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Unitarity Criterion in White Noise Calculus and Nonexistence of Unitary Evolutions Driven by Higher Powers of Quantum White Noises

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Abstract

A normal-ordered white noise differential equation generalizes a quantum stochastic differential equation so as to include in coefficients highly singular noises such as higher powers of quantum white noises. Unique existence of a solution is discussed on the basis of white noise operator theory and its unitarity condition is derived by means of symbol calculus with complex Gaussian integral. It is proved that higher powers of quantum white noises produce no new class of unitary evolutions in the original Fock space where quantum white noises are represented.

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1 Introduction

The white noise theory, initiated by Hida [10], is based on a Gelfand triple called a white noise triple:

$$\mathcal{W} \subset \Gamma(L^2(\mathbf{R})) \cong L^2(E^*, \mu) \subset \mathcal{W}^*, \tag{1}$$

where $\Gamma(L^2(\mathbf{R}))$ is the Boson Fock space over $L^2(\mathbf{R})$ and $L^2(E^*, \mu)$ is its functional realization by means of the Wiener–Itô–Segal isomorphism. Construction of a white

This is the final form of the paper.

noise triple (1) has been extensively discussed, see e.g., [8], [15], [16], [17], [18]. A continuous operator Ξ from $\mathcal{W}$ into $\mathcal{W}^*$ is called a *white noise operator* and the space of such operators is denoted by $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. The white noise operator theory has been established in [4], [18].

A continuous map $t \mapsto L_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ defined on an interval is called a *quantum stochastic process* in the sense of white noise theory. Given a quantum stochastic process $\{L_t\}$, an ordinary differential equation:

$$\frac{dX}{dt} = L_t \diamond X, \qquad X(0) = I, \tag{2}$$

where $\diamond$ stands for the Wick product (also called the normal-ordered product) in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, is generally called a *normal-ordered white noise differential equation* and is our object. Unique existence of a solution in the sense of white noise operators has been studied in the series of papers [5], [20], [21], and its regularity properties in [6], [7], where proper Hilbert spaces in which the solution acts (as unbounded operators in general) are specified.

The normal-ordered white noise differential equation (2) was first brought against quantum stochastic calculus. In the famous paper [12] Hudson and Parthasarathy introduced a quantum stochastic differential equation of Itô type:

$$dX = (L_1 dA_t + L_2 dA_t^* + L_3 d\Lambda_t + L_4 dt)\, X, \qquad X(0) = I, \tag{3}$$

where $\{A_t\}$ is the annihilation process, $\{A_t^*\}$ the creation process, and $\{\Lambda_t\}$ the number process acting in the Fock space $\Gamma(L^2(\mathbf{R}))$. The quantum Itô formula was one of the most important achievements and was used effectively to derive unitarity condition for the solution. In white noise operator theory the three basic processes are all differentiable. In fact,

$$a_t = \frac{dA_t}{dt}, \qquad a_t^* = \frac{dA_t^*}{dt}, \qquad a_t^* a_t = \frac{d\Lambda_t}{dt},$$

hold in the spaces $\mathcal{L}(\mathcal{W}, \mathcal{W})$, $\mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$, and $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, respectively. The operators a_t and a_t^* are respectively called the *annihilation operator* and the *creation operator* at a time point $t \in \mathbf{R}$. It then turns out that (3) is brought into a normal-ordered white noise differential equation (2) with the coefficient given by

$$L_t = L_1 a_t + L_2 a_t^* + L_3 a_t^* a_t + L_4,$$

see [20], [21]. From this observation we are convinced of importance of the white noise approach; for example, higher powers of quantum white noises, which are far beyond the traditional Itô theory, have become within our reach.

In this paper we focus on a normal-ordered white noise differential equation with a particular coefficient of the form:

$$L_t = \sum_{l=0}^{k} \sum_{m=0}^{n} L_{l,m} a_t^{*l} a_t^m.$$

It is proved that the solution acts in the original Fock space $\Gamma(L^2(\mathbf{R}))$ (as unbounded operators in general) only in the case of $0 \le k \le 1$. For a general coefficient, we need another Fock space singular to the original one or a weighted Fock space which is somehow outside the Gaussian analysis. Thus, as for the unitarity, it seems natural to start with the case of

$$L_t = \sum_{m=0}^{n} L_{0,m} a_t^m + \sum_{m=0}^{n} L_{1,m} a_t^* a_t^m.$$

We employ a new approach based on the complex white noise theory recently developed in [23], [24], [22]. By using the unitarity condition in terms of complex Gaussian integral and the operator symbol, we shall show that a unitary solution is obtained only when $0 \le n \le 1$, i.e., when the normal-ordered white noise differential equation is reduced to a quantum stochastic differential equation of Hudson–Parthasarathy type. As a result, the unitarity criterion of Hudson and Parthasarathy [12] is reproduced without using the quantum Itô formula. This is a by-product of our approach though the use of the quantum Itô formula is much simpler.

In conclusion, to obtain a unitary evolution driven by higher powers of quantum white noises one needs a change of inner product of the original Fock space where the quantum white noises act. Relevant questions in higher powers, in particular, in the quadratic powers of quantum white noises have been discussed from a different point of view, see e.g., [1], [2], [3], [25].

