July 25, 2016

For today's lecture, we let V be a finite-dimensional vector space over $\mathbf{R}$, with positive-definite inner product. Let Φ be a root system in V, and let $W = W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$. Fix a simple system Δ in Φ, and let Π be the unique positive system containing Δ.

Lemma 1. *If $\alpha \in \Delta$, then $s_\alpha(\Pi \setminus \{\alpha\}) = \Pi \setminus \{\alpha\}$.*

Define

$$C = \{\lambda \in V \mid (\lambda, \alpha) > 0 \ (\forall \alpha \in \Delta)\},$$
$$D = \{\lambda \in V \mid (\lambda, \alpha) \geq 0 \ (\forall \alpha \in \Delta)\}.$$

For $\alpha \in \Phi$, we define

$$H_\alpha = \{\lambda \in V \mid (\alpha, \lambda) = 0\},$$
$$H_\alpha^+ = \{\lambda \in V \mid (\alpha, \lambda) > 0\},$$
$$H_\alpha^- = \{\lambda \in V \mid (\alpha, \lambda) < 0\},$$

so that $V = H_\alpha^+ \cup H_\alpha \cup H_\alpha^-$ (disjoint). Then

$$C = \bigcap_{\alpha \in \Delta} H_\alpha^+,$$
$$D = \bigcap_{\alpha \in \Delta} (H_\alpha^+ \cup H_\alpha).$$

Lemma 2. *For $w \in W$ and $\alpha \in \Phi$,*

$$wH_\alpha = H_{w\alpha}, \tag{1}$$
$$wH_\alpha^\pm = H_{w\alpha}^\pm. \tag{2}$$

In particular,

$$s_\alpha H_\alpha^\pm = H_\alpha^\mp, \tag{3}$$
$$\bigcup_{\alpha \in \Phi} H_\alpha = w \bigcup_{\alpha \in \Phi} H_\alpha. \tag{4}$$

Definition 3. The members of the family

$$\{wC \mid w \in W\}$$

are called *chambers*.

Proposition 4. *If U is a subset of V, then*

$$\mathrm{Stab}_W(U) = \langle s_\alpha \mid \alpha \in \Phi,\ s_\alpha \in \mathrm{Stab}_W(U) \rangle.$$

Proposition 5. *Let Φ be a root system in V. Then the subgroup*

$$W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$$

of $O(V)$ is a finite reflection group. Moreover, $W(\Phi)$ is essential if and only if Φ spans V. Conversely, for every finite reflection group $W \subset O(V)$, there exists a root system $\Phi \subset V$ such that $W = W(\Phi)$.