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For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product. We also let Φ be a root system in V, and fix a simple system Δ in Φ. Let $\Pi = \Phi \cap \mathbf{R}_{\geq 0}\Delta$ be the unique positive system containing Δ. Recall

$$W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle,$$

which we denote by W for brevity.

Lemma 39. *If $\beta \in \Pi \backslash \Delta$, then there exists $\alpha \in \Delta$ such that $s_\alpha \beta \in \Pi$ and $\mathrm{ht}(\beta) > \mathrm{ht}(s_\alpha \beta)$.*

Proof. Write $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$, where $c_\alpha \in \mathbf{R}_{\geq 0}$ for $\alpha \in \Delta$. Since

$$\begin{aligned} 0 &< (\beta, \beta) \\ &= \sum_{\alpha \in \Delta} c_\alpha (\alpha, \beta), \end{aligned}$$

there exists $\alpha \in \Delta$ such that $c_\alpha(\alpha, \beta) > 0$. In particular, as $c_\alpha \geq 0$, we have

$$c = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} > 0.$$

Since

$$\begin{aligned} s_\alpha \beta &= \beta - c\alpha \\ &= \sum_{\gamma \in \Delta \backslash \{\alpha\}} c_\gamma \gamma + (c_\alpha - c)\alpha, \end{aligned}$$

we have $\mathrm{ht}(s_\alpha \beta) = \mathrm{ht}(\beta) - c < \mathrm{ht}(\beta)$. Since $\beta \in \Pi \setminus \Delta \subset \Pi \setminus \{\alpha\}$, Lemma 34 implies $s_\alpha \beta \in \Pi$. □

Lemma 40. *If $\beta \in \Phi$, then there exists a sequence $\alpha_1, \ldots, \alpha_m$ of elements in Δ such that $s_{\alpha_1} \cdots s_{\alpha_m} \beta \in \Delta$.*

Proof. We first prove the assertion for $\beta \in \Pi$. Suppose there exists $\beta \in \Pi$ such that the assertion does not hold. Then clearly $\beta \notin \Delta$. We may assume that β has minimal height among such elements. By Lemma 39, there exists $\alpha \in \Delta$ such that $s_\alpha \beta \in \Pi$ and $\mathrm{ht}(\beta) > \mathrm{ht}(s_\alpha \beta)$. By the minimality of $\mathrm{ht}(\beta)$, there exists a sequence $\alpha_1, \ldots, \alpha_m$ of elements of Δ such that $s_{\alpha_1} \cdots s_{\alpha_m}(s_\alpha \beta) \in \Delta$. This is a contradiction.

If $\beta \in -\Pi$, then $-\beta \in \Pi$, so there exist $\alpha, \alpha_1, \ldots, \alpha_m \in \Delta$ such that

$$\alpha = s_{\alpha_1} \cdots s_{\alpha_m}(-\beta).$$

Then

$$s_\alpha s_{\alpha_1} \cdots s_{\alpha_m} \beta = -s_\alpha s_{\alpha_1} \cdots s_{\alpha_m}(-\beta)$$

$$
\begin{aligned}
&= -s_\alpha \alpha \\
&= \alpha \\
&\in \Delta.
\end{aligned}
$$

□

Theorem 41. *If Δ is a simple system in a root system Φ, then $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$.*

Proof. Let $\beta \in \Phi$. By Lemma 40, there exist $\alpha_0, \alpha_1, \dots, \alpha_m \in \Delta$ such that $s_{\alpha_1} \cdots s_{\alpha_m} \beta = \alpha_0$. Then

$$
\begin{aligned}
s_\beta &= s_{(s_{\alpha_1} \cdots s_{\alpha_m})^{-1} \alpha_0} \\
&= (s_{\alpha_1} \cdots s_{\alpha_m})^{-1} s_{\alpha_0} s_{\alpha_1} \cdots s_{\alpha_m} && \text{(by Lemma 12)} \\
&= s_{\alpha_m} \cdots s_{\alpha_1} s_{\alpha_0} s_{\alpha_1} \cdots s_{\alpha_m} \\
&\in \langle s_\alpha \mid \alpha \in \Delta \rangle.
\end{aligned}
$$

□

Definition 42. For $w \in W$, we define the *length* of w, denoted $\ell(w)$, to be

$$
\ell(w) = \min\{r \in \mathbf{Z} \mid r \geq 0,\ \exists \alpha_1, \dots, \alpha_r \in \Delta,\ w = s_{\alpha_1} \cdots s_{\alpha_r}\}.
$$

By convention, $\ell(1) = 0$.

Clearly, $\ell(w) = 1$ if and only if $w = s_\alpha$ for some $\alpha \in \Delta$. It is also clear that $\ell(w) = \ell(w^{-1})$.

