Heat Equation Associated with Lévy Laplacian

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Abstract

A solution to the heat equation associated with the Lévy Laplacian is studied by means of nuclear spaces of infinite dimensional entire functions. In particular, evolution of positive distributions and relation to the quadratic quantum white noise are discussed in a unified manner.

1 Introduction

In the famous books [17], [18] P. Lévy introduced and studied an infinite dimensional generalization of the classical Laplace operator:

$$\Delta_L = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2}. \tag{1}$$

This operator, called the *Lévy Laplacian*, possesses many peculiar properties and has been studied by many authors from various aspects. For example, formulating as a differential operator acting on functions on a Hilbert space, Feller [10], Polishchuk [28] and others (see the references cited therein) studied differential equations such as boundary problems in detail and Obata [19] gave a group-theoretical characterization. In recent years, more attention has been paid to the Lévy Laplacian acting on functions on a nuclear space for its rich structure. A somehow unexpected relation to the Gross Laplacian was found by Kuo–Obata–Saito [15]. A connection between Yang–Mills equations and heat

*Supported in part by JSPS Grant-in-Aid for Scientific Research No. 12440036.

equations associated with the Lévy Laplacian, first pointed out by Aref'eva–Volovich [6], has become an important research topic, see e.g., Accardi [1], Accardi–Bogachev [2], Accardi–Gibilisco–Volovich [3]. Using a particular domain constructed from Lévy's normal functions, Chung–Ji–Saitô [7] solved a heat equation associated with the Lévy Laplacian by means of an analytic one-parameter group $e^{z\Delta_L}$, see also Saitô [29]. Recently, using an idea of Poisson analysis, Saitô–Tsoi [31] found a new space where the Lévy Laplacian is formulated as a selfadjoint operator. In this direction further progress has been made by Saitô [30] and Kuo–Obata–Saitô [16].

In this paper, we focus on the heat equation associated with the Lévy Laplacian acting on functions on a real nuclear space E. Thus we are interested in the Cauchy problem:

$$\frac{\partial}{\partial t} F(t,\xi) = \Delta_L F(t,\xi), \qquad F(0,\xi) = F_0(\xi), \tag{2}$$

where the initial condition F_0 is a certain function on a nuclear space E. When $F_0(\xi)$ is the Fourier transform of a measure μ on E' which is invariant under a certain shift operator, a solution to (2) was explicitly obtained by Accardi–Roselli–Smolyanov [5]. Another interesting function $F(t,\xi)$ satisfying the heat equation was constructed by Obata [22] from a normal-ordered white noise equation involving the quadratic quantum white noises. The main purpose of this paper is to show that the above two classes of solutions are obtained in a unified manner without assuming that $\{x_1, x_2, \dots\}$ is an orthogonal coordinate system, which is a traditional assumption in the definition of the Lévy Laplacian (1). Furthermore, employing the recent framework of infinite dimensional holomorphic functions due to Gannoun–Hachaichi–Ouerdiane–Rezgui [11], we obtain an evolution of a positive distribution driven by the Lévy Laplacian. It is noted that our approach is independent of Gaussian analysis and seems appropriate for analysis of the Lévy Laplacian.

2 Preliminaries

In this section we assemble some basic notation and results on entire functions on nuclear spaces, for more details see [11].

2.1 Entire functions with θ-exponential growth

We begin with a general notation. For a complex Banach space $(B, \|\cdot\|)$ we denote by $H(B)$ the space of entire functions on B, i.e., continuous functions $B \to \mathbb{C}$ whose restrictions to every affine line of B are entire holomorphic on $\mathbb{C}$. We classify such entire functions by growth rates. Let θ be a Young function, i.e., $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, convex, increasing function such that $\theta(0) = 0$ and

$$\lim_{x\to\infty} \frac{\theta(x)}{x} = \infty. \tag{3}$$

For $m > 0$ we define the *space of entire functions on B with θ-exponential growth of finite type m* by

$$\mathrm{Exp}\,(B, \theta, m) = \left\{ f \in H(B)\,;\, \|f\|_{\theta,m} \equiv \sup_{u \in B} |f(u)| e^{-\theta(m\|u\|)} < \infty \right\}.$$

If θ is a Young function,

$$\theta^*(x) = \sup_{t \geq 0} (tx - \theta(t)), \qquad x \geq 0, \tag{4}$$

becomes also a Young function. This is called the *polar function* of θ and plays a role in duality argument.

