cap P_i \neq \emptyset\}| = |\{i \mid \emptyset \neq B \cap P_i \neq B' \cap P_i \neq \emptyset\}|.$$

Let W_1 and W_2 be weighing matrices of the same order and weight. We say that $\mathcal{D}(W_1)$ and $\mathcal{D}(W_2)$ are equivalent if there is a permutation σ of $\mathcal{P}$ which maps $\mathcal{B}(W_1)$ to $\mathcal{B}(W_2)$. Obviously, the equivalence of W_1 and W_2 implies that of $\mathcal{D}(W_1)$ and $\mathcal{D}(W_2)$. Conversely, the following lemma gives a criterion under which the equivalence of $\mathcal{D}(W_1)$ and $\mathcal{D}(W_2)$ implies that of W_1 and W_2.

Lemma 4. *Let W be a weighing matrix of order n, and let $\mathcal{D}(W)$ be the square incidence structure defined by W. Suppose that*

$$\tau = \tau_0 = (1, -1)(2, -2) \cdots (n, -n)$$

is the unique fixed-point-free involutive automorphism of $\mathcal{D}(W)$ satisfying the conditions (i) and (ii) above, up to conjugacy in $\mathrm{Aut}(\mathcal{D}(W))$*. If U is a weighing matrix such that $\mathcal{D}(U)$ is equivalent to $\mathcal{D}(W)$, then U is equivalent to W.*

Proof. Let σ denote a map from $\mathcal{D}(U)$ to $\mathcal{D}(W)$ giving an equivalence. This means that σ is a permutation of $\mathcal{P}$ which maps $\mathcal{B}(U)$ to $\mathcal{B}(W)$.

We first claim that $\tau = \sigma^{-1}\tau_0\sigma$ satisfies the conditions (i) and (ii) above. Indeed, the orbits on $\mathcal{P}$ under τ are $P_i = \{i, -i\}^\sigma$ $(1 \le i \le n)$. If $B \in \mathcal{B}(W)$, then $B^{\sigma^{-1}} \in \mathcal{B}(U)$, hence $|B \cap P_i| = |B^{\sigma^{-1}} \cap \{i, -i\}| \le 1$. Thus, (i) holds. If $B, B' \in \mathcal{B}(W)$ and $B' \ne B, B^\tau$, then $B'^{\sigma^{-1}} \ne B^{\sigma^{-1}}$ and $B'^{\sigma^{-1}} \ne B^{\tau\sigma^{-1}} = B^{\sigma^{-1}\tau_0}$. Since (ii) holds for $\mathcal{B}(U)$ and τ_0, we have

$$\begin{aligned}
&|\{i \mid B^{\sigma^{-1}} \cap \{i, -i\} = B'^{\sigma^{-1}} \cap \{i, -i\} \ne \emptyset\}| \\
&= |\{i \mid \emptyset \ne B^{\sigma^{-1}} \cap \{i, -i\} \ne B'^{\sigma^{-1}} \cap \{i, -i\} \ne \emptyset\}|.
\end{aligned}$$

Thus, (ii) holds. Therefore, the claim is proved.

By assumption, then, τ is conjugate to τ_0 in $\mathrm{Aut}(\mathcal{D}(W))$. This implies that there exists an automorphism $\pi \in \mathrm{Aut}(\mathcal{D}(W))$ such that $\sigma^{-1}\tau_0\sigma = \pi^{-1}\tau_0\pi$. Replacing σ by $\sigma\pi^{-1}$, we may assume from the beginning that σ commutes with τ_0. Then there exists a permutation $\rho \in S_n$ and $q_j \in \{\pm 1\}$ such that $(\pm j)^\sigma = \pm q_j j^\rho$. Let

$$\begin{aligned}
\mathcal{B}(W) &= \{B_i^\varepsilon \mid 1 \le i \le n,\ \varepsilon = \pm 1\}, \\
\mathcal{B}(U) &= \{C_i^\varepsilon \mid 1 \le i \le n,\ \varepsilon = \pm 1\},
\end{aligned}$$

where

$$\begin{aligned}
B_i^\varepsilon &= \{\varepsilon w_{ij} j \mid 1 \le j \le n,\ w_{ij} \ne 0\}, \\
C_i^\varepsilon &= \{\varepsilon u_{ij} j \mid 1 \le j \le n,\ u_{ij} \ne 0\}.
\end{aligned}$$

