Grover walk on finite graphs with infinite tails

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1 Introduction

Quantum walks are the quantum version of classical random walks. One of the widely known quantum walks is the Grover walk. In our study, we analyse the Grover walk on a finite graph in which semi-infinite length paths are connected. We derive an explicit expression of the stationary state on a distance regular graph. Moreover, we change the time evolution operator in order to develop a model for a quantum search.

2 Definition of Grover walks

Let $G = (V, E)$ be a connected graph. Denote by A, the arc set induced by E. Then the state space of the walker is $l^2(A)$. If the time evolution operator of the walker is U, for a given initial state $\Psi_0 \in l^2(A)$, the state of the walker at t time step is

$$\Psi_t = U\Psi_{t-1}, t \in \mathbb{Z}_{>0},$$

where $\mathbb{Z}_{>0}$ is the set of positive integers.

Definition 1 The Grover dynamics on A is defined by for all $a, b \in A$,

$$U_{ab} = \begin{cases} \dfrac{2}{\deg(t(b))} - \delta_{a\bar{b}} & \text{if } o(a) = t(b), \\ 0 & \text{otherwise.} \end{cases}$$

where $U_{ab} = \langle a| U |b\rangle$. Here $o(a)$ and $t(a)$ are origin and the terminal vertices respectively. Then the state of the walker at time step t is given by

$$\Psi_t(a) = \sum_{b:t(b)=o(a)} \left(\frac{2}{\deg(t(b))} - \delta_{a\bar{b}}\right)\Psi_{t-1}(b), a \in A.$$

3 Tailed Grover walks

Let $G_0 = (V_0, E_0)$ be a finite connected graph. The arc set induced by E_0 is denoted by A_0. Let $\delta V \subset V_0$ be a non-empty subset and put $\delta V = \{u_1, u_2, ..., u_r\}$. Let $\{\mathbb{P}_j : j = 1, ..., r\}$ be the set of semi-infinite length tails connected to the finite graph G_0 such that $V(\mathbb{P}_j) = \left\{u_j = v_j^{(0)} \sim v_j^{(1)} \sim v_j^{(2)} \sim ...\right\}$. Now denote the graph constructed by $\tilde{G} = (\tilde{V}, \tilde{E})$ and the arc set induced by $\tilde{E}$ by $\tilde{A}$. We consider the Grover walk on $\tilde{G}$ whose time evolution is U and the initial state is defined as follows.

$$\Psi_0(a) = \begin{cases} \gamma_s z^j & \substack{\text{if } o(a)=v_s^{(j+1)}, t(a)=v_s^{(j)}, s=1,...,r; \\ j=0,1,2,...}, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Here $z \in \mathbb{C}$ with $|z| = 1$. Then the state of the walker at time step t is

$$\Psi_t = U^t\Psi_0.$$

Then we have the following theorem.

Theorem 1 $\lim_{t\to\infty} z^t\Psi_t$ *exists*

Theorem 2 *Let* $\Psi_\infty := \lim_{t\to\infty} z^t\Psi_t$ *and let* $\alpha_j := \Psi_\infty(a)$ *where* $o(a) \in V(\mathbb{P}_j) \setminus V_0$ *and* $t(a) \in V_0$*. Then for any* $a \in \tilde{A}$*,*

$$\Psi_\infty(a) + \Psi_\infty(\overline{a}) = \frac{2}{r}\sum_{j=1}^{r} \alpha_j.$$

4 Main results

4.1 Explicit expression of stationary state

Let $G_0 = (V_0, E_0)$ be a distance regular graph with the diameter D and the valancy r. Choose a vertex $u_+ \in V_0$. Define the equitable partitions Γ_j as follows.

$$\Gamma_j = \{v \in V_0 : \mathrm{dist}(v, u_+) = j\}, j = 0, ..., D.$$

Now choose a sequence $0 = j_0 < j_1 < ... < j_m \le D$. Now set $\delta V = \bigcup_{k=0}^{m} \Gamma_{j_k}$. Let The tail connected to u_+ be $\mathbb{P}_+$ such that $V(\mathbb{P}_+) = \left\{u_+ = v_+^{(0)} \sim v_+^{(1)} \sim v_+^{(2)} \sim ...\right\}$. Define the initial state Ψ_0 as follows.

Definition 2

$$\Psi_0(a) = \begin{cases} 1 & \text{if } o(a) = v_+^{(j+1)}, t(a) = v_+^{(j)}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the explicit expression of Ψ_∞ is given by the following theorem.

Theorem 3 *For all* $a \in A_0$*,*

$$\Psi_\infty(a) = \begin{cases} \dfrac{1}{|\delta V|}\left\{1 + \dfrac{\sum_{k=j+1}^{D} |\Gamma_k \cap \delta V|}{b_j|\Gamma_j|}\right\} & \substack{\text{if } o(a)\in\Gamma_j, \\ t(b)\in\Gamma_{j+1}}, \\ \dfrac{1}{|\delta V|}\left\{1 - \dfrac{\sum_{k=j+1}^{D} |\Gamma_k \cap \delta V|}{b_j|\Gamma_j|}\right\} & \substack{\text{if } o(a)\in\Gamma_{j+1}, \\ t(b)\in\Gamma_j}, \\ \dfrac{1}{|\delta V|} & \text{if } o(a), t(b) \in \Gamma_j. \end{cases}$$