2.2 Nuclear spaces of entire functions

Let N be a complex nuclear Fréchet space whose topology is defined by a family of increasing Hilbertian norms $\{|\cdot|_p\,,\, p \in \mathbb{N}\}$. The space N can be represented as $N = \cap_{p\in\mathbb{N}} N_p$, where N_p is the Hilbert space obtained by completing N with respect the norm $|\cdot|_p$. Denote by N_{-p} the topological dual space of N_p. Then by general duality theory the dual space N' can be expressed as $N' = \cup_{p\in\mathbb{N}} N_{-p}$. Because of the nuclearity of the space N, the strong topology of N' coincides with the inductive limit topology.

It is easily verified that $\{\mathrm{Exp}\,(N_{-p}, \theta, m)\}$ forms a projective system of Banach spaces as $p \to \infty$ and $m \downarrow 0$. We then define the *space of entire functions on N' with θ-exponential growth of minimal type* by

$$\mathcal{F}_\theta(N') = \bigcap_{p \in \mathbb{N}, m > 0} \mathrm{Exp}\,(N_{-p}, \theta, m). \tag{5}$$

Similarly, $\{\mathrm{Exp}\,(N_p, \theta, m)\}$ forms an inductive system of Banach spaces as $p \to \infty$ and $m \to \infty$, and we define the *space of entire functions on N with θ-exponential growth of (arbitrarily) finite type* by

$$\mathcal{G}_\theta(N) = \bigcup_{p \in \mathbb{N}, m > 0} \mathrm{Exp}\,(N_p, \theta, m). \tag{6}$$

If θ and φ are two Young functions which are equivalent at infinity, i.e., $\lim_{x\to\infty} \theta(x)/\varphi(x) = 1$, we have $\mathcal{F}_\theta(N') = \mathcal{F}_\varphi(N')$ and $\mathcal{G}_\theta(N) = \mathcal{G}_\varphi(N)$.

2.3 Taylor series map

Each $f \in \mathcal{F}_\theta(N')$ and $g \in \mathcal{G}_\theta(N)$ admit Taylor series expansions:

$$f(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, f_n \rangle, \qquad x \in N',$$

$$g(\xi) = \sum_{n=0}^{\infty} \langle g_n, \xi^{\otimes n} \rangle, \qquad \xi \in N.$$

Characterization of these spaces in terms of Taylor expansion is useful. The correspondences $f \mapsto \vec{f} = (f_n)_{n\geq 0}$ and $g \mapsto \vec{g} = (g_n)_{n\geq 0}$ are called the *Taylor series map* (at zero) and denoted by $\mathcal{T}$.

Given a Young function θ, we put

$$\theta_n = \inf_{r>0} \frac{e^{\theta(r)}}{r^n}.$$

Then we define the Hilbert space $F_{\theta,m}(N_p)$ by

$$F_{\theta,m}(N_p) = \left\{ \vec{f} = (f_n)_{n\geq 0} \,;\, f_n \in N_p^{\odot n},\, \|\vec{f}\|_{\theta,p,m} < \infty \right\},$$

where $N_p^{\odot n}$ is the n-fold symmetric tensor power of N_p and

$$\|\vec{f}\|^2_{\theta,p,m} = \sum_{n=0}^{\infty} \theta_n^{-2} m^{-n} |f_n|_p^2, \qquad \vec{f} = (f_n).$$

Then, equipped with the projective limit topology,

$$F_\theta(N) = \bigcap_{p\in\mathbb{N},m>0} F_{\theta,m}(N_p)$$

becomes a nuclear Fréchet space. In a similar manner, one defines

$$G_{\theta,m}(N_{-p}) = \{\vec{\Phi} = (\Phi_n)_{n\geq 0} \,;\, \Phi_n \in N_{-p}^{\odot n}, \|\vec{\Phi}\|_{\theta,-p,m} < \infty\},$$

where

$$\|\vec{\Phi}\|^2_{\theta,-p,m} = \sum_{n=0}^{\infty} (n!\theta_n)^2 m^n |\Phi_n|^2_{-p}.$$

Then we put

$$G_\theta(N') = \bigcup_{p\in\mathbb{N},m>0} G_{\theta,m}(N_{-p}),$$

which is equipped with the inductive limit topology. By definition the power series spaces $F_\theta(N)$ and $G_\theta(N')$ are dual each other with the canonical bilinear form defined by

$$\langle\!\langle \vec{\Phi}, \vec{f} \rangle\!\rangle = \sum_{n=0}^{\infty} n!\, \langle \Phi_n, f_n \rangle. \tag{7}$$

Theorem 1 (Gannoun–Hachaichi–Ouerdiane–Rezgui [11]) *The Taylor series map $\mathcal{T}$ induces two topological isomorphisms:*

$$\mathcal{T} : \mathcal{F}_\theta(N') \longrightarrow F_\theta(N) \quad \textit{and} \quad \mathcal{T} : \mathcal{G}_{\theta^*}(N) \longrightarrow G_\theta(N'), \tag{8}$$

where θ^* is the polar function of θ.

