Asymptotic Spectral Analysis of Growing Graphs: A Quantum Probabilistic Approach

Nobuaki Obata
Graduate School of Information Sciences
Tohoku University
Sendai 980-8579 Japan
E-Mail: obata@math.is.tohoku.ac.jp

Abstract: We review the recently developed method for spectral analysis of growing graphs on the basis of quantum (or noncommutative or algebraic) probability theory. The asymptotic spectral distribution of a growing regular graph is derived from the quantum central limit theorem for quantum components of the adjacency matrix.

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Introduction

Quantum probability theory is based on an algebraic probability space $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital $*$-algebra and $\varphi : \mathcal{A} \to \mathbf{C}$ is a state, i.e., a linear function which is positive ($\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$) and normalized ($\varphi(1_{\mathcal{A}}) = 1$). As is well known, a classical random variable X on a probability space $(\Omega, \mathcal{F}, P)$ is canonically considered as an algebraic random variable and is decomposed into a sum of its quantum components. This quantum decomposition brings X into a noncommutative regime, where quantum probabilistic techniques are available. In particular, a crucial role is played by the profound relation between interacting Fock spaces and orthogonal polynomials, see Accardi–Bożejko [2]. We apply this idea to asymptotic spectral analysis of the adjacency matrix of a growing graph. Once the adjacency matrix is considered as an algebraic random variable, the asymptotic spectral distribution is derived along with various quantum central limit theorems. Our approach releases cumbersome combinatorial questions about the graph structure and demands instead computation of some statistics related to its local structure.

Quantum probabilistic approach to asymptotic spectral analysis of a growing graph traces back to Hora [9], where growing distance-regular graphs were studied. His method was not based upon the quantum decomposition but required some classical results, see also Hora [10, 11]. Hashimoto–Obata–Tabei [7] applied the method of quantum decomposition to Hamming graphs and obtained the limit distributions (Gaussian and Poisson distributions) without combinatorial arguments required in the classical method. Hashimoto [5] applied the same idea to Cayley graphs and developed a general theory. The idea of quantum decomposition is so naive that similar consideration is found in various contexts; however, as a method of analyzing an adjacency matrix or a classical random variable within quantum probability theory, the term "quantum decomposition" was introduced first by Hashimoto [5]. Later on, Hashimoto–Hora–Obata [6] studied the limit distributions for distance-regular graphs in general. In particular, the exponential distributions (Laguerre polynomials) and the geometric distributions (Meixner polynomials) were derived from Johnson graphs. A general theory for growing regular graphs was established to some extent by Hora–Obata [15, 16]. In this paper we unify and generalize the above mentioned works. Theorems 7.1 and 7.2 are our main results.

Relevant topics are found in Accardi–Ben Ghorbal–Obata [1], Hashimoto [4], Hora [12, 13, 14], Obata [19, 20]. For a comprehensive account of our approach, see the forthcoming monograph Hora–Obata [17].

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1 Main Problem

Let $\mathcal{G} = (V, E)$ be a graph, where V is a non-empty set of *vertices* and E is a set of *edges*, i.e., $E \subset \{\{x, y\}\,;\, x, y \in V,\, x \neq y\}$. Two vertices $x, y \in V$ are called *adjacent* if $\{x, y\} \in E$ and, in this case we write $x \sim y$. The graph structure is fully contained in the adjacency matrix $A = (A_{xy})$ defined by

$$A_{xy} = \begin{cases} 1, & x \sim y, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Apparently, A is symmetric. We are interested in spectral properties of A. If the graph is finite (i.e., $|V| < \infty$), the complete list of eigenvalues of A with multiplicities answers to our question. If not, A becomes an infinite matrix and analytical consideration is required. Below our question will be formulated in a more concrete form.

We start with some notions of graph theory. A finite sequence $x_0, x_1, \dots, x_n \in V$ is called a *walk* of length n (connecting x_0 and x_n) if $x_i \sim x_{i+1}$ for $i = 0, 1, \dots, n-1$. In a walk some points of $x_0, x_1, \dots, x_n$ may occur repeatedly. A graph is called *connected* if any pair of distinct points is connected by a walk. The *degree* or *valency* of a vertex $x \in V$ is defined by $\kappa(x) = |\{y \in V\,;\, y \sim x\}|$. A graph is called *locally finite* if $\kappa(x) < \infty$ for all $x \in V$, *uniformly locally finite* if $\sup\{\kappa(x)\,;\, x \in V\} < \infty$, and *regular* if $\kappa(x) = \kappa < \infty$ for all $x \in V$.

