July 11, 2016

For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product.

Definition 1. Let Φ be a nonempty finite set of nonzero vectors in V. We say that Φ is a *root system* if

(R1) $\Phi \cap \mathbf{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$,

(R2) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

Lemma 2. *Let G be a finite group acting transitively on a set Ω. Let G_α denote the stabilizer of α in G, that is,*

$$G_\alpha = \{g \in G \mid g.\alpha = \alpha\}.$$

Then the following are equivalent:

(i) *G acts simply transitively on Ω,*

(ii) *for every $\alpha \in \Omega$, $G_\alpha = \{1\}$,*

(iii) *for some $\alpha \in \Omega$, $G_\alpha = \{1\}$,*

(iv) $|G| = |\Omega|$.

Let Φ be a root system in V, and let $W = W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$. Recall that $\mathcal{S}(\Phi)$ denotes the set of simple systems in Φ. Fix $\Delta \in \mathcal{S}(\Phi)$, and define

$$C = \{\lambda \in V \mid (\lambda, \alpha) > 0 \ (\forall \alpha \in \Delta)\},$$
$$D = \{\lambda \in V \mid (\lambda, \alpha) \geq 0 \ (\forall \alpha \in \Delta)\}.$$

Notation 3. For a subset U of V, define

$$\mathrm{Stab}_W(U) = \{w \in W \mid w\lambda = \lambda \ (\forall \lambda \in U)\}.$$

Lemma 4. (i) *If $\lambda \in D$, then*

$$\mathrm{Stab}_W(\{\lambda\}) = \langle s_\alpha \mid \alpha \in \Delta,\ s_\alpha\lambda = \lambda \rangle.$$

(ii) *If $\lambda, \mu \in D$, $w \in W$ and $w\lambda = \mu$, then $\lambda = \mu$.*

(iii) *If $\lambda \in C$, then* $\mathrm{Stab}_W(\{\lambda\}) = \{1\}$.

(iv) *If $\lambda \in V$, then*

$$\mathrm{Stab}_W(\{\lambda\}) = \langle s_\alpha \mid \alpha \in \Phi,\ s_\alpha\lambda = \lambda \rangle.$$

Theorem 5. *For each $\lambda \in V$, $|W\lambda \cap D| = 1$.*