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2 White Noise Triple

Up to now several different spaces of white noise distributions have been introduced keeping a common spirit in the characterization theorems of S-transform and of operator symbols. Here we adopt the CKS-space [8] for explicit computation. As usual we start with the real Gelfand triple:

$$E = \mathcal{S}(\mathbf{R}) \subset H = L^2(\mathbf{R}, dt) \subset E^* = \mathcal{S}'(\mathbf{R}), \tag{4}$$

where $\mathcal{S}(\mathbf{R})$ is the space of rapidly decreasing functions, and $\mathcal{S}'(\mathbf{R})$ the space of tempered distributions. The canonical bilinear form on $E^* \times E$ and the inner product of H are denoted by the common symbol $\langle \cdot, \cdot \rangle$ since they are compatible. The norm of H is denoted by $|\cdot|_0$. We shall introduce the canonical topology of $E = \mathcal{S}(\mathbf{R})$ by means of Hilbertian norms. For $p \in \mathbf{R}$ we put

$$|\xi|_p = |A^p \xi|_0, \qquad \xi \in H, \quad A = 1 + t^2 - \frac{d^2}{dt^2}.$$

Then, for $p \geq 0$ the set

$$E_p = \{\xi \in H; |\xi|_p < \infty\} \tag{5}$$

becomes a Hilbert space with norm $|\cdot|_p$. While, E_{-p} denotes the completion of H with respect to the norm $|\cdot|_{-p}$. Note that E_p and E_{-p} are dual each other. With these notations we have

$$E = \mathcal{S}(\mathbf{R}) = \operatorname*{proj\,lim}_{p\to\infty} E_p, \qquad E^* = \mathcal{S}'(\mathbf{R}) = \operatorname*{ind\,lim}_{p\to\infty} E_{-p}. \tag{6}$$

For $n \geq 0$ let $H^{\widehat{\otimes} n}$ be the n-fold symmetric tensor power of H and their norms are denoted by the common symbol $|\cdot|_0$. For a sequence $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ of positive numbers we put

$$\Gamma_\alpha(H) = \left\{ \phi \sim (f_n)_{n=0}^{\infty}; f_n \in H^{\widehat{\otimes} n}, \| \phi \|_{0,+}^2 \equiv \sum_{n=0}^{\infty} n!\, \alpha(n) \, |f_n|_0^2 < \infty \right\}.$$

Then $\Gamma_\alpha(H)$ becomes a Hilbert space and is called a *weighted Fock space*. The (Boson) Fock space is the special case of $\alpha(n) \equiv 1$ and is denoted by $\Gamma(H)$. From now on we assume that $\alpha = \{\alpha(n)\}$ fulfills the following five conditions:

(A1) $1 = \alpha(0) \leq \alpha(1) \leq \alpha(2) \leq \cdots$;

(A2) the generating function

$$G_\alpha(t) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} t^n$$

has an infinite radius of convergence;

(A3) the power series

$$\widetilde{G}_\alpha(t) = \sum_{n=0}^{\infty} \frac{n^{2n}}{n!\alpha(n)} \left\{ \inf_{s>0} \frac{G_\alpha(s)}{s^n} \right\} t^n$$

has a positive radius of convergence $R_\alpha > 0$;

(A4) there exists a constant $C_{1\alpha} > 0$ such that $\alpha(m)\alpha(n) \leq C_{1\alpha}^{m+n} \alpha(m+n)$ for all m, n;

(A5) there exists a constant $C_{2\alpha} > 0$ such that $\alpha(m+n) \leq C_{2\alpha}^{m+n}\alpha(m)\alpha(n)$ for all m, n.

Now we construct the space of white noise distributions. For the Hilbert space E_p defined in (5) consider the weighted Fock space $\Gamma_\alpha(E_p)$, and according to (6) define

$$\Gamma_\alpha(E) = \operatorname{proj}\lim_{p\to\infty} \Gamma_\alpha(E_p).$$

It is easily shown that $\Gamma_\alpha(E)$ is a nuclear space whose topology is given by the family of norms:

$$\| \phi \|_{p,+}^2 = \sum_{n=0}^{\infty} n!\, \alpha(n) \, | f_n |_p^2 , \qquad \phi \sim (f_n), \quad p \geq 0.$$

By a standard argument we see that

$$\Gamma_\alpha(E)^* \cong \operatorname{ind}\lim_{p\to\infty} \Gamma_{\alpha^{-1}}(E_{-p}),$$

where $\Gamma_\alpha(E)^*$ carries the strong dual topology and $\cong$ means a topological isomorphism. Then, by taking the complexification, we obtain a complex Gelfand triple:

$$\mathcal{W}_\alpha \equiv \Gamma_\alpha(E_{\mathbf{C}}) \subset \Gamma(H_{\mathbf{C}}) \subset \Gamma_\alpha(E_{\mathbf{C}})^* \equiv \mathcal{W}_\alpha^*, \tag{7}$$

where the middle space is the usual Fock space over $H_{\mathbf{C}}$. The above Gelfand triple is referred to as the *Cochran–Kuo–Sengupta space* (or *CKS-space* for short) with weight sequence $\alpha = \{\alpha(n)\}$. When there is no danger of confusion, we write $\mathcal{W} = \mathcal{W}_\alpha$ for simplicity. The canonical bilinear form on $\mathcal{W}^* \times \mathcal{W}$ is denoted by $\langle\!\langle \cdot , \cdot \rangle\!\rangle$. Then

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \, \langle F_n, f_n \rangle , \qquad \Phi \sim (F_n) \in \mathcal{W}^*, \quad \phi \sim (f_n) \in \mathcal{W},$$

and it holds that

$$| \langle\!\langle \Phi, \phi \rangle\!\rangle | \leq \| \Phi \|_{-p,-} \, \| \phi \|_{p,+} ,$$

where

$$\| \Phi \|_{-p,-}^2 = \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} \, | F_n |_{-p}^2 , \qquad \Phi \sim (F_n).$$

For $0 \leq \beta < 1$ we put $\widetilde{\beta}(n) = (n!)^\beta$. The corresponding CKS-space is called the *Kondratiev–Streit space* [15] and is denoted by $(E)_\beta$. In particular, in case of $\beta = 0$, i.e., $\alpha(n) \equiv 1$, the corresponding CKS-space is called the *Hida–Kubo–Takenaka space* [16] and is denoted by (E) for simplicity. The k-th order Bell numbers $\{B_k(n)\}$ also provide an important class of white noise distributions [8], see also [7] for applications.

3 White Noise Operators

A continuous operator $\Xi : \mathcal{W} \to \mathcal{W}^*$ is called a *white noise operator* and the space of such operators is denoted by $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. Note that $\mathcal{L}(\mathcal{W}, \mathcal{W})$ and $\mathcal{L}(\Gamma(H_{\mathbf{C}}), \Gamma(H_{\mathbf{C}}))$ are subspaces of $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. Moreover, $\mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$ is isomorphic to $\mathcal{L}(\mathcal{W}, \mathcal{W})$ by duality. These spaces are equipped with the topology of uniform convergence on every bounded set. General theory for white noise operators has been extensively developed in [4], [18].