Convention. Throughout the paper, unless otherwise specified, a graph is always assumed to be connected and locally finite.

The adjacency algebra $\mathcal{A}(\mathcal{G})$ is well defined for the local finiteness. An element of $\mathcal{A}(\mathcal{G})$ is expressible in a polynomial in A with complex coefficients. The adjacency algebra is a commutative $*$-algebra (with the identity) and the adjacency matrix is symmetric, i.e., $A = A^*$.

Let $\langle\cdot\rangle$ be a state on $\mathcal{A}(\mathcal{G})$, that is, $a \mapsto \langle a\rangle \in \mathbf{C}$ is a linear function on $\mathcal{A}(\mathcal{G})$, which is positive ($\langle a^*a\rangle \geq 0$ for all $a \in \mathcal{A}(\mathcal{G})$) and normalized ($\langle 1_{\mathcal{A}}\rangle = 1$). It follows from Hamburger's theorem (see e.g., Chihara [3], Shohat–Tamarkin [21]) that there exists a probability distribution μ on $\mathbf{R}$ such that

$$\langle A^m\rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots .$$

Uniqueness of μ does not hold in general (known as the determinate moment problem). We call μ the *spectral distribution* of A in the given state.

Probably a more interesting question is to find the *asymptotic spectral distribution* of the adjacency matrix of a large graph or of a growing graph, namely, to find a probability distribution μ satisfying

$$\langle A^m\rangle \approx \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots ,$$

in a suitable asymptotic sense. The precise formulation will be given in Section 4.

In fact, up to now, our consideration has been restricted to the vacuum state and its deformation. In this paper we deal with the vacuum state only. Let $\mathcal{G} = (V, E)$ be a graph. Let $\ell^2(V)$ be the Hilbert space of square-summable functions on V and $C_0(V)$ the dense subspace of functions with finite supports. The inner product of $\ell^2(V)$ is defined by

$$\langle f, g\rangle = \sum_{x\in V} \overline{f(x)}\, g(x), \qquad f, g \in \ell^2(V).$$

For $x \in V$ define a function δ_x by

$$\delta_x(y) = \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\{\delta_x\,;\, x \in V\}$ becomes a complete orthonormal basis of $\ell^2(V)$ and $C_0(V)$ its linear span. The adjacency algebra $\mathcal{A}(\mathcal{G})$ acts in a natural manner on $C_0(V)$. By analogy of an interacting Fock space we give the following

Definition 1.1 Let $o \in V$ be a fixed origin of a graph $\mathcal{G} = (V, E)$. The vector state on $\mathcal{A}(\mathcal{G})$ defined by

$$\langle a\rangle_o = \langle \delta_o, a\delta_o\rangle, \qquad a \in \mathcal{A}(\mathcal{G}), \tag{1.2}$$

is called the *vacuum state* at $o \in V$.

It is noted that $\langle A^m\rangle_o$ is the number of m-step walks from $o \in V$ to itself. More generally, for $x, y \in V$, we see that $(A^m)_{xy} = \langle \delta_x, A^m\delta_y\rangle$ is the number of m-step walks connecting y and x.

2 Stratification and Quantum Decomposition

Let $\mathcal{G} = (V, E)$ be a graph. As soon as an origin $o \in V$ is chosen, a natural stratification (distance partition) is introduced:

$$V = \bigcup_{n=0}^{\infty} V_n, \qquad V_n = \{x \in V\,;\, \partial(o, x) = n\}. \tag{2.1}$$