Lemma 43. *For $w \in W$, $\det(w) = (-1)^{\ell(w)}$.*

Proof. Since $\det(s_\alpha) = -1$ for all $\alpha \in \Phi$, the result follows immediately. □

Lemma 44. *For $w \in W$ and $\alpha \in \Delta$, $\ell(s_\alpha w) = \ell(w) + 1$ or $\ell(w) - 1$.*

Proof. It is clear from the definition that $\ell(s_\alpha w) \leq \ell(w) + 1$. Switching the role of w and $s_\alpha w$, we obtain $\ell(s_\alpha w) \geq \ell(w) - 1$. Thus

$$
\ell(s_\alpha w) \in \{\ell(w) - 1, \ell(w), \ell(w) + 1\}.
$$

Since

$$
\begin{aligned}
(-1)^{\ell(s_\alpha w)} &= \det(s_\alpha w) && \text{(by Lemma 43)} \\
&= -\det w \\
&= -(-1)^{\ell(w)} && \text{(by Lemma 43).}
\end{aligned}
$$

This implies $\ell(s_\alpha w) \neq \ell(w)$. □

Notation 45. For $w \in W$, we write

$$n(w) = |\Pi \cap w^{-1}(-\Pi)|.$$

Lemma 46. *For $w \in W$, $n(w^{-1}) = n(w)$.*

Proof.

$$\begin{aligned} n(w^{-1}) &= |\Pi \cap w(-\Pi)| \\ &= |w^{-1}\Pi \cap (-\Pi)| \\ &= |w^{-1}(-\Pi) \cap \Pi| \\ &= n(w). \end{aligned}$$

□

Lemma 47. *For $w \in W$ and $\alpha \in \Delta$, the following statements hold:*

(i) $w\alpha > 0 \implies n(ws_\alpha) = n(w) + 1.$

(ii) $w\alpha < 0 \implies n(ws_\alpha) = n(w) - 1.$

(iii) $w^{-1}\alpha > 0 \implies n(s_\alpha w) = n(w) + 1.$

(iv) $w^{-1}\alpha < 0 \implies n(s_\alpha w) = n(w) - 1.$

Proof. (i) Since $w\alpha \in \Pi$, we have $\alpha \in w^{-1}\Pi$. Thus

$$\alpha \notin w^{-1}(-\Pi), \tag{69}$$

and

$$\begin{aligned} \alpha &= -s_\alpha \alpha \\ &\in -s_\alpha w^{-1}\Pi \\ &= s_\alpha w^{-1}(-\Pi). \end{aligned} \tag{70}$$

Thus

$$\begin{aligned} n(ws_\alpha) &= |\Pi \cap (ws_\alpha)^{-1}(-\Pi)| \\ &= |\Pi \cap s_\alpha w^{-1}(-\Pi)| \\ &= |(\Pi \setminus \{\alpha\}) \cap s_\alpha w^{-1}(-\Pi)| + 1 && \text{(by (70))} \\ &= |s_\alpha(\Pi \setminus \{\alpha\}) \cap s_\alpha w^{-1}(-\Pi)| + 1 && \text{(by Lemma 34)} \\ &= |(\Pi \setminus \{\alpha\}) \cap w^{-1}(-\Pi)| + 1 \\ &= |\Pi \cap w^{-1}(-\Pi)| + 1 && \text{(by (69))} \\ &= n(w) + 1. \end{aligned}$$

(ii) Since $w\alpha \in -\Pi$, we have

$$\alpha \in w^{-1}(-\Pi), \tag{71}$$

and $\alpha \notin w^{-1}\Pi$, so

$$\begin{aligned}\alpha &= -s_\alpha \alpha \\ &\notin -s_\alpha w^{-1}\Pi \\ &= s_\alpha w^{-1}(-\Pi).\end{aligned} \tag{72}$$

Thus

$$\begin{aligned}
n(ws_\alpha) &= |\Pi \cap (ws_\alpha)^{-1}(-\Pi)| \\
&= |\Pi \cap s_\alpha w^{-1}(-\Pi)| \\
&= |(\Pi \setminus \{\alpha\}) \cap s_\alpha w^{-1}(-\Pi)| && \text{(by (72))} \\
&= |s_\alpha(\Pi \setminus \{\alpha\}) \cap s_\alpha w^{-1}(-\Pi)| && \text{(by Lemma 34)} \\
&= |(\Pi \setminus \{\alpha\}) \cap w^{-1}(-\Pi)| \\
&= |\Pi \cap w^{-1}(-\Pi)| - 1 && \text{(by (71))} \\
&= n(w) - 1.
\end{aligned}$$

(iii) and (iv)

$$\begin{aligned}
n(s_\alpha w) &= n((s_\alpha w)^{-1}) && \text{(by Lemma 46)} \\
&= n(w^{-1}s_\alpha) \\
&= \begin{cases} n(w^{-1}) + 1 & \text{if } w^{-1}\alpha > 0, \\ n(w^{-1}) - 1 & \text{if } w^{-1}\alpha < 0 \end{cases} \\
&= \begin{cases} n(w) + 1 & \text{if } w^{-1}\alpha > 0, \\ n(w) - 1 & \text{if } w^{-1}\alpha < 0 \end{cases} && \text{(by Lemma 46).}
\end{aligned}$$

□

Theorem 48. *Let Δ be a simple system in a root system Φ. Let $\alpha_1, \ldots, \alpha_r \in \Delta$ and $w = s_1 \cdots s_r \in W$, where $s_i = s_{\alpha_i}$ for $1 \le i \le r$. If $n(w) < r$, then there exist i, j with $1 \le i < j \le r$ satisfying the following conditions:*

(i) $\alpha_i = s_{i+1} \cdots s_{j-1}\alpha_j$,

(ii) $s_{i+1}s_{i+2} \cdots s_j = s_i s_{i+1} \cdots s_{j-1}$,

(iii) $w = s_1 \cdots s_{i-1}s_{i+1} \cdots s_{j-1}s_{j+1} \cdots s_r$.