Since $\mathcal{B}(U)^\sigma = \mathcal{B}(W)$, for any i, there exists i' and and $p_i \in \{\pm 1\}$ such that $(C_i^+)^\sigma = B_{i'}^{p_i}$. Since σ commutes with τ_0, we have $(C_i^-)^\sigma = B_{i'}^{-p_i}$. This implies that there exists a permutation $\pi \in S_n$ such that $(C_i^+)^\sigma = B_{i^\pi}^{p_i}$. Thus, $q_j u_{ij} = p_i w_{i^\pi, j^\rho}$.

Now, define monomial matrices $P = (p_i \delta_{i^\pi, j})$, $Q = (q_i \delta_{i^\rho, j})$. Then we obtain $PWQ^{-1} = U$. Therefore, W is equivalent to U. $\square$

4 Weighing matrices of weight 5

In the course of reproducing a classification of weighing matrices of order 12 (see Section 5), we discovered errors in the classification of weighing matrices of weight 5 given in [4, Theorem 5]. In this section, we give a revised classification of weighing matrices of weight 5.

In the proof of [4, Theorem 5], the authors of [4] divide the classification into the following three cases:

(a) at least two other rows intersect the first row in four places or,

(b) no rows intersect any other row in four places or,

(c) exactly one row intersects the first row in four places.

Then all weighing matrices of weight 5 for the three cases (a), (b) and (c) were classified in [4, Theorem 5]. In the proof of [4, Theorem 5], $D(16,5)$ is claimed to be the unique weighing matrix of weight 5 satisfying (b). However, we found more weighing matrices of weight 5 satisfying (b). In Figure 1, we give such weighing matrices $W_{12,5}$ and $W_{14,5}$ of orders 12 and 14, respectively.

Lemma 5. *Let W be a weighing matrix of order $2n$ and weight 5 satisfying the condition (b). Then W contains $W_{12,5}$, $W_{14,5}$ or $D(16,5)$ as a direct summand.*

Proof. From the condition (b), we may assume without loss of generality that the first 5 rows of W have the following form:

$$M_1 = \left(\begin{array}{cc|ccc|ccc|ccc|c}
+ & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots0 \\
+ & - & 0 & 0 & 0 & + & + & + & 0 & 0 & 0 & 0\cdots0 \\
+ & 0 & - & 0 & 0 & - & 0 & 0 & + & + & 0 & 0\cdots0 \\
+ & 0 & 0 & - & 0 & 0 & - & 0 & - & 0 & + & 0\cdots0 \\
+ & 0 & 0 & 0 & - & 0 & 0 & - & 0 & - & - & 0\cdots0
\end{array}\right),$$

where $+, -$ denote $1, -1$, respectively. In addition, we may assume without loss of generality that the next three rows have the following form:

$$M_2 = \left(\begin{array}{cc|ccc|c|c|c}
0 & + & - & 0 & 0 & & & \\
0 & + & 0 & - & 0 & A & B & C \\
0 & + & 0 & 0 & - & & &
\end{array}\right),$$

$$W_{12,5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$W_{14,5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & -1 & 0 \end{pmatrix}$$

Figure 1: Weighing matrices of orders $12, 14$ and weight 5

where A is a 3×3 permutation matrix, B is some 3×3 matrix and C is some $3 \times (2n - 11)$ matrix. Let $M(A, B, C)$ denote the matrix

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}.$$

If $A' = P_1 A P_1^{-1}$ for some 3×3 permutation matrix P_1, then

$$PM(A, B, C)P^{-1} = M(A', B, C),$$

where

$$P = \begin{pmatrix} I_2 & & & \\ & P_1 & & \\ & & P_1 & \\ & & & I_{2n-8} \end{pmatrix}.$$