If $V_m = \emptyset$ happens for some $m \geq 1$, then $V_n = \emptyset$ for all $n \geq m$. We then define three matrices A^+, A^-, A° by

$$(A^\epsilon)_{yx} = \begin{cases} A_{yx} = 1, & \text{if } y \sim x \text{ and } \partial(y, o) = \partial(x, o) + \epsilon, \\ 0, & \text{otherwise,} \end{cases} \qquad x, y \in V,$$

where ϵ takes values $+1, -1, 0$ according as $\epsilon = +, -, \circ$ (see Figure 1). Obviously,

$$(A^+)^* = A^-, \qquad (A^\circ)^* = A^\circ,$$

and

$$A = A^+ + A^- + A^\circ. \tag{2.2}$$

This is called the *quantum decomposition* of A and A^ϵ a *quantum component.* A quantum decomposition depends on the stratification (2.1), and hence on the choice of an origin $o \in V$. Let $\tilde{\mathcal{A}}(\mathcal{G})$ be the $*$-algebra generated by $\{A^+, A^-, A^\circ\}$. (Since the quantum components A^ϵ are locally finite matrices, their products are defined entry-wise.) Apparently, $\tilde{\mathcal{A}}(\mathcal{G})$ is a non-commutative extension of the adjacency algebra $\mathcal{A}(\mathcal{G})$ and is called the *extended adjacency algebra.*

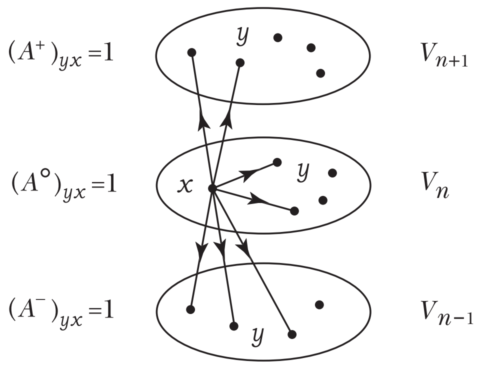

Figure 1: Quantum decomposition: $A = A^+ + A^- + A^\circ$

Given a stratification (2.1), we next define an orthonormal set in $\ell^2(V)$. For each $n \geq 0$ with $V_n \neq \emptyset$ we set

$$\Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} \delta_x. \tag{2.3}$$

Let $\Gamma(\mathcal{G}) \subset \ell^2(V)$ be the subspace spanned by $\{\Phi_n\}$. Let us observe that $\Gamma(\mathcal{G})$ is not necessarily kept invariant under the actions of the quantum components. For $x \in V$ and $\epsilon \in \{+, -, \circ\}$ we define

$$\omega_\epsilon(x) = \{y \in V\,;\, y \sim x,\ \partial(o,y) = \partial(o,x) + \epsilon\}. \tag{2.4}$$

In other words, $\omega_\epsilon(x)$ is the set of vertices which are adjacent to x and lie in the upper, lower or level stratum according as $\epsilon = +, -, \circ$. Obviously,

$$\kappa(x) = |\omega_+(x)| + |\omega_-(x)| + |\omega_\circ(x)|, \qquad x \in V.$$

It follows from definition that

$$|V_n|^{1/2} A^+ \Phi_n = \sum_{x \in V_n} A^+ \delta_x = \sum_{y \in V_{n+1}} |\omega_-(y)|\, \delta_y,$$

and hence

$$A^+ \Phi_n = |V_n|^{-1/2} \sum_{y \in V_{n+1}} |\omega_-(y)|\, \delta_y. \tag{2.5}$$

In a similar fashion we obtain

$$A^- \Phi_n = |V_n|^{-1/2} \sum_{y \in V_{n-1}} |\omega_+(y)|\, \delta_y, \tag{2.6}$$

$$A^\circ \Phi_n = |V_n|^{-1/2} \sum_{y \in V_n} |\omega_\circ(y)|\, \delta_y. \tag{2.7}$$

It is then obvious from (2.5)–(2.7) that $\Gamma(\mathcal{G})$ is invariant if and only if $|\omega_\epsilon(y)|$ is constant on each stratum V_n. This paper is devoted to the following two cases:

(i) $\Gamma(\mathcal{G})$ is invariant under the quantum components of A;

(ii) $\Gamma(\mathcal{G})$ is asymptotically invariant under the quantum components of A.

3 Case of $\Gamma(\mathcal{G})$ Being Invariant

The following definition is useful.