2.4 Laplace Transform

Let $\mathcal{F}_\theta^*(N')$ denote the strong dual space of $\mathcal{F}_\theta(N')$. We shall obtain its concise description.

Let $\Phi \in \mathcal{F}_\theta^*(N')$. By the adjoint map $\mathcal{T}^* : F_\theta^*(N) \to \mathcal{F}_\theta^*(N')$, which is also an isomorphism by (8), we obtain $\mathcal{T}^{*-1}\Phi \in F_\theta^*(N)$. On the other hand, $F_\theta^*(N)$ is identified with $G_\theta(N')$ through (7). Let $\vec{\Phi} = (\Phi_n) \in G_\theta(N')$ be the element corresponding to $\mathcal{T}^{*-1}\Phi$. Then, for $f \in \mathcal{F}_\theta(N')$ we have

$$\Phi(f) = \langle\!\langle \vec{\Phi}, \vec{f} \rangle\!\rangle = \sum_{n=0}^{\infty} n!\, \langle \Phi_n, f_n \rangle, \tag{9}$$

where $\vec{f} = (f_n) = \mathcal{T} f$.

For $\xi \in N$ we define the exponential function $e^\xi : N' \to \mathbf{C}$ by $e^\xi(x) = e^{\langle x, \xi \rangle}$, $x \in N'$. It is proved with the help of (3) that $e^\xi \in \mathcal{F}_\theta(N')$ for all $\xi \in N$. The *Laplace transform* of $\Phi \in \mathcal{F}_\theta^*(N')$ is defined by

$$(\mathcal{L}\Phi)(\xi) \equiv \hat{\Phi}(\xi) = \Phi(e^\xi), \qquad \xi \in N. \tag{10}$$

Since the Taylor expansion of e^ξ is given by

$$e^\xi(x) = e^{\langle x, \xi \rangle} = \sum_{n=0}^{\infty} \left\langle x^{\otimes n}, \frac{\xi^{\otimes n}}{n!} \right\rangle,$$

we have

$$\mathcal{T}(e^\xi) = \left(1, \xi, \dots, \frac{\xi^{\otimes n}}{n!}, \dots \right).$$

Therefore, by (9) we see that (10) becomes

$$(\mathcal{L}\Phi)(\xi) = \hat{\Phi}(\xi) = \sum_{n=0}^{\infty} n! \left\langle \Phi_n, \frac{\xi^{\otimes n}}{n!} \right\rangle = \sum_{n=0}^{\infty} \langle \Phi_n, \xi^{\otimes n} \rangle.$$

Thus, taking Theorem 1 into account, we come to the following

Theorem 2 (Gannoun–Hachaichi–Ouerdiane–Rezgui [11]) *The Laplace transform induces a topological isomorphism*

$$\mathcal{L} : \mathcal{F}_\theta^*(N') \to \mathcal{G}_{\theta^*}(N), \tag{11}$$

where θ^* is the polar function of θ.

2.5 Integral representation of positive distributions

We assume that $N = E + iE$, where E is a real nuclear Fréchet space. Then an element $f \in \mathcal{F}_\theta(N')$ is called *positive* if $f(x + i0) \geq 0$ for all $x \in E'$. We denote by $\mathcal{F}_\theta(N')_+$ the set of positive functions. An element $\Phi \in \mathcal{F}_\theta^*(N')$ is called *positive* if $\Phi(f) \geq 0$ for all $f \in \mathcal{F}_\theta(N')_+$. The cone of positive elements in $\mathcal{F}_\theta^*(N')$ is denoted by $\mathcal{F}_\theta^*(N')_+$. We always assume that E' is equipped with the Borel σ-field.