Let a_t and a_t^* be the annihilation and creation operators at a point $t \in \mathbf{R}$. Then, for $\phi \in \mathcal{W}$ we have

$$a_t\phi(x) = \lim_{\theta \to 0} \frac{\phi(x + \theta\delta_t) - \phi(x)}{\theta},$$

where the limit always exists for all $t \in \mathbf{R}$ and $x \in E^*$. It is known that $a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ and $a_t^* \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$. Moreover, the maps $t \mapsto a_t$ and $t \mapsto a_t^*$ are both infinitely many times differentiable. The pair $\{a_t, a_t^*\}$ is referred to as the *quantum white noise process*. The annihilation, creation and number processes are respectively defined by

$$A_t = \int_0^t a_s\, ds, \qquad A_t^* = \int_0^t a_s^*\, ds, \qquad \Lambda_t = \int_0^t a_s^* a_s\, ds, \tag{8}$$

where the integrals are understood as integral kernel operators, see below. These are white noise operators and

$$\frac{d}{dt} A_t = a_t, \qquad \frac{d}{dt} A_t^* = a_t^*, \qquad \frac{d}{dt} \Lambda_t = a_t^* a_t,$$

hold in $\mathcal{L}(\mathcal{W}, \mathcal{W})$, in $\mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$, and in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, respectively. The following result illustrates a precise norm estimate which follows by combination of [4, Lemma 2.1] and a norm estimate of

$$\frac{d}{dt} 1_{[0,t]} = \delta_t.$$

Proposition 1 *For any $p \geq 0$ and $q > 0$ with $p + q > 11/12$ there exists a constant $C > 0$ depending only on $p + q$ such that*

$$\left\| \left(\frac{A_{t+h} - A_t}{h} - a_t \right) \phi \right\|_{p,+} \leq C \left(\frac{2^q}{2qe \log 2} \right)^{1/2} |h| \, \| \phi \|_{p+q,+},$$

for all $\phi \in \mathcal{W}$ and $t, h \in \mathbf{R}$.

Since the composition $a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m}$ is well-defined as an operator in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, it is quite natural to consider an operator of the form:

$$\int_{\mathbf{R}^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m.$$

In fact, extending the Lebesgue integral by duality, we may define an *integral kernel operator* with an arbitrary distribution kernel $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ and denote it by $\Xi_{l,m}(\kappa)$. More precisely, for $\phi \sim (f_n) \in \mathcal{W}$ we define $\Xi_{l,m}(\kappa)\phi \sim (g_n)$ by

$$g_n = 0, \quad 0 \le n < l; \qquad g_n = \frac{(n-l+m)!}{(n-l)!}\,\kappa \otimes_m f_{n-l+m}, \quad n \ge l,$$

where $\otimes_m$ denotes the right contraction of tensor products. Then, by a direct verification we see that $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ and the map $\kappa \mapsto \Xi_{l,m}(\kappa)$ is continuous, for more details see [18, Chapter 4]. It is noted that every white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is decomposed into an infinite series:

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}),$$

where the right hand side converges in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. In that case we put

$$\deg \Xi = \max\{l+m\,;\, \Xi_{l,m}(\kappa_{l,m}) \neq 0\}.$$

The white noise operators with $\deg \Xi < \infty$, i.e., which are finite sums of integral kernel operators belong to $\mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ for any choice of α.

An *exponential vector* or a *coherent vector* is defined by

$$\phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \cdots, \frac{\xi^{\otimes n}}{n!}, \cdots\right), \qquad \xi \in E_{\mathbf{C}}. \tag{9}$$

Because the exponential vectors $\{\phi_\xi\,;\, \xi \in E_{\mathbf{C}}\}$ span a dense subspace of $\mathcal{W}$, every white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is uniquely specified by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in \mathcal{W}.$$

The above is called the *symbol* of Ξ. A remarkable outcome of white noise theory is found in the following

Theorem 2 [5] (Characterization for operator symbols) *A* $\mathbf{C}$*-valued function* Θ *on* $E_{\mathbf{C}} \times E_{\mathbf{C}}$ *is the symbol of an operator* $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ *if and only if*

(O1) *for any* $\xi, \xi_1, \eta, \eta_1 \in E_{\mathbf{C}}$, *the function* $(z, w) \mapsto \Theta(z\xi + \xi_1, w\eta + \eta_1)$ *is entire holomorphic on* $\mathbf{C} \times \mathbf{C}$;

(O2) *there exist constant numbers* $C \ge 0$ *and* $p \ge 0$ *such that*

$$|\Theta(\xi, \eta)|^2 \le C G_\alpha(|\xi|_p^2) G_\alpha(|\eta|_p^2), \qquad \xi, \eta \in E_{\mathbf{C}}.$$

Several refinements have been so far obtained. Let $K^{\pm}$ be Hilbert spaces interpolating the Gelfand triple (4) in such a way that

$$E \subset K^{+} \subset H \subset K^{-} \subset E^{*}, \tag{10}$$

where all the embeddings are continuous and have dense images, $K^{+} \to H$ is a contraction, and $K^{\pm}$ are dual each other. With this fine structure we have

Theorem 3 [7] *Let $\beta = \{\beta(n)\}$ be another weight sequence satisfying conditions (A1)–(A3). Let $\Theta : E_{\mathbf{C}} \times E_{\mathbf{C}} rightarrow \mathbf{C}$ be a function satisfying condition (O1). Assume that there exist $C \geq 0$, $p \geq 0$ and a bounded, non-negative sesquilinear form Q on $K_{\mathbf{C}}^{+}$ with $\operatorname{Tr} Q < R_{\beta}$ such that*

$$|\Theta(\xi,\eta)|^2 \leq C G_{\alpha}(|\xi|_p^2) G_{\beta}(Q(\eta,\eta)), \qquad \xi, \eta \in E_{\mathbf{C}}.$$

Then there exists a unique $\Xi \in \mathcal{L}(\mathcal{W}_{\alpha}, \Gamma_{\beta^{-1}}(K_{\mathbf{C}}^{-}))$ such that $\Theta = \widehat{\Xi}$.