Definition 3.1 A pair of sequences $(\{\omega_n\}, \{\alpha_n\})$ is called a *Jacobi coefficient* if (i) $\{\omega_n\,;\, n = 1, 2, \dots\}$ is an infinite sequence of positive numbers and $\{\alpha_n\,;\, n = 1, 2, \dots\}$ is an infinite sequence of real numbers; or (ii) there exists $m_0 \ge 1$ such that $\{\omega_n\,;\, n = 1, 2, \dots, m_0 - 1\}$ is a finite sequence of positive numbers (or an empty sequence if $m_0 = 1$) and $\{\alpha_n\,;\, n = 1, 2, \dots, m_0\}$ is a finite sequence of real numbers.

Let $\mathcal{G} = (V, E)$ be a graph. Given a fixed origin $o \in V$, we consider the quantum decomposition of the adjacency matrix $A = A^+ + A^- + A^\circ$ and a dense subspace $\Gamma(\mathcal{G}) \subset \ell^2(V)$ spanned by the unit vectors $\{\Phi_n\}$ defined in (2.3).

Lemma 3.2 *If $\Gamma(\mathcal{G})$ is invariant under the quantum components A^ϵ, $\epsilon \in \{+, -, \circ\}$, there exists a Jacobi coefficient $(\{\omega_n\}, \{\alpha_n\})$ such that*

$$A^+\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1}, \quad n = 0, 1, 2, \dots, \tag{3.1}$$

$$A^-\Phi_0 = 0, \quad A^-\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1}, \quad n = 1, 2, \dots, \tag{3.2}$$

$$A^\circ\Phi_n = \alpha_{n+1}\Phi_n, \quad n = 0, 1, 2, \dots. \tag{3.3}$$

In particular, $(\Gamma(\mathcal{G}), \{\Phi_n\}, A^+, A^-)$ is an interacting Fock space associated with $\{\omega_n\}$.

The proof is easy from (2.5)–(2.7). In fact,

$$\omega_1 = \kappa(o), \qquad \alpha_1 = 0. \tag{3.4}$$

Then, the spectral distribution of A in the vacuum state is obtained directly from the general theory of an interacting Fock space.

Theorem 3.3 *Notations and assumptions being as in Lemma 3.2, let μ be the spectral distribution of A in the vacuum state, i.e.,*

$$\langle A^m \rangle_o = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \dots.$$

Then, the orthogonal polynomials $\{P_n(x) = x^n + \dots\}$ associated with μ obey the following three-term recurrence relation:

$$\begin{aligned}
&P_0(x) = 1, \\
&P_1(x) = x - \alpha_1, \\
&xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x), \qquad n = 1, 2, \dots.
\end{aligned}$$

Moreover, if μ is the solution of a determinate moment problem, the Stieltjes transform of μ admits a continued fraction expansion:

$$\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z - x} = \frac{1}{z - \alpha_1 -} \frac{\omega_1}{z - \alpha_2 -} \frac{\omega_2}{z - \alpha_3 -} \frac{\omega_3}{z - \alpha_4 - \cdots}, \tag{3.5}$$

which converges in $\{\operatorname{Im} z \neq 0\}$.

Remark 3.4 With a probability distribution μ on $\mathbf{R}$ which has finite moments of all orders we associate a Jacobi coefficient defined through the orthogonal polynomials. Then we have a surjective map from such probability distributions onto the set of Jacobi coefficients. But this map is not injective. If the counter image of a Jacobi coefficient consists of a single μ, we say that μ is the solution of a determinate moment problem. A simple sufficient condition for this is that $\omega_n = O((n \log n)^2)$ and $\alpha_n = O(n \log n)$. When a Jacobi coefficient is of finite type, the moment problem is determinate and μ is a finite sum of δ-measures. If μ has a compact support, it is the solution of a determinate moment problem.

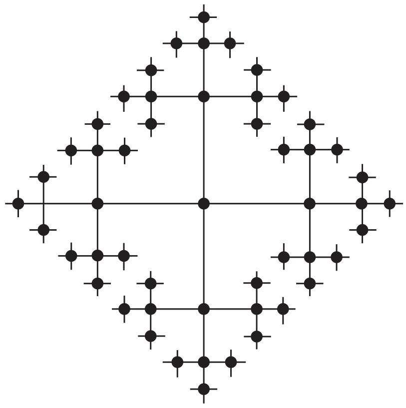

Figure 2: $\mathcal{T}_4$: Homogeneous tree of degree 4

A homogeneous tree is an instructive example of our method. Let $\kappa \geq 2$ and consider a homogeneous tree $\mathcal{T}_\kappa$ of degree κ. Let $A = A_\kappa$ be the adjacency matrix.

