Extremal Configurations in Dimension 48

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Motivation

The binary perfect Golay code is a set of 2^{12} elements of the vector space $\mathbb{F}_2^{23}$, giving a partition of $\mathbb{F}_2^{23}$ into 2^{12} spheres of volume

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The Golay code G_{23} can be made linear, and unique up to isometry.
Construction: by quadratic residues modulo 23.

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Surprisingly, Mathieu groups were discovered first.

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As a graded ring, R has Poincaré series

$$\sum_{j=0}^{\infty} \dim R_j t^j = \frac{1}{(1-t^8)(1-t^{12})}$$

The Poincaré series

The Poincaré series of the graded ring of the invariant ring containing the weight enumerator polynomials of doubly-even self-dual codes is

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$$+ 3t^{48} + 2t^{52} + 3t^{56} + \cdots$$

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Roughly speaking, $\dim R_n$ is the degree of freedom for minimum weight bound.

General inequality for 3-distance set

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$$n = \ 23 \text{ (Leech}\cap\mathbb{R}^{23}\text{)}, \ 47 \text{ (?)}, 79\text{(?)},$$

$$\ldots, (2m+1)^2 - 2, \ldots$$

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An antipodal 3-distance set in the unit sphere in $\mathbb{R}^{47}$ could have $n(n+1) = 47 * 48.$

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An antipodal 3-distance set in the unit sphere in $\mathbb{R}^{47}$ could have $47 * 48$, but it was shown recently (Bannai–M.–Venkov) that such a set does not exist.

Construction A

Let $\varphi : \mathbb{Z}^n \to (\mathbb{Z}/m\mathbb{Z})^n$ be the canonical homomorphism. If $C \subset (\mathbb{Z}/m\mathbb{Z})^n$ is a self-dual code, then the lattice

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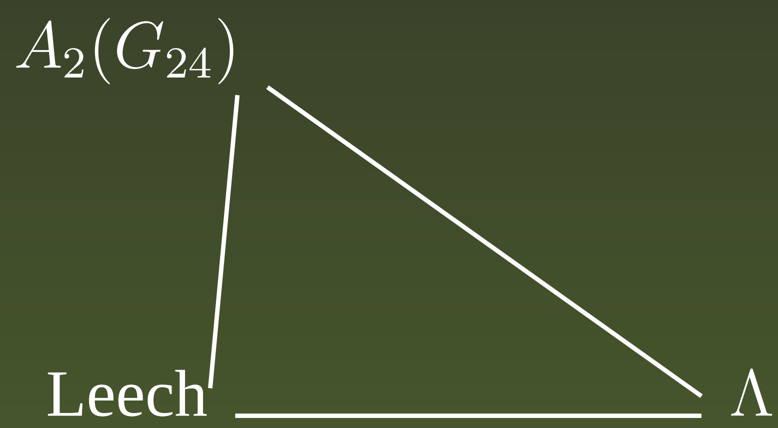
The lattice $A_2(G_{24})$ has a 2-frame (hence is not the Leech lattice).

Neighbor relations

Two unimodular lattices Γ, Γ' are said to be neighbors if $\Gamma \cap \Gamma'$ has index 2 in Γ (and also in Γ').

Dimension 24:



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Dimension 48: (Harada–Kitazume–M.–Venkov)

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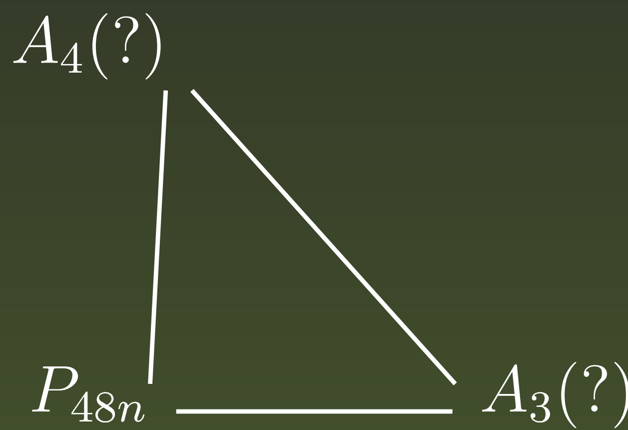