Where $b_j = |\{w \in \Gamma_{j+1} : w \sim v\}|$ for any given $j \in \mathbb{Z}$ such that $0 \le j \le D$ and $v \in \Gamma_j$. Remark that $\sum_{k=j+1}^{D} |\Gamma_k \cap \delta V|$ is the number of tails on $\bigcup_{k=1}^{D} \Gamma_k$ and $b_j|\Gamma_j|$ is the number of edges from Γ_j to Γ_{j+1}. From Theorem 3 we compute

$$f(x) = \sum_{a:t(a)\in\Gamma_x} |\Psi_\infty(a)|^2$$

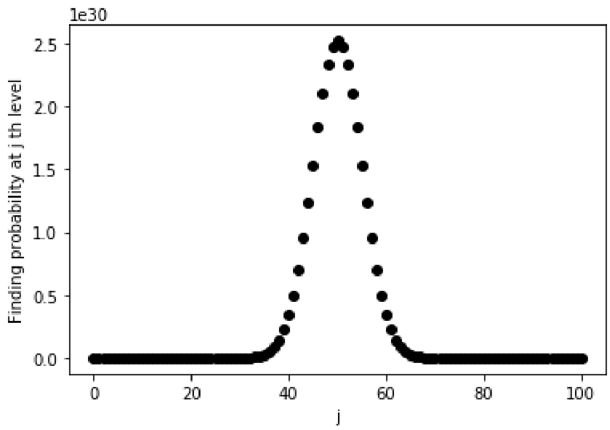
for some special cases where $a \in A_0$. Observe that the finding probability of the walker at Γ_x is,

$$\mu_\infty(x) = \frac{f(x)}{\sum_{y=0}^{D} f(y)}.$$

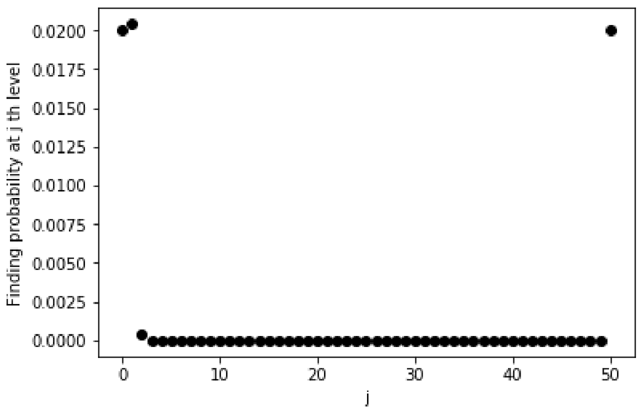
Example 1 Consider the hypercube $H(d,2)$ with $\delta V = \Gamma_0 \cup \Gamma_d$. Then $f(x)$ is given by

$$f(x) = \begin{cases} \frac{1}{4}\left(1 - \frac{1}{d}\right)^2 & \text{if } x = 0, \\ \frac{1}{4}\left(d\binom{d}{x} + \frac{1}{\binom{d}{x}}\left(\frac{1}{x} + \frac{1}{d-x}\right)\right) & \text{if } 0 < x < d, \\ \frac{1}{4}\left(1 + \frac{1}{d}\right)^2 & \text{if } x = d. \end{cases}$$

In particular, for $d = 100$, the graph of f is as follows,



Example 2 Consider the hypercube $H(d,2)$ with $\delta V = \bigcup_{j=0}^{d} \Gamma_j$. Then for $d = 50$, the graph of f is as follows.



4.2 Quantum search derived by the tailed model

Let $G_0 = (V_0, E_0)$ be a finite connected graph with the vertex set V_0 and the edge set E_0. Let A_0 be the arc set induced by E_0. Now choose $\delta V = V_0$. We choose a set of defective vertices $\emptyset \ne M \subset V_0$. Now define the time evolution operator U on A_0 as follows.

Definition 3

$$U_{ab} = \begin{cases} (-1)^{\mathbb{1}_M(t(b))}\left(\frac{2}{\deg(t(b))} - \delta_{a\bar{b}}\right) & \text{if } o(a) = t(b), \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbb{1}_M$ is the characteristic function of M. Then we have the tails $\mathbb{P}_1, ..., \mathbb{P}_n$ where $n = |V_0|$ and we choose the initial state of the walker as,

$$\Psi_0(a) = \begin{cases} 1 & \text{if } o(a) = v_s^{(j+1)}, t(b) = v_s^{(j)}, s = 1, ..., n, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following lemma.

Lemma 1 *For a fixed $u \in \tilde{G}$, for any $a \in \tilde{A}$ such that $o(a) = u$,*
$\psi_\infty(a) + \psi_\infty(\bar{a})$ is a constant if $u \notin M$.
$\psi_\infty(a) - \psi_\infty(\bar{a})$ is a constant if $u \in M$.

In this model we observe that the quantum search works well for the complete graph.

Example 3 Let G_0 be the complete graph K_n. Choose $M = \{u\}$ where $u \in V(K_n)$. Then, in the long run, the finding probability of the walker at the vertex v is given by $\mu_\infty(\{u\}) = \mu_\infty(V_0 \setminus \{u\}) = \frac{1}{2}$.

5 Conclusion and future work

In general, the Grover walk on a finite graph does not necessarily converge. We have modified our walk so that we obtain the stationarity. We have computed an explicit expression of the stationary state on distance regular graphs and we derive the finding probabilities of the walker. Moreover, in our study, we are interested in obtaining higher finding probabilities by modifying the walk. We have introduce a new defective tailed model and we have demonstrated the quantum search on complete graphs.

In future, we would like to modify our model to get higher finding probabilities and we would like to study the spectrum to compute the convergence speed of our model.

References

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