Following the general theory mentioned above, we fix an origin o and define a Hilbert space $\Gamma(\mathcal{T}_\kappa)$ with a complete orthonormal basis $\{\Phi_n \,;\, n = 0, 1, 2, \dots\}$. By direct observation we obtain

$$\begin{aligned} A\Phi_0 &= \sqrt{\kappa}\,\Phi_1, \\ A\Phi_1 &= \sqrt{\kappa}\,\Phi_0 + \sqrt{\kappa-1}\,\Phi_2, \\ A\Phi_n &= \sqrt{\kappa-1}\,\Phi_{n-1} + \sqrt{\kappa-1}\,\Phi_{n+1}. \quad n = 2, 3, \dots, \end{aligned}$$

In other words,

$$A^+\Phi_0 = \sqrt{\kappa}\,\Phi_1, \quad A^+\Phi_n = \sqrt{\kappa-1}\,\Phi_{n+1}, \quad n = 1, 2, \dots, \tag{3.6}$$

$$A^-\Phi_0 = 0, \quad A^-\Phi_1 = \sqrt{\kappa}\,\Phi_0, \quad A^-\Phi_n = \sqrt{\kappa-1}\,\Phi_{n-1}, \quad n = 2, 3, \dots, \tag{3.7}$$

$$A^\circ\Phi_n = 0, \quad n = 0, 1, 2, \dots. \tag{3.8}$$

Therefore $\Gamma(\mathcal{T}_\kappa)$ is invariant under the quantum components of A and Lemma 3.2 is applicable. The corresponding Jacobi coefficient is

$$\omega_1 = \kappa, \quad \omega_2 = \omega_3 = \cdots = \kappa - 1; \qquad \alpha_1 = \alpha_2 = \cdots = 0.$$

For this Jacobi coefficient the moment problem is determinate (see Remark 3.4). Let μ be the spectral distribution of A in the vacuum state and $G_\mu(z)$ its Stieltjes transform. It then follows from the second half of Theorem 3.3 that

$$\begin{aligned} G_\mu(z) &= \frac{1}{z-}\,\frac{\kappa}{z-}\,\frac{\kappa-1}{z}\,-\,\frac{\kappa-1}{z}\,-\cdots \\ &= -\frac{1}{2}\,\frac{-(\kappa-2)z + \kappa\sqrt{z^2 - 4(\kappa-1)}}{\kappa^2 - z^2}. \end{aligned}$$

Then, applying the Stieltjes inversion formula, the density function of the absolutely continuous part of μ is obtained:

$$\rho_\kappa(x) = -\frac{1}{\pi} \lim_{y \to +0} \operatorname{Im} G(x+iy) = \frac{\kappa}{2\pi} \frac{\sqrt{4(\kappa-1)-x^2}}{\kappa^2 - x^2}, \quad |x| \le 2\sqrt{\kappa-1}. \tag{3.9}$$

We can check easily that $\rho_\kappa(x)dx$ is a probability distribution. (In general, a pole of $G_\mu(z)$ may correspond to an atom.) Thus,

Proposition 3.5 *Let $\mathcal{T}_\kappa$ be a homogeneous tree of degree $\kappa \ge 2$. For the adjacency matrix A_κ we have*

$$\langle A_\kappa^m \rangle_o = \int_{-2\sqrt{\kappa-1}}^{+2\sqrt{\kappa-1}} x^m \rho_\kappa(x)\, dx, \qquad m = 1, 2, \dots, \tag{3.10}$$

where $\rho_\kappa(x)$ is given in (3.9).

Kesten [18] studied the Cayley graph of a free group on N generators (i.e., a homogeneous tree of degree $2N$) and obtained the distribution of the transition matrix $P_N = (2N)^{-1} A_{2N}$ in δ_e. This is a simple scaling transformation of (3.10). The probability distribution $\rho_\kappa(x)dx$ is called the *Kesten distribution* with parameter $\kappa, \kappa - 1$.

