Product Structures of Networks and Their Spectra

Mathematical System Analysis II
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July 21, 2011

0 Introduction

In the real world, there are many complex networks, such as traffic networks, social networks, biological networks, etc. Such networks are expressed in terms of graph theory. In this paper we are interested in spectral properties of graphs G and digraphs $\overrightarrow{G}$, which are expected to possess important structural information of networks. Though matrix theory helps to understand complex networks easily, it is hard to compute such matrix if the number of vertices is large. Therefore, it is desirable to develop a method for computing spectrum of a large graph G by means of its smaller components. For example, certain product structures are within our focus. This paper, we focus the Comb product of graphs and Manhattan product of digraphs.

1 Preliminaries

Definition 1.1 [Graphs]
A *graph* $G = (V, E)$ is a pair of sets, where V is a set of vertices and E is a set of unordered pairs of vertices of V. We say that v_i and v_j are adjacent and write $v_i \sim v_j$ if an edge $\{v_i, v_j\}$ belongs to E.

Definition 1.2 [Digraphs]
A *directed graph* (or *digraph*) $\vec{G} = (V, E)$ is a pair of sets, where V is a set of vertices and E is a set of ordered pairs of vertices. We say that v_i and v_j are adjacent from v_i to v_j and write $v_i \to v_j$ if an arc (v_i, v_j) belongs to E.

Definition 1.3 [Adjacency matrix of G]
The adjacency matrix $A = (a_{ij})$ of a graph G is defined by

$$a_{ij} = \begin{cases} 1, \text{ if } v_i \sim v_j \ ; \\ 0, \text{ otherwise.} \end{cases}$$

Definition 1.4 [Adjacency matrix of $\vec{G}$]
The adjacency matrix $A = (a_{ij})$ of a digraph $\vec{G}$ is defined by

$$a_{ij} = \begin{cases} 1, \text{ if } v_i \to v_j \ ; \\ 0, \text{ otherwise.} \end{cases}$$

Let A be the adjacency matrix of a graph G. We can obtain the eigenvalues of A by solving the eigenvalue problem $Ax = \lambda x$, or the characteristic equation $\det(\lambda I - A) = 0$.

Definition 1.5 [Spectrum]
Let A be the adjacency matrix of G. The spectrum of a graph G, Spec G, is the table of numbers which are eigenvalues and the multiplicities of the eigenvalues of A. If the distinct eigenvalues of A are $\lambda_1 > \lambda_2 > \ldots > \lambda_s$, and their multiplicities are $m(\lambda_1)$, $m(\lambda_2)$, $\ldots$, $m(\lambda_s)$, respectively, we write

$$\text{Spec } G = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_s \\ m(\lambda_1) & m(\lambda_2) & \ldots & m(\lambda_s) \end{pmatrix}.$$

The spectrum of a digraph $\overrightarrow{G}$, Spec $\overrightarrow{G}$, is defined in a similar way.

2 Comb Product of Graphs

In this section we discuss the comb product of graphs, which is a relatively new concept introduced in the context of quatum physics. We refer to the mathematical formulation given in [9]. We derive the spectrum of $G \triangleright P_n$, where G is an arbitrary graph and P_n is the path with n vertices.

Definition 2.1 [Comb product]
Let G_1 and G_2 be two graphs and assume that the second graph is given a distinguished vertex $o \in V(G_2)$. The *comb product graph* G is defined as a subgraph of $G_1 \times G_2$, obtained by grafting a copy of G_2 at the vertex o into each vertex of G_1. The comb product is denoted by $G = G_1 \triangleright G_2$.