4 Analytic Definition of Wick Product

We start with some elementary properties of the generating function $G_{\alpha}(t)$ defined in (A2).

Lemma 4 *Let $\alpha = \{\alpha(n)\}$ be a positive sequence as before and $G_{\alpha}(t)$ the generating function. Then,*

(1) $G_{\alpha}(0) = 1$ *and* $G_{\alpha}(s) \leq G_{\alpha}(t)$ *for* $0 \leq s \leq t$;

(2) $e^s G_{\alpha}(t) \leq G_{\alpha}(s+t)$ *and* $e^t \leq G_{\alpha}(t)$ *for* $s, t \geq 0$;

(3) $\gamma[G_{\alpha}(t) - 1] \leq G_{\alpha}(\gamma t) - 1$ *for any* $\gamma \geq 1$ *and* $t \geq 0$.

(4) $G_{\alpha}(s) G_{\alpha}(t) \leq G_{\alpha}(C_{1\alpha}(s+t))$ *for* $s, t \geq 0$.

(5) $G_{\alpha}(s+t) \leq G_{\alpha}(C_{2\alpha} s) G_{\alpha}(C_{2\alpha} t)$ *for* $s, t \geq 0$.

Lemma 5 *For two white noise operators $\Xi_1, \Xi_2 \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ there exists a unique operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ such that*

$$\widehat{\Xi}(\xi,\eta) = \widehat{\Xi}_1(\xi,\eta)\widehat{\Xi}_2(\xi,\eta) e^{-\langle \xi, \eta \rangle}, \qquad \xi, \eta \in E_{\mathbf{C}}. \tag{11}$$

Proof. For simplicity we write $\Theta(\xi,\eta)$ for the right hand side of (11). By virtue of Theorem 2 we need only to show that Θ satisfies conditions (O1) and (O2). In fact, (O1) is obvious. As for (O2), we take $C_j \geq 0$ and $p_j \geq 0$ in such a way that

$$|\widehat{\Xi}_j(\xi,\eta)|^2 \leq C_j G_{\alpha}(|\,\xi\,|_{p_j}^2) G_{\alpha}(|\,\eta\,|_{p_j}^2), \qquad j = 1, 2.$$

Since $|\xi|_p \le \rho^q |\xi|_{p+q} \le |\xi|_{p+q}$, $\rho = \|A^{-1}\|_{\mathrm{OP}} = 1/2$, we have

$$|\widehat{\Xi}_1(\xi,\eta)\widehat{\Xi}_2(\xi,\eta)|^2 \le CG_\alpha^2(|\xi|_p^2)G_\alpha^2(|\eta|_p^2),$$

where $C = C_1C_2$ and $p = \max\{p_1, p_2\}$. Moreover, combining an obvious inequality:

$$|e^{-\langle\xi,\eta\rangle}|^2 \le e^{2|\xi|_0|\eta|_0} \le e^{|\xi|_0^2+|\eta|_0^2} \le e^{|\xi|_p^2+|\eta|_p^2},$$

we obtain

$$|\Theta(\xi,\eta)|^2 \le Ce^{|\xi|_p^2}G_\alpha^2(|\xi|_p^2)e^{|\eta|_p^2}G_\alpha^2(|\eta|_p^2).$$

By using Lemma 4 the above becomes

$$|\Theta(\xi,\eta)|^2 \le CG_\alpha((2C_{1\alpha}+1)|\xi|_p^2)G_\alpha((2C_{1\alpha}+1)|\eta|_p^2). \tag{12}$$

Choose $q \ge 0$ with $(2C_{1\alpha}+1)\rho^{2q} \le 1$ so that (12) becomes

$$|\Theta(\xi,\eta)|^2 \le CG_\alpha(|\xi|_{p+q}^2)G_\alpha(|\eta|_{p+q}^2),$$

which proves (O2). ∎

The operator Ξ defined in (11) is called the *Wick product* or *normal-ordered product* of Ξ_1 and Ξ_2, and is denoted by $\Xi = \Xi_1 \diamond \Xi_2$. We note some simple properties:

$$I \diamond \Xi = \Xi \diamond I = \Xi, \qquad (\Xi_1 \diamond \Xi_2) \diamond \Xi_3 = \Xi_1 \diamond (\Xi_2 \diamond \Xi_3),$$

$$(\Xi_1 \diamond \Xi_2)^* = \Xi_2^* \diamond \Xi_1^*, \qquad \Xi_1 \diamond \Xi_2 = \Xi_2 \diamond \Xi_1.$$

Thus, equipped with the Wick product, $\mathcal{L}(\mathcal{W},\mathcal{W}^*)$ becomes a commutative $*$-algebra. As for the annihilation and creation operators we have

$$a_s \diamond a_t = a_s a_t, \quad a_s^* \diamond a_t = a_s^* a_t, \quad a_s \diamond a_t^* = a_t^* a_s, \quad a_s^* \diamond a_t^* = a_s^* a_t^*. \tag{13}$$

More generally, it follows by direct verification that

$$a_{s_1}^* \cdots a_{s_l}^* \Xi a_{t_1} \cdots a_{t_m} = (a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m}) \diamond \Xi, \quad \Xi \in \mathcal{L}(\mathcal{W},\mathcal{W}^*). \tag{14}$$

Note that the left hand side is well-defined as the composition of white noise operators.