4 Asymptotic Spectral Distribution

Now we formulate the second question in Section 1. Consider a growing family of graphs

$$\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)}),$$

where a growing parameter ν runs over an ordered set. To avoid the trivial case we assume that $|V^{(\nu)}| \ge 2$ for all ν. Let A_ν denote the adjacency matrix of $\mathcal{G}^{(\nu)}$. Suppose that each adjacency algebra $\mathcal{A}(\mathcal{G}_\nu)$ is given a state $\langle \cdot \rangle_\nu$. Then there exists a probability measure μ_ν such that

$$\langle A_\nu^m \rangle_\nu = \int_{-\infty}^{+\infty} x^m \mu_\nu(dx), \qquad m = 1, 2, \dots.$$

Our interest lies in the behavior of μ_ν in the limit. However, as is suggested by limit theorems in probability theory, such a limit does not exist in general without suitable scaling. A natural normalization is given by

$$\frac{A_\nu - \langle A_\nu \rangle}{\Sigma(A_\nu)}, \qquad \Sigma^2(A_\nu) = \langle (A_\nu - \langle A_\nu \rangle)^2 \rangle. \tag{4.1}$$

(The suffix ν is cumbersome and is occasionally dropped .) Our aim is to find a probability measure μ satisfying

$$\lim_\nu \left\langle \left(\frac{A_\nu - \langle A_\nu \rangle}{\Sigma(A_\nu)} \right)^m \right\rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots. \tag{4.2}$$

The above μ is called the *asymptotic spectral distribution* of A_ν in the state $\langle \cdot \rangle_\nu$.

Again a homogeneous tree $\mathcal{T}_\kappa$ is a good illustration for this situation, where $\kappa \to \infty$ is a growing parameter. Let us find the asymptotic spectral distribution of A_κ in the vacuum state. Since

$$\langle A_\kappa \rangle_o = 0, \qquad \Sigma^2(A_\nu) = \langle A_\kappa^2 \rangle_o = \kappa,$$

the proper normalization of A_κ is given by

$$\frac{A_\kappa}{\sqrt{\kappa}} = \frac{A_\kappa^+}{\sqrt{\kappa}} + \frac{A_\kappa^-}{\sqrt{\kappa}}.$$

The actions of these quantum components are immediately obtained from (3.6)–(3.8) as follows:

$$\frac{A_\kappa^+}{\sqrt{\kappa}}\,\Phi_0 = \Phi_1, \quad \frac{A_\kappa^+}{\sqrt{\kappa}}\,\Phi_n = \sqrt{\frac{\kappa-1}{\kappa}}\,\Phi_{n+1}, \quad n = 1, 2, \dots, \tag{4.3}$$

$$\frac{A_\kappa^-}{\sqrt{\kappa}}\,\Phi_0 = 0, \quad \frac{A_\kappa^-}{\sqrt{\kappa}}\,\Phi_1 = \Phi_0, \quad \frac{A_\kappa^-}{\sqrt{\kappa}}\,\Phi_n = \sqrt{\frac{\kappa-1}{\kappa}}\,\Phi_{n-1}, \quad n = 2, 3, \dots, \tag{4.4}$$

We then see, at a formal level, that operators defined by

$$B^\pm = \lim_{\kappa\to\infty} \frac{A_\kappa^\pm}{\sqrt{\kappa}} \tag{4.5}$$

act on a Hilbert space Γ with a complete orthonormal basis $\{\Psi_n\}$ in such a way that

$$B^+\Psi_n = \Psi_{n+1}, \quad n = 0, 1, 2, \dots,$$
$$B^-\Psi_0 = 0, \quad B^-\Psi_n = \Psi_{n-1}, \quad n = 1, 2, \dots.$$

These are nothing but the actions of the annihilation and creation operators of a free Fock space, namely, $(\Gamma, \{\Psi_n\}, B^+, B^-)$ is a free Fock space. Recall that a free Fock space is an interacting Fock space associated with a Jacobi sequence $\{\omega_n \equiv 1\}$.

