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Lemma 50. *With reference to Definition 6, if* $a, b, x, y \in F(X)$ *and* $xN = yN$*, then* $axbN = aybN$.

Proof.

$$\begin{aligned} xN = yN &\implies x^{-1}y \in N \\ &\implies b^{-1}x^{-1}yb \in N \\ &\implies xbN = yb \in N \\ &\implies axbN = aybN. \end{aligned}$$

□

Lemma 51. *With reference to Definition 6, suppose* $t_1, \dots, t_r \in X$*. If there exist* i, j *with* $1 \le i < j \le r$ *such that*

$$t_i \cdots t_{j-1} t_j t_{j-1} \cdots t_{i+1} \in N,$$

then

$$t_1 \cdots t_r N = t_1 \cdots \hat{t}_i \cdots \hat{t}_j \cdots t_r N,$$

where the hat denotes omission.

Proof. Setting $a = t_1 \cdots t_i$, $b = t_{i+1} \cdots t_r$, $x = 1$ and $y = t_i \cdots t_{j-1} t_j t_{j-1} \cdots t_{i+1}$ in Lemma 50 gives the result. □

Theorem 52. *Let* Δ *be a simple system in a root system* Φ*. For* $\alpha, \beta \in \Delta$*, let* $m(\alpha, \beta)$ *denote the order of* $s_\alpha s_\beta$*, that is, the least positive integer* k *such that* $(s_\alpha s_\beta)^k = 1$ *holds. Then the group* $W = W(\Phi)$ *has presentation* $\langle X \mid R \rangle$*, where*

$$\begin{aligned} X &= \{t_\alpha \mid \alpha \in \Delta\} \quad \text{(a set of formal symbols)}, \\ R &= \{(t_\alpha t_\beta)^{m(\alpha,\beta)} \mid \alpha, \beta \in \Delta,\ \alpha \ne \beta\}. \end{aligned}$$

Proof. As in Definition 6, let $F(X)$ denote the free group generated by the set of involutions X. Let N be the subgroup generated by the set

$$\{c^{-1} r^{\pm 1} c \mid c \in F(X),\ r \in R\}. \tag{76}$$

We need to show that W is isomorphic to $F(X)/N$.

Clearly, there is a homomorphism from $F(X)$ to W mapping t_α to s_α for all $\alpha \in \Delta$. By Theorem 41, this homomorphism is surjective. Moreover, since the set (76) is mapped to 1 by this homomorphism, there exists a surjective homomorphism $f : F(X)/N \to W$ satisfying $f(t_\alpha N) = s_\alpha$ for all $\alpha \in \Delta$. We need to show that f is injective. This will follow if

$$t_1, \dots, t_r \in T,\ f(t_1 \cdots t_r N) = 1 \implies t_1 \cdots t_r \in N. \tag{77}$$

We prove this by induction on r. First we note that r is even. Indeed, $f(t_1 \cdots t_r N) = 1$ implies

$$s_1 \cdots s_r = 1, \tag{78}$$

where $s_i = f(t_i N) \in \{s_\alpha \mid \alpha \in \Delta\}$ is a reflection. Thus $\det s_i = -1$, so $(-1)^r = 1$. This implies that r is even. Clearly, (77) holds for $r = 0$. Also, if $r = 2$, then $s_1 s_2 = 1$. This implies $s_1 = s_2$, so $t_1 = t_2$. Thus $t_1 t_2 = 1 \in N$.

Now assume $r = 2q$, where $q \geq 2$. We first prove the special case where

$$t_1 = t_3 = \cdots = t_{2q-1},\ t_2 = t_4 = \cdots = t_{2q}. \tag{79}$$

In this case, let $t_1 = t_\alpha$ and $t_2 = t_\beta$. then (78) implies $(s_\alpha s_\beta)^q = 1$, which in turn implies $m(\alpha, \beta) \mid q$. Thus

$$t_1 \cdots t_{2q} = ((t_\alpha t_\beta)^{m(\alpha,\beta)})^{q/m(\alpha,\beta)} \in N.$$

Next we prove another special case where

$$1 \leq \exists i < \exists j \leq 2q,\ j - i < q,\ s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_{2q} = 1. \tag{80}$$

Indeed, comparing this with (78) yields

$$s_i \cdots s_j = s_{i+1} \cdots s_{j-1},$$

or equivalently,

$$f(t_i \cdots t_{j-1} t_j t_{j-1} \cdots t_{i+1} N) = 1.$$

Since $j - i < q$, we can apply the inductive hypothesis to conclude

$$t_i \cdots t_{j-1} t_j t_{j-1} \cdots t_{i+1} \in N.$$

Using Lemma 51, we obtain

$$t_1 \cdots t_{2q} N = t_1 \cdots \hat{t}_i \cdots \hat{t}_j \cdots t_{2q} N. \tag{81}$$

Together with the assumption of (77), we obtain

$$f(t_1 \cdots \hat{t}_i \cdots \hat{t}_j \cdots t_{2q} N) = 1,$$

which, by the inductive hypothesis, shows

$$t_1 \cdots \hat{t}_i \cdots \hat{t}_j \cdots t_{2q} \in N.$$

The result then follows from (81).