5 Normal-Ordered White Noise Differential Equations

A continuous map $t \mapsto L_t \in \mathcal{L}(\mathcal{W},\mathcal{W}^*)$ defined on a time interval is called a *quantum stochastic process* in the sense of white noise theory [19]. Then, $\{A_t\}$, $\{A_t^*\}$, $\{\Lambda_t\}$ are all quantum stochastic processes as well as their derivatives $\{a_t\}$, $\{a_t^*\}$,

$\{a_t^* a_t\}$. Given a quantum stochastic process $\{L_t\}$ defined on an interval containing 0, we shall focus on a linear equation for unknown quantum stochastic process $\{\Xi_t\}$ of the following type:

$$\frac{d\Xi}{dt} = L_t \diamond \Xi, \qquad \Xi(0) = I. \tag{15}$$

The above equation is generally called a *normal-ordered white noise differential equation*. Since the equation (15) is linear and $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is a commutative algebra with the Wick product, the formal solution to (15) is obtained by the Wick exponential:

$$\Xi_t = \text{wexp}\left(\int_0^t L_s\, ds\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\int_0^t L_s\, ds\right)^{\diamond n}. \tag{16}$$

A serious question is convergence of the above infinite series and is answered, for example, in the following

Theorem 6 [7] *Let $\{L_t\} \subset \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ be a quantum stochastic process. Let ω be another weight sequence satisfying condition (A1)–(A5) and assume the relation:*

$$G_\omega(t) = \exp \gamma\{G_\alpha(t) - 1\},$$

where $\gamma > 0$ is a certain constant. Then, the series (16) converges in $\mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$ and is a unique solution to (15).

Relevant results are proved in [5], [6], [20], [21]; see also [13] for a nonlinear case. The next question then arises: in which Hilbert space the solution $\{\Xi_t\}$ acts. Here we recall

Theorem 7 [7] *Let $K^\pm$ be Hilbert spaces defined as in (10). Assume that $\{L_t\}$ is given by*

$$L_t = \sum_{l=0}^{1}\sum_{m=0}^{n} \Xi_{l,m}(\lambda_{l,m}(t)), \quad \kappa_{l,m}(t) \equiv \int_0^t \lambda_{l,m}(s)\, ds \in (K_{\mathbf{C}}^-)^{\otimes l} \otimes (E_{\mathbf{C}}^{\otimes m})^*.$$

Then, the unique solution to (15) lies in $\mathcal{L}((E), \Gamma(K_{\mathbf{C}}^-))$ if $0 \le n \le 1$; and in $\mathcal{L}((E)_\beta, \Gamma(K_{\mathbf{C}}^-))$ with $\beta = 1 - 1/n$ if $n \ge 1$.

As a simple consequence we have

Theorem 8 *Let $n \ge 1$. The initial value problem (15) with*

$$L_t = \sum_{m=0}^{n} L_{0,m} a_t^m + \sum_{m=0}^{n} L_{1,m} a_t^* a_t^m, \qquad L_{0,m}, L_{1,m} \in \mathbf{C},$$

has a unique solution in $\mathcal{L}((E)_\beta, \Gamma(H_{\mathbf{C}}))$, where $\beta = 1 - 1/n$.

Proof. For any $\xi, \eta \in E_{\mathbf{C}}$, we have

$$\begin{aligned} e^{-\langle \xi, \eta \rangle} \int_0^t \widehat{L_s}(\xi, \eta) ds &= \sum_{m=0}^{n} L_{0,m} \int_0^t \xi(s)^m ds + \sum_{m=0}^{n} L_{1,m} \int_0^t \eta(s)\xi(s)^m ds \\ &= \sum_{m=0}^{n} L_{0,m} \langle 1_{[0,t]}, \xi^m \rangle + \sum_{m=0}^{n} L_{1,m} \langle M_{[0,t]} \xi^m, \eta \rangle \\ &\equiv \sum_{m=0}^{n} \langle \kappa_{0,m}(t), \xi^{\otimes m} \rangle + \sum_{m=0}^{n} \langle \kappa_{1,m}(t), \eta \otimes \xi^{\otimes m} \rangle, \end{aligned}$$

where $M_{[0,t]}$ is the multiplication operator by the indicator function $1_{[0,t]}$ of $[0, t]$. Since the pointwise multiplication $(\xi_1, \cdots, \xi_m) \mapsto \xi_1 \cdots \xi_m \in E_{\mathbf{C}}$ is a continuous m-linear map, there is a unique continuous linear map $T \in \mathcal{L}(E_{\mathbf{C}}^{\otimes m}, E_{\mathbf{C}})$ such that $T(\xi_1 \otimes \cdots \otimes \xi_m) = \xi_1 \cdots \xi_m$. Then, $\kappa_{0,m}(t) = L_{0,m} T^* 1_{[0,t]} \in (E_{\mathbf{C}}^{\otimes m})^*$. On the other hand, $\kappa_{1,m}(t)$ corresponds to $L_{1,m} M_{[0,t]} T \in \mathcal{L}(E_{\mathbf{C}}^{\otimes m}, H_{\mathbf{C}})$ and hence $\kappa_{1,m}(t) \in H_{\mathbf{C}} \otimes (E_{\mathbf{C}}^{\otimes m})^*$. With these observation it follows from Theorem 7 that the unique solution lies in $\mathcal{L}((E)_\beta, \Gamma(H_{\mathbf{C}}))$. ∎

In the above theorem, the degree of the creation operators contained in $\{L_t\}$ is at most one. If $\{L_t\}$ involves higher powers of creation operators, the solution no longer acts in the original Fock space $\Gamma(H_{\mathbf{C}})$. In that case the solution acts (as unbounded operators in general) in another Fock space or in a weighted Fock space interpolating the CKS-space, for more details see e.g., [7].

To end with, we mention briefly the classical–quantum correspondence in white noise theory. A continuous map $t \mapsto \Phi_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is called a *classical stochastic process* in the sense of white noise theory. The pointwise multiplication by a white noise distribution $\Phi \in \mathcal{W}^*$ gives rise to a white noise operator, which is denoted by $\widetilde{\Phi} \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. Thus, a classical stochastic process $\{\Phi_t\}$ yields a quantum stochastic process $\{\widetilde{\Phi}_t\}$ possessing the relation $\widetilde{\Phi}_t \phi_0 = \Phi_t$, where ϕ_0 is the Fock vacuum. The fundamental relations $W_t = a_t + a_t^*$ for the classical white noise process and $B_t = A_t + A_t^*$ for the Brownian motion are understood in this sense.