Theorem 3 (Ouerdiane–Rezgui [27]) *For each $\Phi \in \mathcal{F}_\theta^*(N')_+$ there exists a unique positive Radon measure $\mu = \mu_\Phi$ on the space E' such that*

$$\Phi(f) = \int_{E'} f(x + i0)\, d\mu(x), \qquad f \in \mathcal{F}_\theta(N'). \tag{12}$$

In that case there exist $q > 0$ and $m > 0$ such that the measure μ is carried by the space E_{-q} and

$$\int_{E_{-q}} e^{\theta(m|x|_{-q})}\, d\mu(x) < \infty. \tag{13}$$

Conversely, such a positive finite measure μ on the space E' defines a positive distribution $\Phi \in \mathcal{F}_\theta^*(N')_+$ by formula (12).

Note that the Fourier transform of μ_Φ and the Laplace transform of Φ is related as

$$\mathcal{F}\mu_\Phi(\xi) = \int_{E'} e^{i\langle x, \xi\rangle}\, d\mu_\Phi(x) = \Phi(e^{i\xi}) = \mathcal{L}\Phi(i\xi), \qquad \xi \in E. \tag{14}$$

3 The Lévy Laplacian

3.1 Definition in general

Let E be a real nuclear Fréchet space as before. A function $F : E \to \mathbf{R}$ is called of class $C^2(E)$ if there exist two continuous maps $\xi \mapsto F'(\xi) \in E'$ and $\xi \mapsto F''(\xi) \in \mathcal{L}(E, E')$, $\xi \in E$, such that

$$F(\xi + \eta) = F(\xi) + \langle F'(\xi), \eta\rangle + \frac{1}{2}\langle F''(\xi)\eta, \eta\rangle + \epsilon(\eta), \qquad \xi, \eta \in E,$$

where the error term satisfies:

$$\lim_{t\to 0} \frac{\epsilon(t\eta)}{t^2} = 0, \qquad \eta \in E.$$

In view of the nuclear kernel theorem $\mathcal{L}(E, E') \cong (E \otimes E)' \cong \mathcal{B}(E, E)$ we use the common symbol $F''(\xi)$ for all:

$$\langle F''(\xi)\eta, \eta\rangle = \langle F''(\xi), \eta \otimes \eta\rangle = F''(\xi)(\eta, \eta) = D_\eta D_\eta F(\xi),$$

where D_η is the Fréchet derivative in the direction η, i.e.,

$$(D_\eta F)(\xi) = \lim_{\lambda \to 0} \frac{F(\xi + \lambda\eta) - F(\xi)}{\lambda}.$$

A $\mathbb{C}$-valued function $F : E \to \mathbb{C}$ is a member of $C^2(E)$ if so are its real and imaginary parts. In that case, $F'(\xi) \in N'$ and $F''(\xi) \in (N \otimes N)'$.

Fix an arbitrary infinite sequence $\{e_n\}_{n=1}^{\infty} \subset E$. We shall assume additional properties later, though. The *Lévy Laplacian* is defined for $F \in C^2(E)$ by

$$\Delta_L F(\xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle F''(\xi)e_n, e_n\rangle, \qquad \xi \in E,$$

whenever the limit exists. Let $\mathcal{D}_L(E)$ be the space of all $F \in C^2(E)$ for which $\Delta_L F(\xi)$ exists for all $\xi \in E$. It is noted that the Lévy Laplacian depends on the choice of the sequence $\{e_n\}$ as well as its arrangement.

3.2 Cesàro mean

We prepare a notation. Let $\{e_n\}_{n=1}^{\infty} \subset E$ be an arbitrary sequence as in the previous subsection. Recall that $N = E + iE$. We denote by $(E \otimes E)'_L$ (resp. $(N \otimes N)'_L$) the set of all $f \in (E \otimes E)'$ (resp. $f \in (N \otimes N)'$) which admit the limit

$$\langle f \rangle_L = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle f, e_n \otimes e_n\rangle.$$

Although not explicitly written, $(E \otimes E)'_L$ and $(N \otimes N)'_L$ depend on the choice of $\{e_n\}$. Obviously, $f \in (N \otimes N)'_L$ if and only if its real and imaginary parts belong to $(E \otimes E)'_L$.