Before proceeding to the general case, observe

$$s_1 \cdots s_r = 1 \iff s_i \cdots s_r s_1 \cdots s_{i-1} = 1,$$
$$t_1 \cdots t_r \in N \iff t_i \cdots t_r t_1 \cdots t_{i-1} \in N.$$

Define $s_{r+i} = s_i$ for $1 \le i \le r$ and $t_{r+i} = t_i$ for $1 \le i \le r$. Then the second special case treated above actually takes care of the case:

$$1 \le \exists i < \exists j \le 4q,\ j - i < q,\ s_i \cdots s_j = s_{i+1} \cdots s_{j-1}. \tag{82}$$

Also, since the first special case has already been established, we may assume that there exists i with $1 \le i \le 2q$ such that $t_i \ne t_{i+2}$. Without loss of generality, we may assume $t_1 \ne t_3$, so

$$s_1 \ne s_3. \tag{83}$$

Since

$$s_k s_{k+1} \cdots s_{k+q} = s_{k+2q-1} s_{k+2q-2} \cdots s_{k+q+1} \quad (1 \le k \le 2q),$$

we have

$$\ell(s_k s_{k+1} \cdots s_{k+q}) \le q - 1 < q + 1.$$

Theorem 48(iii) implies that there exist i, j with $k \le i < j \le k + q$ such that

$$s_k s_{k+1} \cdots s_{k+q} = s_k \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_{k+q},$$

or equivalently,

$$s_i \cdots s_j = s_{i+1} \cdots s_{j-1}.$$

Since the second special case includes (82), we may assume $k = i$ and $j = k + q$, that is,

$$s_k s_{k+1} \cdots s_{k+q} = s_{k+1} \cdots s_{k+q-1} \quad (1 \le k \le 2q).$$

In particular, as $q \ge 2$,

$$\begin{aligned} s_1 s_2 \cdots s_{q+1} &= s_2 \cdots s_q, \\ s_2 s_3 \cdots s_{q+2} &= s_3 \cdots s_{q+1}, \\ s_3 s_4 \cdots s_{q+3} &= s_4 \cdots s_{q+2}, \end{aligned} \tag{84}$$

or equivalently,

$$s_1 s_2 \cdots s_q = s_2 \cdots s_{q+1},$$

$$s_2 s_3 \cdots s_{q+1} = s_3 \cdots s_{q+2}, \tag{85}$$

$$s_3 s_4 \cdots s_{q+2} = s_4 \cdots s_{q+3}. \tag{86}$$

By (85), we have

$$s_3 (s_2 \cdots s_{q+1})(s_{q+2} \cdots s_4) = 1. \tag{87}$$

In particular,

$$\ell(s_3(s_2 \cdots s_{q+1})) \le q - 1 < q + 1.$$

If

$$s_3(s_2 \cdots s_{q+1}) = s_2 \cdots s_q, \tag{88}$$

then (84) implies $s_1 = s_3$, contradicting (83). Thus $s_3(s_2 \cdots s_{q+1}) \neq s_2 \cdots s_q$, and hence Theorem 48(iii) implies that we are in the second special case for the relation (87), and hence

$$t_3(t_2 \cdots t_{q+1})(t_{q+2} \cdots t_4) \in N.$$

This implies

$$t_2 \cdots t_{q+1} t_{q+2} t_{q+1} \cdots t_3 \in N.$$

By Lemma 51, we obtain

$$t_1 \cdots t_{2q} N = t_1 \hat{t}_2 \cdots \hat{t}_{q+2} \cdots t_{2q} N. \tag{89}$$

Together with the assumption of (77), we obtain

$$f(t_1 \hat{t}_2 \cdots \hat{t}_{q+2} \cdots t_{2q} N) = 1,$$

which, by the inductive hypothesis, shows

$$t_1 \hat{t}_2 \cdots \hat{t}_{q+2} \cdots t_{2q} \in N.$$

The result then follows from (89). □