Moreover, we note the following

Proposition 9 *Let $\{L_t\} \subset \mathcal{W}^*$ be a classical stochastic process and assume that a quantum stochastic process $\{\Xi_t\} \subset \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ satisfies the normal-ordered white noise differential equation:*

$$\frac{d\Xi}{dt} = \widetilde{L}_t \diamond \Xi, \qquad \Xi(0) = I.$$

Then the classical stochastic process defined by $\Phi_t = \Xi_t \phi_0$ satisfies the normal-ordered white *noise differential equation of classical type:*

$$\frac{d\Phi}{dt} = L_t \diamond \Phi, \qquad \Phi(0) = \phi_0.$$

Proof. For the Wick product of white noise distributions we refer to e.g., [14], [17]. As is easily verified by definition, for two white noise operators $\Xi_1, \Xi_2 \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ we have

$$(\Xi_1 \diamond \Xi_2)\phi_0 = \Xi_1\phi_0 \diamond \Xi_2\phi_0,$$

where the right hand side is the Wick product of white noise distributions. Hence

$$(\text{wexp}\,\Xi)\phi_0 = \text{wexp}\,(\Xi\phi_0), \qquad \Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*),$$

where the right hand side is the Wick exponential of a white noise distribution. With the above observation the proof is obvious. ∎

6 Complex Gaussian Integral and Unitarity Criterion

Let μ' be the Gaussian measure on $E^* = \mathcal{S}'(\mathbf{R})$ with variance 1/2, namely, a probability measure on E^* determined uniquely by the characteristic function:

$$e^{-|\xi|_0^2/4} = \int_{E^*} e^{i\langle x,\, \xi\rangle} \mu'(dx), \qquad \xi \in E.$$

In view of the topological isomorphism $E^*_{\mathbf{C}} \cong E^* \times E^*$, we define a probability measure $\nu = \mu' \times \mu'$ on $E^*_{\mathbf{C}}$ by

$$\nu(dz) = \mu'(dx)\mu'(dy), \qquad z = x + iy \in E^*_{\mathbf{C}}.$$

The probability space $(E^*_{\mathbf{C}}, \nu)$ is then called the *complex Gaussian space* after Hida [11, Chapter 6]. The "reproducing property" is essential:

$$\int_{E^*_{\mathbf{C}}} e^{\langle \bar{z},\, \xi\rangle + \langle z,\, \eta\rangle}\, \nu(dz) = e^{\langle \xi,\, \eta\rangle}, \qquad \xi, \eta \in E_{\mathbf{C}}, \tag{17}$$

where $\bar{z} = x - iy$ for $z = x + iy \in E^* + iE^*$, and $\langle \cdot, \cdot\rangle$ is the canonical $\mathbf{C}$-bilinear form on $E^*_{\mathbf{C}} \times E_{\mathbf{C}}$.

We next study unitarity of a white noise operator. The hermitian inner product of $\Gamma(H_{\mathbf{C}})$ is defined by

$$\langle\!\langle\!\langle \phi, \psi \rangle\!\rangle\!\rangle = \langle\!\langle \bar{\phi}, \psi \rangle\!\rangle.$$

For an operator Ξ we denote by $\Xi^\dagger$ its adjoint with respect to the above hermitian inner product. As is easily verified, it holds that

$$\Xi^\dagger \phi = \overline{\Xi^* \bar{\phi}}, \qquad \phi \in \mathcal{W}.$$

By definition $\Xi \in \mathcal{L}(\mathcal{W}, \Gamma(H_{\mathbf{C}}))$ is called an *isometry* on $\Gamma(H_{\mathbf{C}})$ if $\Xi^{\dagger}\Xi = I$; and is called a *unitary operator* if both Ξ and $\Xi^{\dagger}$ are isometries. Since the exponential vectors $\{\phi_{\xi}\,;\, \xi \in E_{\mathbf{C}}\}$ span a dense subspace of $\mathcal{W}$ and hence of $\Gamma(H_{\mathbf{C}})$, the condition $\Xi^{\dagger}\Xi = I$ is equivalent to

$$\langle\!\langle\!\langle \Xi\phi_{\xi}, \Xi\phi_{\eta} \rangle\!\rangle\!\rangle = \langle\!\langle\!\langle \phi_{\xi}, \phi_{\eta} \rangle\!\rangle\!\rangle, \qquad \xi, \eta \in E_{\mathbf{C}},$$

or in terms of the original $\mathbf{C}$-bilinear form:

$$\langle\!\langle \overline{\Xi\phi_{\bar{\xi}}}, \Xi\phi_{\eta} \rangle\!\rangle = \langle\!\langle \phi_{\xi}, \phi_{\eta} \rangle\!\rangle = e^{\langle \xi, \eta \rangle}, \qquad \xi, \eta \in E_{\mathbf{C}}. \tag{18}$$

Similarly, under the assumption that $\Xi^* \in \mathcal{L}(\mathcal{W}, \Gamma(H_{\mathbf{C}}))$, the condition $\Xi\Xi^{\dagger} = I$ is equivalent to

$$\langle\!\langle \overline{\Xi^*\phi_{\bar{\xi}}}, \Xi^*\phi_{\eta} \rangle\!\rangle = \langle\!\langle \phi_{\xi}, \phi_{\eta} \rangle\!\rangle = e^{\langle \xi, \eta \rangle}, \qquad \xi, \eta \in E_{\mathbf{C}}. \tag{19}$$

We shall derive an equivalent condition by means of complex Gaussian integral. By the same formula as in (9) we define $\phi_z \in \mathcal{W}^*$ also for $z \in E_{\mathbf{C}}^*$, which is again called an exponential vector.

Lemma 10 *For $\Phi \in \Gamma(H_{\mathbf{C}})$ the S-transform*

$$S\Phi(\xi) = \langle\!\langle \Phi, \phi_{\xi} \rangle\!\rangle, \qquad \xi \in E_{\mathbf{C}},$$

is extended to a unique L^2-function on $E_{\mathbf{C}}^*$ with respect to ν.