By definition we have the following

Lemma 4 *A function $F \in C^2(E)$ belongs to $\mathcal{D}_L(E)$ if and only if $F''(\xi) \in (N \otimes N)'_L$ for all $\xi \in E$. In that case,*

$$\Delta_L F(\xi) = \langle F''(\xi) \rangle_L.$$

Let E'_L (resp. N'_L) denote the set of all $a \in E'$ (resp. $a \in N'$) such that $a \otimes a \in (E \otimes E)'_L$ (resp. $a \otimes a \in (N \otimes N)'_L$), i.e., the limit

$$\langle a \otimes a \rangle_L = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle a, e_n\rangle^2$$

exists. For a real $a \in E'_L$ we also write $\|a\|_L^2 = \langle a \otimes a \rangle_L$. It is clear that $E'_L \subset N'_L$ but $N'_L = E'_L + iE'_L$ does not necessarily hold.

Lemma 5 *For $a, b \in E'_L$ it holds that*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\langle a \otimes b, e_n \otimes e_n\rangle| \le \|a\|_L \|b\|_L.$$

Proof. Note the Schwartz inequality

$$\sum_{n=1}^{N} |\langle a\otimes b, e_n\otimes e_n\rangle| \le \left(\sum_{n=1}^{N}\langle a, e_n\rangle^2\right)^{1/2} \left(\sum_{n=1}^{N}\langle b, e_n\rangle^2\right)^{1/2},$$

from which the assertion follows immediately. ■

Lemma 6 *Let $a, b \in E_L'$.*
(1) *If $a+b \in E_L'$, then $\|a+b\|_L \le \|a\|_L + \|b\|_L$.*
(2) *If $\|b\|_L = 0$, then $a+b \in E_L'$ and $\|a+b\|_L = \|a\|_L$.*

Proof. We note the obvious identity:

$$\sum_{n=1}^{N}\langle a+b, e_n\rangle^2 = \sum_{n=1}^{N}\langle a, e_n\rangle^2 + \sum_{n=1}^{N}\langle b, e_n\rangle^2 + 2\sum_{n=1}^{N}\langle a\otimes b, e_n\otimes e_n\rangle.$$

Then the assertions are immediate from Lemma 5. ■

3.3 Eigenfunctions

Lemma 7 *Let $p \in \mathcal{D}_L(E)$ with $p'(\xi) \in N_L'$ for all $\xi \in E$. Then $e^p \in \mathcal{D}_L(E)$ and*

$$\Delta_L e^{p(\xi)} = (\langle p''(\xi)\rangle_L + \langle p'(\xi)\otimes p'(\xi)\rangle_L)\, e^{p(\xi)}.$$

Proof. The assertion is immediate from

$$D_{e_n}^2 e^{p(\xi)} = \left\{\langle p''(\xi)e_n, e_n\rangle + \langle p'(\xi), e_n\rangle^2\right\} e^{p(\xi)},$$

which is verified by a direct computation. ■

Now we show two typical classes of eigenfunctions of Δ_L.

Proposition 8 (1) *For $a \in N_L'$ it holds that*

$$\Delta_L e^{\langle a,\xi\rangle} = \langle a\otimes a\rangle_L\, e^{\langle a,\xi\rangle}. \tag{15}$$

(2) *Let $f \in (N\otimes N)_L'$ be symmetric. If $\langle (f\otimes_1 \xi)\otimes(f\otimes_1\xi)\rangle_L = 0$ for all $\xi \in E$, it holds that*

$$\Delta_L e^{\langle f,\xi\otimes\xi\rangle} = 2\langle f\rangle_L e^{\langle f,\xi\otimes\xi\rangle}.$$

Proof. (1) is immediate from Lemma 7. We prove (2). Put $p(\xi) = \langle f, \xi\otimes\xi\rangle$. Note that

$$\langle p'(\xi), e_n\rangle = 2\langle f, \xi\otimes e_n\rangle = 2\langle f\otimes_1 \xi, e_n\rangle,$$

where $f\otimes_1\xi$ is constraction of degree one and is defined as above. Then

$$\langle p'(\xi)\otimes p'(\xi)\rangle_L = 4\langle (f\otimes_1\xi)\otimes(f\otimes_1\xi)\rangle_L = 0$$

by assumption. On the other hand, $\langle p''(\xi)\rangle_L = 2\langle f\rangle_L$. Hence the assertion follows from Lemma 7. ■

3.4 Derivation property

It is widely known as one of the peculiar properties that the Lévy Lapalacian is a derivation, i.e., behaves like a first order differential operator. This property, however, depends on the domain as shown in the next proposition. A similar fact was already pointed out by Accardi–Obata [4].