This means that it is sufficient to consider the matrix A up to conjugacy in the symmetric group of degree 3, so we assume $A = I_3, A_2$ or A_3, where

$$A_2 = \begin{pmatrix} + & 0 & 0 \\ 0 & 0 & + \\ 0 & + & 0 \end{pmatrix} \text{ and } A_3 = \begin{pmatrix} 0 & 0 & + \\ + & 0 & 0 \\ 0 & + & 0 \end{pmatrix}.$$

- Case $A = I_3$:
  From the orthogonality of rows,

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } C = \left(\begin{array}{ccc|c} + & + & 0 & 0\cdots0 \\ - & 0 & - & 0\cdots0 \\ 0 & - & + & 0\cdots0 \end{array}\right).$$

  Moreover, the matrix $M(A,B,C)$ is uniquely extended to

$$W = \left(\begin{array}{c|c} D(16,5) & O \\ \hline O & * \end{array}\right),$$

  up to equivalence, where O is the zero matrix.

- Case $A = A_2$:
  From the orthogonality of rows,

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & - \\ 0 & 0 & + \end{pmatrix} \text{ and } C = \left(\begin{array}{cc|c} + & + & 0\cdots0 \\ 0 & - & 0\cdots0 \\ - & 0 & 0\cdots0 \end{array}\right).$$

  Moreover, the matrix $M(A,B,C)$ is uniquely extended to

$$W = \left(\begin{array}{c|c} W_{14,5} & O \\ \hline O & * \end{array}\right),$$

  up to equivalence, where $W_{14,5}$ is given in Figure 1.

- Case $A = A_3$:
  From the orthogonality of rows, B must be one of the following three matrices:

$$\begin{pmatrix} 0 & - & 0 \\ 0 & + & - \\ 0 & 0 & + \end{pmatrix}, \begin{pmatrix} - & 0 & - \\ + & 0 & 0 \\ 0 & 0 & + \end{pmatrix} \text{ and } \begin{pmatrix} 0 & - & 0 \\ + & 0 & 0 \\ - & + & 0 \end{pmatrix}.$$

  Then C can be considered as

$$\left(\begin{array}{c|c} + & 0\cdots 0 \\ 0 & 0\cdots 0 \\ - & 0\cdots 0 \end{array}\right), \left(\begin{array}{c|c} 0 & 0\cdots 0 \\ + & 0\cdots 0 \\ - & 0\cdots 0 \end{array}\right) \text{ and } \left(\begin{array}{c|c} + & 0\cdots 0 \\ - & 0\cdots 0 \\ 0 & 0\cdots 0 \end{array}\right),$$

  respectively. Moreover, for each case the matrix $M(A, B, C)$ is uniquely extended to

$$W = \left(\begin{array}{c|c} W_{12,5} & O \\ \hline O & * \end{array}\right),$$

  up to equivalence, where $W_{12,5}$ is given in Figure 1.

Therefore, W contains $W_{12,5}$, $W_{14,5}$ or $D(16, 5)$ as a direct summand. □

Remark 6. For order 14, it follows from [4, Theorem 5] that there are two inequivalent weighing matrices of weight 5, namely, $E(14, 5)$ and $W(6, 5) \oplus W(8, 5)$ in [4]. On the other hand, the table in [4, Appendix B] lists the number of inequivalent weighing matrices of weight 5 to be three, and the missing matrix is denoted by $D(14, 5)$ which, however, is not defined in [4].