This is a consequence of the Segal–Bargmann transform (see e.g., [9]). Thus, for $\Xi \in \mathcal{L}(\mathcal{W}, \Gamma(H_{\mathbf{C}}))$ the symbol $\widehat{\Xi}(\xi, \eta)$ being originally a $\mathbf{C}$-valued function on $E_{\mathbf{C}} \times E_{\mathbf{C}}$, admits an extension to a function on $E_{\mathbf{C}} \times E_{\mathbf{C}}^*$ as a function in $L^2(E_{\mathbf{C}}^*, \nu)$ with respect to the second argument. The extension is denoted by $\widehat{\Xi}(\xi, z) = \langle\!\langle \Xi\phi_{\xi}, \phi_z \rangle\!\rangle$ for simplicity.

Theorem 11 *A white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \Gamma(H_{\mathbf{C}}))$ is an isometry on $\Gamma(H_{\mathbf{C}})$, i.e., $\Xi^{\dagger}\Xi = I$ if and only if*

$$\int_{E_{\mathbf{C}}^*} \overline{\widehat{\Xi}(\bar{\xi}, z)}\, \widehat{\Xi}(\eta, z)\, \nu(dz) = e^{\langle \xi, \eta \rangle}, \qquad \xi, \eta \in E_{\mathbf{C}}. \tag{20}$$

In addition, assume $\Xi^* \in \mathcal{L}(\mathcal{W}, \Gamma(H_{\mathbf{C}}))$. Then, Ξ is a unitary operator on $\Gamma(H_{\mathbf{C}})$, i.e., $\Xi^{\dagger}\Xi = \Xi\Xi^{\dagger} = I$ if and only if (20) and

$$\int_{E_{\mathbf{C}}^*} \overline{\widehat{\Xi}(z, \bar{\xi})}\, \widehat{\Xi}(z, \eta)\, \nu(dz) = e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E_{\mathbf{C}}. \tag{21}$$

Proof. For $z \in E_{\mathbf{C}}^*$ we define

$$Q_z\phi = \langle\langle \phi_{\bar{z}}, \phi \rangle\rangle \, \phi_z, \qquad \phi \in \mathcal{W}.$$

Then $Q_z \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ and the map $z \mapsto Q_z$ is continuous. Moreover, the resolution of the identity:

$$I = \int_{E_{\mathbf{C}}^*} Q_z \, \nu(dz)$$

holds, where the right hand side is understood through the canonical bilinear form [23]. The left hand side of (18) thereby becomes

$$\begin{aligned}
\langle\langle \overline{\Xi\phi_{\bar{\xi}}}, \Xi\phi_\eta \rangle\rangle &= \int_{E_{\mathbf{C}}^*} \langle\langle \overline{\Xi\phi_{\bar{\xi}}}, \phi_{\bar{z}} \rangle\rangle \langle\langle \phi_z, \Xi\phi_\eta \rangle\rangle \, \nu(dz) \\
&= \int_{E_{\mathbf{C}}^*} \overline{\langle\langle \Xi\phi_{\bar{\xi}}, \phi_z \rangle\rangle} \langle\langle \Xi\phi_\eta, \phi_z \rangle\rangle \, \nu(dz) \\
&= \int_{E_{\mathbf{C}}^*} \overline{\widehat{\Xi}(\bar{\xi}, z)} \, \widehat{\Xi}(\eta, z) \, \nu(dz),
\end{aligned}$$

which proves (20). The rest is proved similarly using the fact that ν is invariant under the complex conjugation $z \mapsto \bar{z}$. ∎

7 Unitarity of Solutions

We go back to the normal-ordered white noise differential equation:

$$\frac{d\Xi}{dt} = \left(\sum_{m=0}^{n} L_{0,m} a_t^m + \sum_{m=0}^{n} L_{1,m} a_t^* a_t^m \right) \diamond \Xi, \qquad \Xi(0) = I, \tag{22}$$

where $L_{0,m}, L_{1,m} \in \mathbf{C}$.

Theorem 12 *The solution to (22) is isometric if and only if there exist $\theta \in \mathbf{R}$, $\alpha \in \mathbf{R}$ and $\beta \in \mathbf{C}$ such that*

$$L_{0,0} = -\frac{1}{2}|\beta|^2 + i\alpha, \quad L_{0,1} = \beta, \quad L_{1,0} = -e^{i\theta}\overline{\beta}, \quad L_{1,1} = e^{i\theta} - 1,$$

$$L_{0,m} = L_{1,m} = 0, \qquad m \geq 2.$$

In that case the solution is automatically unitary.

Proof. The symbol of the solution to (22) is given by

$$\widehat{\Xi}_t(\xi, \eta) = \exp\left\{ \langle \xi, \eta \rangle + \sum_{m=0}^{n} L_{0,m} \left\langle 1_{[0,t]}, \xi^m \right\rangle + \sum_{m=0}^{n} L_{1,m} \left\langle 1_{[0,t]}, \eta\xi^m \right\rangle \right\}. \tag{23}$$

Obviously, the function $\eta \mapsto \widehat{\Xi}_t(\xi, \eta)$ is extended to an L^2-function on the complex Gaussian space $(E^*_{\mathbf{C}}, \nu)$. Then, a simple computation yields

$$\overline{\widehat{\Xi}_t(\bar{\xi}, z)}\,\widehat{\Xi}_t(\eta, z)$$
$$= \exp\left\{ (L_{0,0} + \overline{L_{0,0}})t + \sum_{m\geq 1} \overline{L_{0,m}} \left\langle 1_{[0,t]},\, \xi^m \right\rangle + \sum_{m\geq 1} L_{0,m} \left\langle 1_{[0,t]},\, \eta^m \right\rangle \right\}$$
$$\times \exp\left\{ \left\langle \bar{z},\, \xi + \sum_{m\geq 0} \overline{L_{1,m}} 1_{[0,t]} \xi^m \right\rangle + \left\langle z,\, \eta + \sum_{m\geq 0} L_{1,m} 1_{[0,t]} \eta^m \right\rangle \right\}.$$