Proposition 9 *Let* $F_1, F_2 \in \mathcal{D}_L(E)$. *If* $\langle F'(\xi) \otimes G'(\xi) \rangle_L = 0$ *for all* $\xi \in E$, *then*

$$\Delta_L(F_1F_2) = (\Delta_L F_1)F_2 + F_1(\Delta_L F_2).$$

The proof is straightforward. A function $H \in C^2(E)$ is called *Lévy-harmonic* if $\Delta_L H(\xi) = 0$ for all $\xi \in E$. Then we have immediately the following

Corollary 10 *Let* $F \in \mathcal{D}_L(E)$ *and* H *a Lévy-harmonic function. If* $\langle F'(\xi) \otimes H'(\xi) \rangle_L = 0$ *for all* $\xi \in E$, *then*

$$\Delta_L(FH) = (\Delta_L F)H.$$

4 Heat Equation

4.1 Cauchy problem

In general, the Cauchy problem associated with the Lévy Laplacian is stated as follows:

$$\frac{\partial F}{\partial t} = \gamma \Delta_L F, \qquad F(0, \xi) = F_0(\xi), \tag{16}$$

where $\gamma \in \mathbb{C}$ is a constant, the initial condition F_0 is a suitable function on E and t runs over an interval including 0. Note that (16) involves both heat type and Schrödinger type equations associated with the Lévy Laplacian.

The formal solution F of (16) is given by

$$F(t, \xi) = (e^{t\gamma \Delta_L} F_0)(\xi) = \sum_{n=0}^{\infty} \frac{(t\gamma)^n}{n!} (\Delta_L^n F_0)(\xi).$$

However, the convergence is always in question. For particular initial conditions the convergence is proved by Chung–Ji–Saitô [7]. We do not go into this direction.

As a general remark we only mention the following

Proposition 11 *Let* $p, q \in \mathcal{D}_L(E)$ *and assume that*

$$\Delta_L p(\xi) \equiv \alpha, \qquad \Delta_L q(\xi) \equiv 0,$$

and

$$\langle p'(\xi) \otimes p'(\xi) \rangle_L = \langle p'(\xi) \otimes q'(\xi) \rangle_L \equiv 0, \qquad \langle q'(\xi) \otimes q'(\xi) \rangle_L \equiv \beta,$$

where $\alpha, \beta \in \mathbb{C}$ are constant numbers. Let $\gamma \in \mathbb{C}$ be another constant. Then,

$$F_t(\xi) = F(t, \xi) = e^{t\gamma(\alpha+\beta)} e^{p(\xi)+q(\xi)}, \qquad t \in \mathbb{R}, \quad \xi \in E, \tag{17}$$

satisfies the Cauchy problem associated with the Lévy Laplacian:

$$\frac{\partial}{\partial t} F(t, \xi) = \gamma \Delta_L F(t, \xi), \qquad F(0, \xi) = e^{p(\xi)+q(\xi)}. \tag{18}$$

Proof. By Lemma 7 we have

$$\Delta_L e^{p(\xi)+q(\xi)}$$
$$= \left(\langle p''(\xi)\rangle_L + \langle q''(\xi)\rangle_L + \langle (p'(\xi) + q'(\xi)) \otimes (p'(\xi) + q'(\xi))\rangle_L\right) e^{p(\xi)+q(\xi)}.$$

By assumption we have

$$\Delta_L e^{p(\xi)+q(\xi)} = (\alpha + \beta)\, e^{p(\xi)+q(\xi)},$$

from which the assertion is immediate. ■

In order to check the condition in the above theorem such results as in Lemmas 5 and 6 are useful. Typically we take

$$p(\xi) = \langle f, \xi \otimes \xi \rangle, \qquad q(\xi) = \langle a, \xi \rangle,$$

and consider (17), see Proposition 8. From the next subsection on, we shall show that superposition of (17) gives a solution to the Cauchy problem (16) with an interesting initial condition.

4.2 Shift-invariance

Recall that the Lévy Laplacian Δ_L depends on an arbitrarily fixed sequence $\{e_n\}_{n=1}^{\infty} \subset E$. We now consider the shift operator S associated with this sequence. Assume that there exists a continuous operator $S : E \to E$ such that $Se_n = e_{n+1}$ for all n. It would be more natural to do the converse. Given a continuous operator S and a fixed $e_1 \in E$, we may construct the sequence $\{e_n\}$ by $e_n = S^{n-1} e_1$.