Theorem 2.2 [Spectrum of $G \triangleright P_n$]
Suppose the spectrum of a graph G is given by

$$\text{Spec } G = \begin{pmatrix} \cdots & \mu_k & \cdots \\ \cdots & m_k & \cdots \end{pmatrix}.$$

Then the spectrum of $G \triangleright P_n$ is

$$\text{Spec}\,(G \triangleright P_n) = \begin{pmatrix} \cdots & \lambda_1(\mu_k) & \cdots & \lambda_n(\mu_k) & \cdots \\ \cdots & m_k & \cdots & m_k & \cdots \end{pmatrix},$$

where $\lambda_1(\mu) < \ldots < \lambda_n(\mu)$ are the solution of

$$\mu = \frac{\varphi_n(\lambda)}{\varphi_{n-1}(\lambda)},$$

where $\varphi_n(\lambda)$ is the characteristic polynomial of P_n. (In fact, essentially the Chebyshev polynomials of the second kind.)

3 Manhattan Product of Digraphs

In this section, we focus on a product of digraphs. Simply by G we denote digraph. We introduce the Manhattan product of digraphs derive the spectrum of $C_2 \sharp P_n$, where C_2 is a directed cycle with two vertices.

Definition 3.1 Given a digraph $G = (V, E)$, its converse digraph $G^\vee = (V, E^\vee)$ is obtained from G by reversing all the orientations of the arcs in E; that is $(v_i, v_j) \in E^\vee$ if and only if $(v_j, v_i) \in E$.

Definition 3.2 [Manhattan product]
Let $G_i = (V_i, E_i)$ be bipartite digraphs with independent sets $V_i = V_{i0} \cup V_{i1}$, $N_i = |V_i|$, $i = 1, 2, \dots, n$. Let π be the characteristic function of $V_{i1} \subset V_i$ for any i; that is,

$$\pi(u) = \begin{cases} 0, \text{ if } u \in V_{i0}, \\ 1, \text{ if } u \in V_{i1}. \end{cases}$$

Then, the *Manhattan product* $M_n = G_1 \sharp G_2 \sharp \cdots \sharp G_n$ is the digraph with vertex set $V(M_n) = V_1 \times V_2 \times \cdots \times V_n$, and each vertex $(u_1, \dots, u_i, \dots, u_n)$ is adjacent to vertices $(u_1, \dots, v_i, \dots, u_n)$, $1 \le i \le n$, when

(1) $v_i \in \Gamma^+(u_i)$ if $\sum_{j \neq i} \pi(u_j)$ is even,

(2) $v_i \in \Gamma^-(u_i)$ if $\sum_{j \neq i} \pi(u_j)$ is odd.

Where $\Gamma^+(u_i)$ be the set of vertices which are adjacent form i, and $\Gamma^-(u_i)$ be the set of vertices which are adjacent to i.

Lemma 3.3 *Let $\varphi_n(\lambda)$ be the characteristic polynomial of the $G = C_2 \sharp P_n$. Then it holds that*

$$\varphi_0(\lambda) = 1\,, \ \ \varphi_1(\lambda) = \lambda^2 - 1\,, \ \ \varphi_2(\lambda) = \lambda^4 - 2\lambda^2$$
$$\varphi_n(\lambda) = \lambda^2(\varphi_{n-1}(\lambda) - \varphi_{n-2}(\lambda)), \ \ n \ge 2$$

Theorem 3.4 *Let $\tilde{U_n}(\lambda) = U_n(\lambda/2)$, where $U_n(x)$ is the Chebyshev polynomial of the second kind. Then the characteristic polynomial $\varphi_n(\lambda)$ hold that*

$$\varphi_n(\lambda) = \lambda^{n-1} \tilde{U}_{n+1}(\lambda), \ \ n \ge 1.$$

Theorem 3.5 [Sepctrum of $C_2 \sharp P_n$]
The spectrum of $C_2 \sharp P_n$ is given by

$$\text{Spec } C_2 \sharp P_n = \begin{pmatrix} 0 & 2\cos\frac{k\pi}{n+2} \\ n-1 & 1 \end{pmatrix},$$
$$k = 1, 2, \dots, n+1.$$

Theorem 3.6 *The asymptotic spectral distribution of $C_2 \sharp P_n$ as $n \to \infty$ is given by*

$$\frac{1}{2}\delta_0 + \frac{1}{2}\rho(x)dx,$$

where

$$\rho(x) = \begin{cases} \dfrac{1}{\pi\sqrt{4 - x^2}}, & -2 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

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