Remark 7. Let $R = (r_{ij})$ be the square matrix of order n with $r_{ij} = 1$ if $i + j - 1 = n$ and 0 otherwise. If A_1 and A_2 are circulant matrices of order n with entries $0, \pm 1$ satisfying $A_1 A_1^T + A_2 A_2^T = kI$, then the matrices

$$W_1 = \begin{pmatrix} A_1 & A_2 \\ -A_2^T & A_1^T \end{pmatrix} \text{ and } W_2 = \begin{pmatrix} A_1 & A_2 R \\ -A_2 R & A_1 \end{pmatrix}$$

are weighing matrices of order $2n$ and weight k [6, Proposition 4.46]. Kotsireas and Koukouvinos [13] claim that all weighing matrices of the form W_1 or W_2 are found by an exhaustive search for $n \leq 11$. Although the results of their search are not given, this means that they must have found the weighing matrix $W_{14,5}$, since it is equivalent to the weighing matrix W_1 where A_1 and A_2 are the circulant matrices with first rows

$$(1,0,0,0,0,0,0) \text{ and } (-1,1,1,0,1,0,0),$$

respectively. We verified that no weighing matrices W_1, W_2 constructed from two circulant matrices A_1 and A_2 are equivalent to $W_{12,5}$. This was done by finding all weighing matrices of the form W_1 and W_2 by an exhaustive search.

Remark 8. Let W be any of $W_{12,5}, W_{14,5}$ and $D(16,5)$. Then W^T also satisfies (b). Let $\overline{W}$ be the $(1,0)$-matrix obtained from W by changing -1 to 1 in the entries. Then $\overline{W}$ is the incidence matrix of a semibiplane (see [25] for the definition of a semibiplane). The three semibiplanes obtained in this way are given in [25, Proposition 15].

By the above lemma, we have the following revised classification for weight 5. See [4] for the definitions of the weighing matrices $W(6,5)$, $W(8,5)$, $E(4t_i+2,5)$ and $F(4t_j+4,5)$.

Theorem 9. *Any weighing matrix of order* $2n$ *and weight* 5 *is equivalent to*

$$\bigoplus_{i_1} W(6,5) \bigoplus_{i_2} W(8,5) \bigoplus_{i_3} W_{12,5} \bigoplus_{i_4} W_{14,5} \\ \bigoplus_{i_5} D(16,5) \bigoplus_{t_i} \Big(\bigoplus_{i_6} E(4t_i+2,5)\Big) \bigoplus_{t_j} \Big(\bigoplus_{i_8} F(4t_j+4,5)\Big),$$

where $t_i \geq 2$ *and* $t_j \geq 2$.

Table 4 is a revised table of a classification of weighing matrices of order $2n \leq 20$ and weight 5 in [4, Appendix B].

5 Weighing matrices of order 12

The classification of weighing matrices of order 12 was done in [4] and [17]. In this section, we give a revised list of weighing matrices of weights $6, 8, 10$. These classifications were done by considering self-dual $\mathbb{Z}_k$-codes of length 12, where $k = 3, 4$ and 5, respectively, using the method in Section 3. These approaches are similar, and we give details only for weight 6.

Table 4: Weighing matrices of weight 5

$2n$	#	Matrices
6	1	$W(6,5)$
8	1	$W(8,5)$
10	1	$E(10,5)$
12	3	$W_{12,5}$, $F(12,5)$, $W(6,5) \oplus W(6,5)$
14	3	$W_{14,5}$, $E(14,5)$, $W(6,5) \oplus W(8,5)$
16	4	$D(16,5)$, $F(16,5)$, $W(8,5) \oplus W(8,5)$, $W(6,5) \oplus E(10,5)$
18	5	$E(18,5)$, $W(6,5) \oplus W_{12,5}$, $W(6,5) \oplus F(12,5)$, $W(6,5) \oplus W(6,5) \oplus W(6,5)$ $W(8,5) \oplus F(10,5)$
20	7	$F(20,5)$, $W(6,5) \oplus W_{14,5}$, $W(6,5) \oplus E(14,5)$, $W(6,5) \oplus W(6,5) \oplus W(8,5)$ $W(8,5) \oplus W_{12,5}$, $W(8,5) \oplus F(12,5)$, $E(10,5) \oplus E(10,5)$

5.1 Weight 6

As described in Section 3, any weighing matrix of order 12 and weight 6 can be regarded as 12 codewords of weight 6 in some ternary self-dual code of length 12. There are three inequivalent ternary self-dual codes of length 12 [15, Table 1], and these codes are denoted by G_{12}, $4C_3(12)$ and $3E_4$. The code G_{12} has minimum weight 6 and the other codes have minimum weight 3, and the numbers of codewords of weight 6 in these codes are 264, 240 and 192, respectively. By considering sets of 12 codewords of weight 6 in these codes, we have the following classification of weighing matrices of order 12 and weight 6, using the method in Section 3.