Proposition 12 *The Lévy Laplacian is invariant under the shift S, i.e.,*

$$\Delta_L(F \circ S) = (\Delta_L F) \circ S, \qquad F \in \mathcal{D}_L(E).$$

Proof. By a direct computation we have

$$\langle (F \circ S)''(\xi) e_n, e_n \rangle = \langle F''(S\xi) S e_n, S e_n \rangle = \langle F''(S\xi) e_{n+1}, e_{n+1} \rangle.$$

Since the Cesàro mean is invariant under the shift, the assertion follows. ■

4.3 Evolution of positive distributions

For $x \in E'$ we put $q_x(\xi) = e^{i\langle x,\xi\rangle}$. Then, if $x \in E'_L$ we have

$$\Delta_L q_x(\xi) = -\langle x \otimes x\rangle_L q_x(\xi) = -\|x\|_L^2 q_x(\xi).$$

We shall consider a superposition of such q_x.

Note that the adjoint S^* is a continuous operator from E' into itself.

Theorem 13 *Let $\Phi_0 \in \mathcal{F}_\theta^*(N')_+$ and μ the corresponding Radon measure on E', see Theorem 3. If μ is invariant under S^*, then $x \in E'_L$ for μ-a.e. x and*

$$F_t(\xi) = F(t,\xi) = \int_{E'} e^{-t\|x\|_L^2} e^{i\langle x,\xi\rangle} d\mu(x), \qquad \xi \in E, \quad t \geq 0, \tag{19}$$

is a solution to the Cauchy problem:

$$\frac{\partial F}{\partial t} = \Delta_L F, \qquad F(0,\xi) = \mathcal{L}\Phi_0(i\xi) = \int_{E'} e^{i\langle x,\xi\rangle} d\mu(x). \tag{20}$$

Proof. For simplicity we put

$$G(x) = \langle x, e_1\rangle^2, \qquad x \in E'.$$

Then $G \in L^1(E',\mu)$. In fact, taking Theorem 3 into account, we note that

$$\begin{aligned}\int_{E'} |G(x)|\, d\mu(x) &= \int_{E_{-q}} \langle x, e_1\rangle^2 e^{-\theta(m|x|_{-q})} e^{\theta(m|x|_{-q})}\, d\mu(x) \\ &\leq \int_{E_{-q}} |x|_{-q}^2 |e_1|_q^2 e^{-\theta(m|x|_{-q})} e^{\theta(m|x|_{-q})}\, d\mu(x). \end{aligned}\tag{21}$$

Since $\sup_{t\geq 0} t^2 e^{-\theta(mt)} < \infty$ by the assumptions on θ, (21) is finite as desired. Now we recall the assumption that μ is invariant under the measurable transformation S^*. Then applying the ergodic theorem (see e.g., [9, Chapter VIII]), we see that

$$\tilde{G}(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} G(S^{*(n-1)}x)$$

converges for μ-a.e. $x \in E'$. Moreover, the convergence holds also in the L^1-sense and $\tilde{G} \in L^1(E',\mu)$. On the other hand, since

$$\sum_{n=1}^{N} G(S^{*(n-1)}x) = \sum_{n=1}^{N} \langle S^{*(n-1)}x, e_1\rangle^2 = \sum_{n=1}^{N} \langle x, S^{n-1}e_1\rangle^2 = \sum_{n=1}^{N} \langle x, e_n\rangle^2,$$

we have $\tilde{G}(x) = \langle x \otimes x\rangle_L = \|x\|_L^2$. Consequently, a measurable function $x \mapsto \|x\|_L^2$ is defined μ-a.e. $x \in E'$ and belongs to $L^1(E',\mu)$. Then, one can check easily that (19) is a solution to (20) by the Lebesque convergence theorem. ∎

In the usual definition of Δ_L, the sequence $\{e_n\} \subset E$ is assumed to have some particular properties, typically, to be a complete orthonormal basis for a certain Hilbert space. We note that in Theorem 13 such additional assumptions are not required. However, the idea of proof is essentially due to Accardi–Roselli–Smolyanov [5] and Accardi–Obata [4].