In particular, $n(w) \ge \ell(w)$.

Proof. (i) Setting $w = 1$ in Lemma 47(i), we find $n(s_\alpha) = 1$ for every $\alpha \in \Delta$. This implies that, if $r = 1$, then $n(w) = 1$. Therefore, we may assume $r \geq 2$.

We claim that there exists j with $2 \leq j \leq r$ such that $s_1 \cdots s_{j-1}\alpha_j < 0$. Suppose, to the contrary,

$$s_1 \cdots s_{j-1}\alpha_j > 0 \tag{73}$$

for all j with $2 \leq j \leq r$. Since $\alpha_1 > 0$, (73) holds also for $j = 1$. By Lemma 47(i), we obtain $n(s_1 \cdots s_j) = n(s_1 \cdots s_{j-1}) + 1$ for $1 \leq j \leq r$. By using induction, we obtain $n(w) = r$, contrary to our hypothesis.

Since $\alpha_j > 0$, there exists i with $1 \leq i < j$ such that

$$\begin{aligned} s_{i+1} \cdots s_{j-1}\alpha_j &> 0, \\ s_i s_{i+1} \cdots s_{j-1}\alpha_j &< 0. \end{aligned}$$

Thus

$$\begin{aligned} s_i s_{i+1} \cdots s_{j-1}\alpha_j &\in s_i\Pi \cap (-\Pi) \\ &= s_{\alpha_i}((\Pi \setminus \{\alpha_i\}) \cup \{\alpha_i\}) \cap (-\Pi) \\ &= ((\Pi \setminus \{\alpha_i\}) \cup \{-\alpha_i\}) \cap (-\Pi) && \text{(by Lemma 34)} \\ &= \{-\alpha_i\} \\ &= \{s_i(\alpha_i)\}. \end{aligned}$$

This implies $s_{i+1} \cdots s_{j-1}\alpha_j = \alpha_i$.

(ii)

$$\begin{aligned} s_{i+1} \cdots s_j &= s_{i+1} \cdots s_{j-1} s_{\alpha_j} (s_{i+1} \cdots s_{j-1})^{-1} (s_{i+1} \cdots s_{j-1}) \\ &= s_{s_{i+1} \cdots s_{j-1}\alpha_j} (s_{i+1} \cdots s_{j-1}) && \text{(by Lemma 12)} \\ &= s_{\alpha_i} (s_{i+1} \cdots s_{j-1}) && \text{(by (i))} \\ &= s_i s_{i+1} \cdots s_{j-1}. \end{aligned}$$

(iii)

$$\begin{aligned} w &= s_1 \cdots s_r \\ &= s_1 \cdots s_{i-1} (s_i \cdots s_{j-1}) s_j \cdots s_r \\ &= s_1 \cdots s_{i-1} (s_{i+1} \cdots s_j) s_j \cdots s_r && \text{(by (ii))} \\ &= s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} s_{j+1} \cdots s_r. \end{aligned}$$

In particular, $n(w) < r$ implies $r \neq \ell(w)$. Thus $n(w) \geq \ell(w)$. □

Corollary 49. *If $w \in W$, then $n(w) = \ell(w)$.*

Proof. From the last part of Theorem 48, it suffices to prove

$$n(w) \leq \ell(w) \quad (w \in W). \tag{74}$$

By the definition of $\ell(w)$, there exists $\alpha_1, \ldots, \alpha_{\ell(w)} \in \Delta$ such that $w = s_{\alpha_1} \cdots s_{\alpha_{\ell(w)}}$. We prove (74) by induction on $m = \ell(w)$. If $m = 0$, then $w = 1$, and $n(w) = 0 = \ell(w)$. Assume the result holds for up to $m - 1$. Then

$$\begin{aligned} n(s_{\alpha_1} \cdots s_{\alpha_{\ell(w)-1}}) &\leq \ell(s_{\alpha_1} \cdots s_{\alpha_{\ell(w)-1}}) \\ &\leq \ell(w) - 1. \end{aligned} \tag{75}$$

$$\begin{aligned} n(w) &= n((s_{\alpha_1} \cdots s_{\alpha_{\ell(w)-1}}) s_{\alpha_{\ell(w)}}) \\ &\leq n(s_{\alpha_1} \cdots s_{\alpha_{\ell(w)-1}}) + 1 && \text{(by Lemma 47(i),(ii))} \\ &\leq \ell(w) && \text{(by (75)).} \end{aligned}$$

□