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We assume the reader is familiar with linear algebra, for example, finite-dimensional real vector spaces, the standard inner product, subspaces, direct sums, the matrix representation of a linear transformation.

Let $\alpha \in \mathbf{R}^2$ be a nonzero vector. The set of vectors orthogonal to α form a line L, and $\mathbf{R}^2 = \mathbf{R}\alpha \oplus L$ holds. Given $\lambda \in \mathbf{R}^2$ can be expressed as

$$\lambda = c\alpha + \mu \quad \text{for some } c \in \mathbf{R} \text{ and } \mu \in L. \tag{1}$$

Since $(\mu, \alpha) = 0$, we have

$$\begin{aligned} c &= \frac{(c\alpha + \mu, \alpha)}{(\alpha, \alpha)} \\ &= \frac{(\lambda, \alpha)}{(\alpha, \alpha)} \qquad \text{(by (1))}. \end{aligned}$$

The *reflection* of λ with respect to the line L is obtained by negating the $\langle\alpha\rangle$-component of λ in (1), that is,

$$\begin{aligned} -c\alpha + \mu &= \lambda - 2c\alpha \\ &= \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha. \end{aligned}$$

Let $s_\alpha : \mathbf{R}^2 \to \mathbf{R}^2$ denote the mapping defined by the above formula, that is,

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \quad (\lambda \in \mathbf{R}^2). \tag{2}$$

It is clear that s_α is a linear transformation of $\mathbf{R}^2$. This means that there exists a 2×2 matrix S_α such that

$$s_\alpha(\lambda) = S_\alpha \lambda \quad (\lambda \in \mathbf{R}^2). \tag{3}$$

To find S_α, recall that L is the line orthogonal to α. Let

$$\mu = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

be a vector of length 1 in L. The vector

$$\nu = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

is orthogonal to μ, hence in $\mathbf{R}\alpha$. This implies that

$$\begin{aligned} s_\alpha(\mu) &= \mu, \\ s_\alpha(\nu) &= -\nu. \end{aligned}$$

Thus

$$S_\alpha \begin{bmatrix}\mu & \nu\end{bmatrix} = \begin{bmatrix}\mu & -\nu\end{bmatrix},$$

which implies

$$\begin{aligned}
S_\alpha &= \begin{bmatrix}\mu & -\nu\end{bmatrix}\begin{bmatrix}\mu & \nu\end{bmatrix}^{-1} \\
&= \begin{bmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}^{-1} \\
&= \begin{bmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} \\
&= \begin{bmatrix}\cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & -(\cos^2\theta - \sin^2)\end{bmatrix} \\
&= \begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{bmatrix}.
\end{aligned}$$

This is the matrix representation of a reflection on the plane $\mathbf{R}^2$.

We next consider the composition of two reflections. Let s_α and S_α be as before, and let s_β be another reflection, with matrix representation

$$S_\beta = \begin{bmatrix}\cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi\end{bmatrix}.$$

Then

$$\begin{aligned}
S_\alpha S_\beta &= \begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{bmatrix}\begin{bmatrix}\cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi\end{bmatrix} \\
&= \begin{bmatrix}\cos 2(\theta-\varphi) & -\sin 2(\theta-\varphi) \\ \sin 2(\theta-\varphi) & \cos 2(\theta-\varphi)\end{bmatrix} \\
&= \begin{bmatrix}\cos 2(\theta-\varphi) & \cos(2(\theta-\varphi)+\frac{\pi}{2}) \\ \sin 2(\theta-\varphi) & \sin(2(\theta-\varphi)+\frac{\pi}{2})\end{bmatrix}.
\end{aligned}$$

This matrix maps the standard basis vector

$$\begin{bmatrix}1 \\ 0\end{bmatrix} = \begin{bmatrix}\cos 0 \\ \sin 0\end{bmatrix}, \begin{bmatrix}0 \\ 1\end{bmatrix} = \begin{bmatrix}\cos\frac{\pi}{2} \\ \sin\frac{\pi}{2}\end{bmatrix}$$

to

$$\begin{bmatrix}\cos 2(\theta-\varphi) \\ \sin 2(\theta-\varphi)\end{bmatrix}, \begin{bmatrix}\cos(2(\theta-\varphi)+\frac{\pi}{2}) \\ \sin(2(\theta-\varphi)+\frac{\pi}{2})\end{bmatrix},$$

meaning that both vectors are rotated $2(\theta-\varphi)$. Therefore, the product of two reflection is a rotation.