Theorem 10. *There are* 8 *inequivalent weighing matrices of order* 12 *and weight* 6*.*

The number of inequivalent weighing matrices of order 12 and weight 6 was incorrectly reported as 7 in [17, Theorem 5]. The 7 inequivalent matrices in [17, Theorem 5] are denoted by E_1^*, E_2^*, $(E_2^*)^T$, E_5^*, E_{14}^*, $(E_{14}^*)^T$ and G_2^*. The missing matrix $W_{12,6}$ is listed in Figure 2. We remark that $W_{12,6}$ and $W_{12,6}^T$ are equivalent.

Remark 11. It is claimed in the proof of [17, Lemma 31] that there are 4 weighing matrices which are constructed from Case II up to equivalence. The matrix $W_{12,6}$ is also constructed from Case II, and hence there are 5 weighing matrices which are constructed from Case II up to equivalence.

$$
W_{12,6} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\
1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 \\
0 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 1 & -1
\end{pmatrix}
$$

$$
W_{12,10} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\
0 & 1 & 1 & -1 & -1 & 0 & 1 & -1 & 1 & -1 & 1 & 1 \\
-1 & 0 & 1 & 1 & -1 & -1 & 0 & 1 & -1 & 1 & 1 & 1 \\
-1 & -1 & 0 & 1 & 1 & 1 & -1 & 0 & 1 & -1 & 1 & 1 \\
1 & -1 & -1 & 0 & 1 & -1 & 1 & -1 & 0 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 0 & 1 & -1 & 1 & -1 & 0 & 1 & 1 \\
0 & 1 & -1 & 1 & -1 & 0 & -1 & -1 & 1 & 1 & -1 & 1 \\
-1 & 0 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 0 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & -1 & 1 \\
-1 & 1 & -1 & 0 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & 0 & -1 & -1 & 1 & 1 & 0 & -1 & 1
\end{pmatrix}
$$

Figure 2: Weighing matrices $W_{12,6}$ and $W_{12,10}$

In Table 5, we list g-distributions $N(i)$ $(i = 0, 1, \dots, 6)$ for the 7 matrices given in [17, Theorem 5] along with the new matrix $W_{12,6}$. Table 5 also shows that the 8 weighing matrices are inequivalent. By Proposition 3, the ternary codes $C_3(W)$ generated by the rows of these matrices W are self-dual, and the identifications with those appearing in [15] are given in the last column of Table 5.

5.2 Weight 8

According to [17, Theorem 3], there are 6 inequivalent weighing matrices of order 12 and weight 8. However, our method in Section 3, applied to the known classification of self-dual $\mathbb{Z}_4$-codes of length 12 given in [5], leads to

Table 5: Weighing matrices of order 12 and weight 6

W	$N(0)$	$N(1)$	$N(2)$	$N(3)$	$N(4)$	$N(5)$	$N(6)$	$C_3(W)$
E_1^*	396	720	180	240	180	0	0	G_{12}
E_2^*	516	432	420	144	204	0	0	G_{12}
$(E_2^*)^T$	432	528	504	48	192	0	12	$4C_3(12)$
E_5^*	492	432	468	144	180	0	0	G_{12}
E_{14}^*	708	0	756	0	252	0	0	G_{12}
$(E_{14}^*)^T$	384	576	576	0	144	0	36	$3E_4$
G_2^*	432	576	456	0	240	0	12	$4C_3(12)$
$W_{12,6}$	516	360	540	120	180	0	0	G_{12}

the following classification of weighing matrices of order 12 and weight 8.

