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Definition 53. Let G be a group acting on a set Ω. We say that G acts *simply transitively* on Ω if

(i) G acts transitively on Ω,

(ii) for every pair α, β of elements in Ω, there exists a unique element $g \in G$ such that $g.\alpha = \beta$.

Lemma 54. *Let G be a finite group acting transitively on a set Ω. Let G_α denote the stabilizer of α in G, that is,*

$$G_\alpha = \{g \in G \mid g.\alpha = \alpha\}.$$

Then the following are equivalent:

(i) *G acts simply transitively on Ω,*

(ii) *for every $\alpha \in \Omega$, $G_\alpha = \{1\}$,*

(iii) *for some $\alpha \in \Omega$, $G_\alpha = \{1\}$,*

(iv) $|G| = |\Omega|$.

Proof. (i) $\Longrightarrow$ (ii): Immediate from Definition 53(ii) by setting $\alpha = \beta$.

(ii) $\Longrightarrow$ (iii): Trivial.

(iii) $\Longrightarrow$ (iv): The mapping $\phi : G \to \Omega$ defined by $g \mapsto g.\alpha$ is a bijection. Indeed, ϕ is surjective since G is transitive. If $\phi(g) = \phi(h)$, then $g.\alpha = h.\alpha$, hence $g^{-1}h \in G_\alpha = \{1\}$. This implies $g = h$. Thus ϕ is injective.

(iv) $\Longrightarrow$ (i): Let $\alpha \in \Omega$. Then

$$\begin{aligned}
|G| &= |\Omega| \\
&= \sum_{\beta\in\Omega} 1 \\
&\leq \sum_{\beta\in\Omega} |\{g \in G \mid g.\alpha = \beta\}| \\
&= |\bigcup_{\beta\in\Omega} \{g \in G \mid g.\alpha = \beta\}| \\
&= |\{g \in G \mid g.\alpha \in \Omega\}| \\
&= |G|.
\end{aligned}$$

This forces

$$|\{g \in G \mid g.\alpha = \beta\}| = 1 \quad (\forall \beta \in \Omega).$$

Since $\alpha \in \Omega$ was arbitrary, we obtain (i). $\square$

For the remainder of today's lecture, we let Φ be a root system.

Theorem 55. *The group $W(\Phi)$ acts simply transitively on $\mathcal{P}(\Phi)$ and $\mathcal{S}(\Phi)$.*

Proof. By Theorem 36, $W(\Phi)$ acts transitively on $\mathcal{P}(\Phi)$ and $\mathcal{S}(\Phi)$. Let $w \in W(\Phi)$ and $\Pi \in \mathcal{P}(\Phi)$, and suppose $w\Pi = \Pi$. Let Δ be the unique simple system contained in Π. Then by Corollary 49 and Notation 45,

$$\begin{aligned}\ell(w) &= n(w)\\ &= |\Pi \cap w^{-1}(-\Pi)|\\ &= |\Pi \cap (-w^{-1}\Pi)|\\ &= |\Pi \cap (-\Pi)|\\ &= |\emptyset|\\ &= 0.\end{aligned}$$

Thus $w = 1$. Therefore, $W(\Phi)$ acts simply transitively on $\mathcal{P}(\Phi)$.

Next suppose $w\Delta = \Delta$. Then by Lemma 33(i), we obtain $w\Pi = \Pi$, and hence $w = 1$. Therefore, $W(\Phi)$ acts simply transitively on $\mathcal{S}(\Phi)$. □

In what follows, we fix a simple system $\Delta \in \mathcal{S}(\Phi)$. Let $\Pi = \Phi \cap \mathbf{R}_{\geq 0}\Delta$ be the unique positive system in Φ containing Δ.

Notation 56. Let $S = \{s_\alpha \mid \alpha \in \Delta\}$. For $I \subset S$, we define

$$\begin{aligned}W_I &= \langle I \rangle,\\ \Delta_I &= \{\alpha \in \Delta \mid s_\alpha \in I\},\\ V_I &= \mathbf{R}\Delta_I,\\ \Phi_I &= \Phi \cap V_I,\\ \Pi_I &= \Pi \cap V_I.\end{aligned}$$

Lemma 57. *For $w \in \langle s_\alpha \mid \alpha \in \Phi_I \rangle$, we have*

(i) $wV_I = V_I$,

(ii) $w(\Pi \setminus \Pi_I) = \Pi \setminus \Pi_I$.

Proof. It suffices to show (i) and (ii) for $w = s_\alpha$ with $\alpha \in \Phi_I$. Let $\alpha \in \Phi_I$.

(i) For $\beta \in \Delta_I \subset V_I$, $s_\alpha\beta \in \mathbf{R}\alpha + \mathbf{R}\beta \subset V_I$. Thus $s_\alpha\Delta_I \subset V_I$, and this implies $s_\alpha V_I = V_I$.