We are interested in the case where the resulting rotation is of finite order, that is, $2(\theta-\varphi)$ is a rational multiple of 2π. For brevity, write $s = s_\alpha$, $t = s_\beta$ and $\mathrm{id} = 1$. In this case, there exists a positive integer m such that $(st)^m = 1$. We may assume $s \neq t$, so that $st \neq 1$. We may choose minimal such m, so that

$$st, (st)^2, \dots, (st)^{m-1} \neq 1.$$

Writing $r = st$, this implies

$$1, r, r^2, \dots, r^{m-1} \text{ are pairwise distinct.} \tag{4}$$

We aim to determine the set $\langle s, t\rangle$ of all linear transformations expressible as a product of s, t. We have already seen that this set contains at least m distinct elements (4). Since $s^2 = t^2 = 1$, possible product of s, t are one of the following four forms:

$$stst \cdots st, \tag{5}$$
$$stst \cdots sts, \tag{6}$$
$$tsts \cdots ts, \tag{7}$$
$$tsts \cdots tst. \tag{8}$$

Products of the form (5) are precisely described in (4). Products of the form (6) are

$$s, rs, r^2 s, \dots, r^{m-1} s, \tag{9}$$

and these are distinct by (4). Since $ts = t^{-1}s^{-1} = (st)^{-1} = r^{-1}$, products of the form (7) are nothing but those in (4). Finally, since $rt = s$, products of the form (8) are then those in (9). Therefore, $\langle s, t\rangle$ consists of $2m$ elements described in (4) and (9). To show that these $2m$ elements are distinct, it suffices to show that there is no common element in (4) and (9), which follows immediately from the fact that $\det r = 1$ and $\det s = -1$.

It is important to note that this last part of reasoning, except the distinctness, follows only from the transformation rule

$$s^2 = t^2 = 1, \quad (st)^m = 1. \tag{10}$$

Setting $r = st$, we have $r^m = 1$ and $srs = r^{-1}$. Written in terms of r and s, we can also say that the determination of all elements in $\langle s, t\rangle$ follows only from the transformation rule

$$s^2 = r^m = 1, \quad sr = r^{-1}s. \tag{11}$$

Indeed, one can always rewrite sr to $r^{m-1}s$, so every element in $\langle s, r\rangle$ is of the form $r^k s^j$ with $0 \leq k < m$ and $j \in \{0, 1\}$.

In the next lecture, we will discuss a rigorous way of dealing with words in formal symbol subject to relations such as (10) and (11). In addition to this formal aspect, we will discuss explicit realizations of these symbols as linear transformation.

Definition 1. A linear transformation $s : \mathbf{R}^n \to \mathbf{R}^n$ is called a reflection if there exists a nonzero vector α such that $s(\alpha) = -\alpha$ and $s(h) = h$ for all $h \in (\mathbf{R}\alpha)^\perp$.

Note that, since $\mathbf{R}^n = \mathbf{R}\alpha \oplus (\mathbf{R}\alpha)^\perp$, the linear transformation is determined uniquely by the conditions $s(\alpha) = -\alpha$ and $s(h) = h$ for all $h \in (\mathbf{R}\alpha)^\perp$, so we denote this reflection by s_α. Moreover, any nonzero scalar multiple of α defines the same reflection, that is, $s_\alpha = s_{c\alpha}$ for any $c \in \mathbf{R}$ with $c \neq 0$.

Lemma 2. *Let $s : \mathbf{R}^n \to \mathbf{R}^n$ be a reflection. Then the matrix representation S of s is diagonalizable by an orthogonal matrix:*

$$P^{-1}SP = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

for some orthogonal matrix P. Conversely, if the matrix representation of s is of this form for some orthogonal matrix P, then s is a reflection.

Proof. Let $s = s_\alpha$. We may assume without loss of generality $(\alpha, \alpha) = 1$. Let $\beta_2, \ldots, \beta_n$ be an orthonormal basis of $(\mathbf{R}\alpha)^\perp$. Then $\alpha, \beta_2, \ldots, \beta_n$ is an orthonormal basis of $\mathbf{R}^n$. Let

$$P = \begin{bmatrix} \alpha & \beta_2 & \cdots & \beta_n \end{bmatrix}.$$

Then P is an orthogonal matrix, and

$$SP = P \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

To prove the converse, let α be the first column of P. Then clearly $s(\alpha) = -\alpha$ and $s(h) = h$ for any $h \in (\mathbf{R}\alpha)^\perp$. Thus $s = s_\alpha$. $\square$