By integration with the formula (17) we obtain

$$\int_{E^*_{\mathbf{C}}} \overline{\widehat{\Xi}_t(\bar{\xi}, z)}\,\widehat{\Xi}_t(\eta, z)\, \nu(dz)$$
$$= \exp\left\{ (L_{0,0} + \overline{L_{0,0}})t + \sum_{m\geq 1} \overline{L_{0,m}} \left\langle 1_{[0,t]},\, \xi^m \right\rangle + \sum_{m\geq 1} L_{0,m} \left\langle 1_{[0,t]},\, \eta^m \right\rangle \right\}$$
$$\times \exp\left\{ \left\langle \xi + \sum_{m\geq 0} \overline{L_{1,m}} 1_{[0,t]} \xi^m,\, \eta + \sum_{m\geq 0} L_{1,m} 1_{[0,t]} \eta^m \right\rangle \right\}. \tag{24}$$

By Theorem 11, a necessary and sufficient condition for Ξ_t to be isometric is that (24) is equal to $e^{\langle \xi, \eta \rangle}$ for all $\xi, \eta \in E_{\mathbf{C}}$. To this end we may equate coefficients of $\xi^m \eta^n$ inside the exponential function. From the constant terms we obtain

$$L_{0,0} + \overline{L_{0,0}} + \overline{L_{1,0}} L_{1,0} = 0. \tag{25}$$

Observing the coefficients of ξ and η, we obtain

$$\overline{L_{0,1}} + L_{1,0} + \overline{L_{1,1}} L_{1,0} = 0, \tag{26}$$
$$L_{0,1} + \overline{L_{1,0}} + \overline{L_{1,0}} L_{1,1} = 0,$$

which are mutually equivalent. From the coefficients of $\xi\eta$ we have

$$L_{1,1} + \overline{L_{1,1}} + \overline{L_{1,1}} L_{1,1} = 0. \tag{27}$$

From the coefficients of ξ^m and η^m with $m \geq 2$,

$$\overline{L_{0,m}} + \overline{L_{1,m}} L_{1,0} = 0, \tag{28}$$
$$L_{0,m} + \overline{L_{1,0}} L_{1,m} = 0,$$

which are mutually equivalent. From the coefficients of $\xi^m \eta$ and $\xi \eta^m$ with $m \geq 2$,

$$\overline{L_{1,m}} + \overline{L_{1,m}} L_{1,1} = 0, \tag{29}$$
$$L_{1,m} + \overline{L_{1,1}} L_{1,m} = 0,$$

which are mutually equivalent. Finally, from $\xi^m \eta^{m'}$ with $m, m' \geq 2$, we have

$$\overline{L_{1,m}} L_{1,m'} = 0. \tag{30}$$

The desired conditions follow from identities (25)–(30). In that case it is easily verified that the solution Ξ_t satisfies the second identity of Theorem 11, and hence is unitary. ∎

The above result is also valid when the coefficients $L_{m,m'}$ are continuous functions in t. The case where $L_{m,m'}$ are bounded operators acting on an initial Hilbert space can be discussed with slight modification, while Theorem 12 is the case of the initial Hilbert space being $\mathbf{C}$. In that case $\overline{L}$ means the adjoint operator of L. It seems convenient to introduce $W = 1 + L_{1,1}$. It then follows from (27) that $\overline{W}W = 1$, i.e., W is an isometry. Then, arranging identities (25)–(30), we obtain the following

Theorem 13 *Assume that $L_{m,m'}$ are bounded operators on an initial Hilbert space. Then, the solution to (22) is an isometry if and only if there exists an isometry W on the initial Hilbert space and the following relations hold:*

$$\begin{aligned}
&L_{0,0} + \overline{L_{0,0}} + \overline{L_{1,0}} L_{1,0} = 0, \\
&L_{0,1} = -\overline{L_{1,0}} W, \\
&L_{1,1} = W - 1, \\
&\overline{L_{0,m}} + \overline{L_{1,m}} L_{1,0} = 0, \qquad m \geq 2, \\
&\overline{L_{1,m}} W = 0, \qquad m \geq 2, \\
&\overline{L_{1,m}} L_{1,m'} = 0, \qquad m, m' \geq 2.
\end{aligned}$$

By a pararell discussion for $\Xi_t^\dagger$ we see that the solution to (22) is unitary only when W is a unitary operator. In that case, the six conditions in Theorem 13 are reduced to

$$\begin{aligned}
&L_{0,0} + \overline{L_{0,0}} + \overline{L_{1,0}} L_{1,0} = 0, \\
&L_{0,1} = -\overline{L_{1,0}} W, \\
&L_{1,1} = W - 1, \\
&L_{0,m} = L_{1,m} = 0, \qquad m \geq 2.
\end{aligned}$$

Hence we have

Theorem 14 *The solution to (22) is unitary if and only if there exist a unitary operator W, a selfadjoint operator H, and a bounded operator L on the initial Hilbert space such that*

$$L_{0,0} = -\frac{1}{2} L\overline{L} + iH, \quad L_{0,1} = L, \quad L_{1,0} = -W\overline{L}, \quad L_{1,1} = W - 1,$$
$$L_{0,m} = L_{1,m} = 0, \qquad m \geq 2.$$

In particular, the solution to (22) can be unitary (on the original Fock space) only when the coefficient is of the form:

$$L_t = L_{1,1} a_t^* a_t + L_{0,1} a_t + L_{1,0} a_t^* + L_{0,0}.$$

In that case, the corresponding normal-ordered white noise differential equation is equivalent to a quantum stochastic differential equation of Hudson–Parthasarathy type, see also [20], [21]. Thus, Theorem 14 reproduces the famous unitarity condition of Hudson and Parthasarathy [12].

The above argument does not go well when the coefficient $\{L_t\}$ contains higher powers of creation operators. In fact, the symbol of the solution Ξ_t contains a higher powers of η in such a way that

$$\widehat{\Xi}_t(\xi, \eta) = \exp\left\{ \langle \xi, \eta \rangle + \sum_{l,m} L_{l,m} \left\langle 1_{[0,t]}, \eta^l \xi^m \right\rangle \right\}.$$

Hence, contrary to (23), there is no way to extend the function $\eta \mapsto \widehat{\Xi}_t(\xi, \eta)$ to an L^2-function on $E_{\mathbf{C}}^*$. Furthermore, in that case unitarity of the solution should be discussed along with a Hilbert space different from the original Fock space $\Gamma(L^2(\mathbf{R}))$.

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