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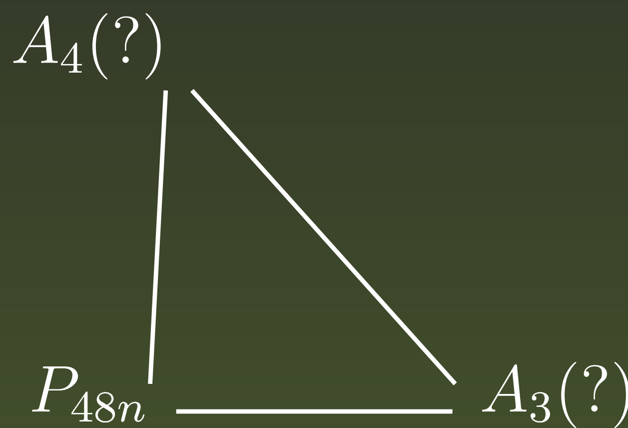

Does Nebe's lattice P_{48n} have a 6-frame?

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Delsarte–Goethals–Seidel Theorem: extremal s-distance set in S^{n-1} $\longrightarrow$ spherical design

Definition of a design

A spherical t-design X is a finite subset of $S^{n-1} \subset \mathbb{R}^n$ s.t.

$$\frac{1}{\int_{S^{n-1}} 1} \int_{S^{n-1}} f(x) = \frac{1}{|X|} \sum_{x \in X} f(x)$$

holds for any polynomial $f(x)$ of degree $\leq t$.

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A (combinatorial) t-design $\mathcal{B}$ is a subset of $\binom{\Omega}{k}$ s.t.

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A (combinatorial) t-design $\mathcal{B}$, or t-(v,k,λ) design is a subset of $\binom{\Omega}{k}$ such that

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If $\mathcal{B}$ is a t-(v, k, λ) design, then

$$\mathcal{B} \ni B \leftrightarrow \text{vector of weight } k \text{ in } \mathbb{F}_2^v$$

and these vector generate a linear code $C \subset \mathbb{F}_2^v$.

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From an extremal doubly-even self-dual code of length 48, one obtains 5-$(48, 12, 8)$ design. Is such a design unique?

If $\mathcal{B}$ is a t-(v, k, λ) design, then

$$\mathcal{B} \ni B \leftrightarrow \text{vector of weight } k \text{ in } \mathbb{F}_2^v$$

and these vector generate a linear code $C \subset \mathbb{F}_2^v$.

Assuming $\mathcal{B}$ is self-orthogonal, i.e.,

$$B, B' \in \mathcal{B} \implies |B \cap B'| : \text{even},$$

we aim to show that C is a (unique) extremal doubly-even self-dual code of length 48.

Characterization Method

Suppose $u \in C^{\perp}$, $A = \operatorname{supp}(u) = \{i_1, \ldots, i_m\}$. In the defining equation of a design, take f to be elementary symmetric functions of degree at most t in $\{x_{i_1}, \ldots, x_{i_m}\}$.

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Then $f(B) = \binom{|A \cap B|}{s}$, and

$$\frac{1}{\binom{v}{k}} \sum_{B \in \binom{\Omega}{k}} f(B) = \frac{1}{|\mathcal{B}|} \sum_{B \in \mathcal{B}} f(B) = \frac{1}{|\mathcal{B}|} \sum_{j=0}^{\infty} \binom{j}{s} n_j$$

where $n_j = |\{B \mid B \in \mathcal{B},\ |A \cap B| = j\}|$.

Designs in dimension 48

For a self-orthogonal 5-$(48, 12, 8)$ design, if $|A| = 8$, then $n_j = 0$ unless $j \in \{0, 2, 4, 6, 8\}$. There are 5 unknowns, 6 $(s = 0, 1, 2, 3, 4, 5)$ equations, no solutions. Therefore $C^\perp$ does not contain a vector of weight 8.

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Theorem 1 (Harada–M.–Tonchev) *A self-orthogonal* 5-$(48, 12, 8)$ *design is unique.*

Nonexistence of an extremal 3-distance set

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$$\frac{1}{\int_{S^{n-1}} 1} \int_{S^{n-1}} f(x) = \frac{1}{|X|} \sum_{x \in X} f(x)$$

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Theorem 2 (Bannai–M.–Venkov) *There is no antipodal 3-distance set of size $47 \cdot 48$ in S^{47}.*