Theorem 12. *There are* 7 *inequivalent weighing matrices of order* 12 *and weight* 8*.*

Remark 13. The 6 inequivalent matrices in [17, Theorem 3] are denoted by A_1, A_3, A_6, A_7, A_8 and A_9. However, A_{11} appeared in the proof of [17, Theorem 3] is inequivalent to any of the matrices A_i ($i = 1, 3, 6, 7, 8, 9$). This is an error in [17, Theorem 3].

Let W be a weighing matrix of order 12 and weight 8. Let $D_4(W)$ be the $\mathbb{Z}_4$-code with generator matrix $(\ I_{12}\ ,\ W\ )$, where the matrix W is regarded as a matrix over $\mathbb{Z}_4$. The numbers $\#D_6$ of codewords of weight 6 of $D_4(W)$, listed in Table 6, were found by the Magma function `NumberOfWords`. These numbers also show that the 7 weighing matrices are inequivalent,

Table 6: Weighing matrices of order 12 and weight 8

W	A_1	A_3	A_6	A_7	A_8	A_9	A_{11}
$\#D_6$	2852	1764	1092	932	1124	1700	1220

5.3 Weight 10

According to [17, Theorem 1], there are 4 inequivalent weighing matrices of order 12 and weight 10. However, our method in Section 3, applied to the known classification of self-dual $\mathbb{F}_5$-codes of length 12 given in [14], leads to the following classification of weighing matrices of order 12 and weight 10.

Theorem 14. *There are* 5 *inequivalent weighing matrices of order* 12 *and weight* 10*.*

The 4 inequivalent matrices in [17, Theorem 1] are denoted by A_1, A_4, A_7 and A_8. The missing matrix $W_{12,10}$ is listed in Figure 2. We remark that $W_{12,10}$ and $W_{12,10}^T$ are equivalent.

Remark 15. It is claimed in the proof of [17, Lemma 11] that there are only 7 vectors such that the matrices Y_i are normal matrices of level 4. We verified that this is incorrect and there is one missing vector, namely, the fourth row of $W_{12,10}$. Moreover, the 7×12 matrix $\bar{Y}_4$ consisting of the first 7 rows of $W_{12,10}$ should be considered in [17, Lemma 12] as a possible matrix of level 7.

In Table 7, we list the self-dual $\mathbb{F}_5$-codes $C_5(W)$ generated by the rows of these matrices W, in the notation of [14]. This shows that $W_{12,6}$ must be inequivalent to any of the other 4 matrices. We consider $\mathbb{F}_5$-codes $D_5(W)$ with generator matrices $(\ I_{12}\ ,\ W\ )$, where the matrices W are regarded as matrices over $\mathbb{F}_5$. The numbers $\#D_8$ of codewords of weight 8 are listed in Table 7, which also shows that the 5 weighing matrices are inequivalent. These numbers were found by the MAGMA function `NumberOfWords`.

Table 7: Weighing matrices of order 12 and weight 10

W	$C_5(W)$	$\#D_8$
A_1	F_6^2	3696
A_4	F_{12}	3000
A_7	F_{12}	4080
A_8	F_6^2	4560
$W_{12,10}$	K_{12}	3792

5.4 Other weights

By Theorem 9 (see Table 4), there are 3 inequivalent weighing matrices of order 12 and weight 5, namely, $W_{12,5}$, $F(12,5)$, and $W(6,5) \oplus W(6,5)$. In

Table 8, we list the self-dual $\mathbb{F}_5$-codes $C_5(W)$ generated by the rows of these matrices W, in the notation of [14].

Table 8: Weighing matrices of order 12 and weight 5

W	$W(6,5) \oplus W(6,5)$	$F(12,5)$	$W_{12,5}$
$C_5(W)$	F_6^2	F_{12}	K_{12}

For weights 7 and 9, we verified that the classifications in [17] are correct, using the classification of self-dual $\mathbb{F}_p$-codes of length 12, where $p = 7$ and 3, respectively. Table 9 summarizes a revised classification of weighing matrices of order 12.