Thus, (4.5) means that the normalized quantum components of A_κ "converges" to the annihilation and creation operators in the free Fock space. Remind that for a different κ, the operators $A_\kappa^\pm$ act on a different space $\Gamma(\mathcal{T}_\kappa)$. So we need to give the precise meaning of the limit in (4.5). According to the standard notion in quantum probability theory, we give the following

Definition 4.1 Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing graph and A_ν the adjacency matrix. Assume that for each ν we are given a normalized linear function $\langle\cdot\rangle_\nu$ and consider the normalized quantum components:

$$C_\nu^\pm = \frac{A_\nu^\pm}{\Sigma(A_\nu)}, \qquad C_\nu^\circ = \frac{A_\nu^\circ - \langle A_\nu\rangle_\nu}{\Sigma(A_\nu)}, \qquad \Sigma^2(A_\nu) = \langle (A_\nu - \langle A_\nu\rangle)^2\rangle.$$

(Here we do not assume the positivity of $\langle\cdot\rangle_\nu$ but require $\Sigma^2(A_\nu) > 0$.) Let $\Gamma_{\{\omega_n\}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ be an interacting Fock space associated with Jacobi sequence $\{\omega_n\}$ and B° a diagonal operator. We say that the normalized quantum components C_ν^ϵ of the adjacency matrix converge stochastically to B^ϵ if

$$\lim_\nu \langle C_\nu^{\epsilon_m} \dots C_\nu^{\epsilon_2} C_\nu^{\epsilon_1}\rangle_\nu = \langle B^{\epsilon_m} \dots B^{\epsilon_2} B^{\epsilon_1}\rangle \tag{4.6}$$

for any choice of $\epsilon_1, \epsilon_2, \dots, \epsilon_m \in \{+, -, \circ\}$, $m = 1, 2, \dots$, where $\langle\cdot\rangle$ in the right hand side of (4.6) is a normalized linear function on $\Gamma_{\{\omega_n\}}$.

If (4.6) holds, as a classical reduction we have

$$\lim_{\nu} \left\langle \left(\frac{A_\nu - \langle A_\nu \rangle}{\Sigma(A_\nu)} \right)^m \right\rangle_\nu = \langle (B^+ + B^- + B^\circ)^m \rangle, \qquad m = 1, 2, \dots .$$

Moreover, if $\langle \cdot \rangle_\nu$ is a state on $\mathcal{A}(\mathcal{G}^{(\nu)})$ for each ν, then the normalized linear function in the limit is also a state on the $*$-algebra generated by $B^+ + B^- + B^\circ$. Hence there exists a probability distribution μ such that

$$\langle (B^+ + B^- + B^\circ)^m \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots .$$

This μ is the asymptotic spectral distribution which we wanted to investigate in (4.2).

Going back to the growing family of homogeneous trees, we can verify easily the convergence of (4.5) in the sense of Definition 4.1.

Proposition 4.2 *Let A_κ be the adjacency matrix of a homogeneous tree $\mathcal{T}_\kappa$ of degree $\kappa \ge 2$ and regard it as an algebraic random variable equipped with the vacuum state at a fixed origin $o_\kappa \in \mathcal{T}_\kappa$. Then, in the sense of stochastic convergence we have*

$$\lim_{\kappa \to \infty} \frac{A_\kappa^\pm}{\sqrt{\kappa}} = B_{\mathrm{free}}^\pm,$$

where $B_{\mathrm{free}}^\pm$ are the annihilation and creation operators of the free Fock space equipped with the vacuum state. In particular,

$$\lim_{\kappa \to \infty} \left\langle \left(\frac{A_\kappa}{\sqrt{\kappa}} \right)^m \right\rangle = \frac{1}{2\pi} \int_{-2}^{+2} x^m \sqrt{4 - x^2}\, dx, \qquad m = 1, 2, \dots, \tag{4.7}$$

where the probability distribution in the right hand side is the Wigner semicircle law.

Remark 4.3 We may obtain the limit distribution (4.7), i.e., the Wigner semicircle law, directly from the explicit form of the spectral distribution of A_κ described in Proposition 3.5. The above argument of stochastic convergence does not require an explicit form of the spectral distribution of A_κ.