4.4 A relation with quadratic quantum white noises

In this subsection we take a concrete nuclear triple:

$$E = \mathcal{S}(\mathbb{R}) \subset H = L^2(\mathbb{R}) \subset E' = \mathcal{S}'(\mathbb{R}).$$

As before, we set $N = E + iE$. For $s \geq 0$ consider

$$p_s(\xi) = \langle 1_{[0,s]}\xi, \xi \rangle = \int_0^s \xi(u)^2 \, du, \qquad \xi \in E.$$

Then,

$$\|p_s'(\xi)\|_L^2 = 4\|1_{[0,s]}\xi\|_L^2 = 4 \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N \left(\int_0^s e_n(u)\xi(u) \, du \right)^2$$

and

$$\Delta_L p_s(\xi) = 2\langle 1_{[0,s]}\tau \rangle_L = 2 \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N \int_0^s e_n(u)^2 \, du,$$

where $\tau \in \mathcal{S}'(\mathbb{R} \times \mathbb{R})$ is the trace. With the help of Proposition 11 we come to the following

Lemma 14 *Let $s \geq 0$. If $\|1_{[0,s]}\xi\|_L = 0$ for all $\xi \in E$ and $\langle 1_{[0,s]}\tau \rangle_L = s$, then*

$$f_t(\xi) = e^{2st} e^{q_s(\xi)}, \qquad t \in \mathbb{R}, \quad \xi \in E,$$

satisfies the heat equation:

$$\frac{\partial}{\partial t} f_t(\xi) = \Delta_L f_t(\xi).$$

Let $\{a_t, a_t^*\}_{t\in\mathbb{R}}$ be the quantum white noise, namely, a_t is a continuous linear operator on $\mathcal{F}_\theta(N')$ defined by

$$a_t e^\xi = \xi(t) e^\xi, \qquad \xi \in N, \quad t \in \mathbb{R},$$

and a_t^* is the dual operator. Consider the normal-ordered white noise differential equation

$$\frac{d\Xi}{dt} = (a_t^2 + a_t^{*2}) \diamond \Xi, \qquad \Xi(0) = I. \tag{22}$$

This is a "singular" quantum stochastic differential equation beyond the traditional Itô theory. By general theory [21] there exists a unique solution to (22) in $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$. Let $\{\Psi_t\} \subset \mathcal{F}_\theta^*(N')$ be the "classical" stochastic process corresponding to the "quantum" stochastic process $\{\Xi_t\}$ defined by $\Psi_t = \Xi_t e^0$, where $e^0(\xi) \equiv 1$. Then by a direct computation we have

$$e^{q_s(\xi)} = \mathcal{L}\Psi_s(\xi), \qquad s \geq 0, \quad \xi \in E.$$

Summing up,

Theorem 15 *Let* $\{\Psi_s\} \subset \mathcal{F}_\theta^*(N')$ *be the classical stochastic process corresponding to the quantum stochastic process determined by* (22). *Let* $s \geq 0$. *Assume that* $\|1_{[0,s]}\xi\|_L = 0$ *for all* $\xi \in E$ *and* $\langle 1_{[0,s]} \rangle_L = s$. *Then*

$$F_t(\xi) = e^{2st}\mathcal{L}\Psi_s(\xi), \qquad t \geq 0, \quad \xi \in E,$$

satisfies the heat equation:

$$\frac{\partial}{\partial t}F = \Delta_L F, \qquad F(0,\xi) = \mathcal{L}\Psi_s(\xi). \tag{23}$$

Let T be a compact interval equipped with a finite measure ν. If the assumption in Theorem 15 is true for all $s \in T$, then

$$F_t(\xi) = \int_T e^{2st}\mathcal{L}\Psi_s(\xi)\,\nu(ds), \qquad t \geq 0, \quad \xi \in E,$$

satisfies the heat equation (23) with an initial condition:

$$F(0,\xi) = \int_T \mathcal{L}\Psi_s(\xi)\,\nu(ds).$$

This draws out an essence of [22, Theorem 6].

Remark 16 Recall that for $\Phi \in \mathcal{F}_\theta^*(N')$, the Laplace transform $\mathcal{L}\Phi$ belongs to $\mathcal{G}_{\theta^*}(N)$. In particular, $\mathcal{L}\Phi \in C^2(E)$. Let $\mathcal{D}_L$ denote the space of all $\Phi \in \mathcal{F}_\theta^*(N')$ such that $\mathcal{L}\Phi \in \mathcal{D}_L(E)$ and $\Delta_L\mathcal{L}\Phi \in \mathcal{G}_{\theta^*}(N)$. Then the *Lévy Laplacian* $\tilde{\Delta}_L$ is defined by

$$\tilde{\Delta}_L\Phi = \mathcal{L}^{-1}\Delta_L\mathcal{L}\Phi, \qquad \Phi \in D_L.$$

This $\tilde{\Delta}_L$ is essentially the same as the Lévy Laplacian formulated within white noise theory, see e.g., Kuo [14] and references cited therein.

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