Table 9: Classification of weighing matrices of order 12

Weight	#	References	Weight	#	References
1	1	[4]	7	3	[17]
2	1	[4]	8	7	Theorem 12
3	1	[4]	9	4	[17]
4	5	[4]	10	5	Theorem 14
5	3	Theorem 9	11	1	[4]
6	8	Theorem 10	12	1	[24]

6 Weighing matrices of orders 14, 15 and 17

We continue a classification of weighing matrices using the method in Section 3. Then we have the following classification of weighing matrices of order n and weight k for

$$(n, k) = (14, 8), (14, 9), (14, 10), (15, 9) \text{ and } (17, 9), \tag{2}$$

using the classification of maximal self-orthogonal $\mathbb{Z}_m$-codes of length n (see Section 2), where $m = 4, 3, 5, 3$ and 3, respectively. Since approaches are similar to that used in Section 5, we only list in Table 10 the numbers # of inequivalent weighing matrices of order n and weight k for (n, k) listed in (2).

Hence, our result completes a classification of weighing matrices of orders up to 15 and order 17.

Table 10: Classification of weighing matrices of orders $14, 15$ and 17

Order	Weight	#	References	Order	Weight	#	References
14	1	1	[4]	15	1	1	[4]
	2	1	[4]		4	6	[4]
	4	3	[4]		9	37	Section 6
	5	3	Theorem 9	17	1	1	[4]
	8	66	Section 6		4	3	[4]
	9	7	Section 6		9	2360	Section 6
	10	19	Section 6		16	1	[4]
	13	1	[4]				

Now, we compare our classifications with the known classifications for $(n, k) = (14, 8)$ and $(17, 9)$. According to [18, Theorem 3.9], there are 65 inequivalent weighing matrices of order 14 and weight 8. Using the classification of self-dual $\mathbb{Z}_4$-codes of length 14, we classified weighing matrices of order 14 and weight 8, and we claim that the classification in [18, Theorem 3.9] misses the matrix $W_{14,8}$, which is listed in Figure 3. We remark that $W_{14,8}$ and $W_{14,8}^T$ are equivalent. Hence, we have the following:

Theorem 16. *There are* 66 *inequivalent weighing matrices of order* 14 *and weight* 8*.*

Remark 17. The intersection patterns of $W_{14,8}$ and $W_{14,8}^T$ are

$$(x_2, x_4, x_6, x_8) = (0, 11, 2, 0)$$

which is the same as $\mathbf{c}_{25}$ in [18, p. 139]. Hence, $W_{14,8}$ is of Type $\mathbf{c}_{25}$ in the sense of [18]. It is claimed in [18, Theorem 3.6] that a matrix of Type $\mathbf{c}_{25}$ is equivalent to some matrix of Type $\mathbf{c}_i$ $(i \neq 25)$. This is an error.

Among the weighing matrices of order 17 and weight 9, Ohmori and Miyamoto [20] claimed to classify those with intersection number 8, and they found exactly 925 such matrices. However, we verified that only 517 of the 2360 weighing matrices of order 17 and weight 9 have intersection number 8. Since their list of 925 weighing matrices is not available, we are unable to compare their result with ours.

$$W_{14,8} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}.$$

Figure 3: Weighing matrix $W_{14,8}$

7 Other orders and weights

For orders $n \geq 13$ and weights $k \geq 6$, we classified weighing matrices of some orders n and weights k listed in Table 11 using the classification of maximal self-orthogonal $\mathbb{F}_p$-codes of length n given in Table 1. Since approaches are similar to that used in Section 5, we only list in Table 11 the numbers # of inequivalent weighing matrices for which we classified, and the primes p. Also, we list in the same table the orders and weights for which we checked the known classifications by our classification method, along with references.

Table 11: Other orders and weights

(n,k)	#	p	References	(n,k)	#	p	References
$(13,9)$	8	3	[19]	$(16,15)$	1	3	[4]
$(16,6)$	30	3		$(18,9)$	11891	3	
$(16,9)$	704	3		$(20,6)$	49	3	
$(16,10)$	670	5		$(20,18)$	53	3	
$(16,12)$	279	3		$(24,6)$	190	3	

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