(ii) Let $\beta \in \Pi \setminus \Pi_I$. Then $\beta \notin V_I = \mathbf{R}\Delta_I$, so there exists $\gamma \in \Delta \setminus \Delta_I$ such that

$$\beta \in \mathbf{R}_{>0}\gamma + \mathbf{R}_{\geq 0}\Delta.$$

Since $\alpha \in \Phi_I \subset V_I = \mathbf{R}\Delta_I$, we have

$$\begin{aligned} s_\alpha\beta &= \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \\ &\in \mathbf{R}_{>0}\gamma + \mathbf{R}_{\geq 0}\Delta + \mathbf{R}\alpha \\ &\subset \mathbf{R}_{>0}\gamma + \mathbf{R}_{\geq 0}\Delta + \mathbf{R}\Delta_I. \end{aligned}$$

Since $\gamma \notin \Delta_I$, the coefficient of γ in the expansion of $s_\alpha\beta$ is positive. This implies $s_\alpha\beta \in \Phi \cap \mathbf{R}_{\geq 0}\Delta = \Pi$. Since $\beta \in \Pi \setminus \Pi_I$ was arbitrary, we obtain $s_\alpha(\Pi \setminus \Pi_I) \subset \Pi$. Since

$$\begin{aligned} s_\alpha(\Pi \setminus \Pi_I) \cap V_I &= s_\alpha(\Pi \setminus V_I) \cap V_I \\ &= s_\alpha(\Pi \setminus V_I) \cap s_\alpha V_I && \text{(by (i))} \\ &= s_\alpha((\Pi \setminus V_I) \cap V_I) \\ &= \emptyset, \end{aligned}$$

we have $s_\alpha(\Pi \setminus \Pi_I) \subset \Pi \setminus V_I = \Pi \setminus \Pi_I$. Since s_α is a bijection, we conclude $s_\alpha(\Pi \setminus \Pi_I) = \Pi \setminus \Pi_I$. □

Proposition 58. *Let $I \subset S$.*

(i) *Φ_I is a root system with simple system Δ_I.*

(ii) *Π_I is the unique positive system of Φ_I containing the simple system Δ_I.*

(iii) *$W(\Phi_I) = W_I$.*

(iv) *Let ℓ be the length function of W with respect to Δ. Then the restriction of ℓ to W_I coincides with the length function ℓ_I of W_I with respect to the simple system Δ_I.*

Proof. (i) For $\alpha \in \Phi_I \subset V_I$,

$$\begin{aligned} \mathbf{R}\alpha \cap \Phi_I &= (\mathbf{R}\alpha \cap \Phi) \cap V_I \\ &= \{\alpha, -\alpha\} \cap V_I \\ &= \{\alpha, -\alpha\}. \end{aligned}$$

Since

$$\begin{aligned} s_\alpha\Phi_I &= s_\alpha\Phi \cap s_\alpha V_I \\ &= \Phi \cap V_I && \text{(by Lemma 57(i))} \\ &= \Phi_I. \end{aligned}$$

we see that Φ_I is a root system. Since Δ is linearly independent, so is Δ_I. Since

$$\begin{aligned} \Phi_I &= \Phi \cap V_I \\ &\subset (\mathbf{R}_{\geq 0}\Delta \cup \mathbf{R}_{\leq 0}\Delta) \cap \mathbf{R}\Delta_I \end{aligned}$$

$$
\begin{aligned}
&= (\mathbf{R}_{\geq 0}\Delta \cap \mathbf{R}\Delta_I) \cup (\mathbf{R}_{\leq 0}\Delta \cap \mathbf{R}\Delta_I) \\
&= (\mathbf{R}_{\geq 0}\Delta_I) \cup (\mathbf{R}_{\leq 0}\Delta_I),
\end{aligned}
$$

we see that Δ_I is a simple system in Φ_I.

(ii) Since

$$
\begin{aligned}
\Pi_I &= \Pi \cap V_I \\
&= \Phi \cap \mathbf{R}_{\geq 0}\Delta \cap V_I \\
&= \Phi \cap V_I \cap \mathbf{R}_{\geq 0}\Delta \cap \mathbf{R}\Delta_I \\
&= \Phi_I \cap \mathbf{R}_{\geq 0}\Delta_I,
\end{aligned}
$$

the result follows from Lemma 29(i).

(iii)

$$
\begin{aligned}
W(\Phi_I) &= \langle s_\alpha \mid \alpha \in \Delta_I \rangle && \text{(by Theorem 41)} \\
&= \langle I \rangle \\
&= W_I.
\end{aligned}
$$

(iv) Let $w \in W_I = W(\Phi)$. Then by Lemma 57(i), we have

$$
w\Phi_I = \Phi_I. \tag{90}
$$

and by Lemma 57(ii), we have $w(\Pi \setminus \Pi_I) = \Pi \setminus \Pi_I \subset \Pi$. This implies $w(\Pi \setminus \Pi_I) \cap (-\Pi) = \emptyset$. Thus

$$
\begin{aligned}
w\Pi \cap (-\Pi) &= w(\Pi_I \cup (\Pi \setminus \Pi_I)) \cap (-\Pi) \\
&= (w\Pi_I \cup w(\Pi \setminus \Pi_I)) \cap (-\Pi) \\
&= (w(\Pi_I) \cap (-\Pi)) \cup (w(\Pi \setminus \Pi_I) \cap (-\Pi)) \\
&= w(\Pi_I) \cap (-\Pi) \\
&= w(\Pi \cap V_I) \cap (-\Pi) \\
&= w\Pi \cap wV_I \cap V_I \cap (-\Pi) \\
&= w(\Pi \cap V_I) \cap (-\Pi \cap V_I) \\
&= w(\Pi_I) \cap (-\Pi_I) && \text{(by (90)).}
\end{aligned} \tag{91}
$$

Therefore,

$$
\begin{aligned}
\ell(w) &= |\Pi \cap w^{-1}(-\Pi)| && \text{(by Corollary 49)} \\
&= |w\Pi \cap (-\Pi)| \\
&= |w(\Pi_I) \cap (-\Pi_I)| && \text{(by (91))} \\
&= |\Pi_I \cap w^{-1}(-\Pi_I)| \\
&= \ell_I(w) && \text{(by Corollary 49).}
\end{aligned}
$$

□