Remark 4.4 Proposition 4.2 is a prototype of the *free central limit theorem* initiated by Voiculescu, see Hiai–Petz [8], Voiculescu–Dykema–Nica [22]. In fact, the adjacency matrix of a homogeneous tree of even degree (the Cayley graph of a free group) admits a natural decomposition into a sum of free independent random variables. There are several different notions of independence in quantum probability theory and their application to spectral analysis of graphs has been developed recently, for a brief review see Obata [20].

5 Case of $\Gamma(\mathcal{G})$ Being Asymptotic Invariant: An Example

We continue the study of asymptotic spectral distribution of the adjacency matrix A_ν of a growing graph $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$. If $\Gamma(\mathcal{G}^{(\nu)})$ is invariant under the quantum components of A_ν, as is illustrated by homogeneous trees, the method of quantum decomposition and stochastic convergence is fully applicable. Recall that, as mentioned in Remark 4.3, for the asymptotic spectral distribution we do not need to use an explicit form of the spectral distribution of A_ν. This suggests that our method is applicable even when $\Gamma(\mathcal{G}_\nu)$ is not invariant under the actions of quantum components of A_ν. We only need to assume "asymptotic invariance" of $\Gamma(\mathcal{G}_\nu)$. We shall illustrate this situation by an example.

For $N = 1, 2, \dots$ consider the N-dimensional integer lattice:

$$\mathbf{Z}^N = \{x = p_1e_1 + \cdots + p_Ne_N \,;\, p_1, \dots, p_N \in \mathbf{Z}\},$$

where $\{e_i\}$ is the canonical basis. Taking $o = (0, 0, \dots, 0)$ to be the origin, we introduce the stratification:

$$\mathbf{Z}^N = \bigcup_{n=0}^{\infty} V_n, \qquad V_n = \{x \in \mathbf{Z}^N \,;\, \partial(x, o) = n\}.$$

Keep in mind that $\partial(x, y)$ is not the Euclidean distance but the graph distance. As usual, for $n = 0, 1, 2, \dots$ we define a unit vector $\Phi_n \in \ell^2(\mathbf{Z}^N)$ by

$$\Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} \delta_x$$

and introduce their linear span $\Gamma(\mathbf{Z}^N)$. On the other hand, according to the stratification the adjacency matrix $A = A_N$ admits a quantum decomposition:

$$A = A^+ + A^-.$$

Note that $A^\circ = 0$ because there is no edge lying in a same stratum V_n, which is verified also from $\partial(o, x) = |p_1| + \cdots + |p_N|$ for $x = p_1e_1 + \cdots + p_Ne_N$.

We shall observe the actions of $A^\pm$ on Φ_n. By definition,

$$|V_n|^{1/2} A^+ \Phi_n = \sum_{x \in V_n} A^+ \delta_x = \sum_{y \in V_{n+1}} |\omega_-(y)| \delta_y. \tag{5.1}$$

It is apparent that $|\omega_-(y)|$ is not constant for all $y \in V_{n+1}$ but so is for "almost all" y. In fact, for a large N, a typical $y \in V_{n+1}$ is obtained by a walk from o by taking a different direction at each step. Namely,

$$|\{y \in V_{n+1} \,;\, \omega_-(y) = n+1\}| = \binom{N}{n+1} 2^{n+1} = \frac{(2N)^{n+1}}{(n+1)!} + O(N^n),$$
$$|\{y \in V_{n+1} \,;\, \omega_-(y) < n+1\}| = O(N^n).$$

On the other hand,

$$|V_n| = \binom{N}{n} 2^n + O(N^{n-1}) = \frac{(2N)^n}{n!} + O(N^{n-1}).$$

Then (5.1) becomes

$$A^+ \Phi_n = \sqrt{2N}\,\sqrt{n+1}\;\Phi_{n+1} + O(N^{-1/2}), \tag{5.2}$$

where $O(N^{-1/2})$ is in the sense of norm. Similarly,

$$A^- \Phi_n = \sqrt{2N}\,\sqrt{n}\;\Phi_{n-1} + O(N^{-1}). \tag{5.3}$$

From (5.2) and (5.3) we obtain

$$\begin{aligned} \frac{A^+}{\sqrt{2N}}\,\Phi_n &= \sqrt{n+1}\;\Phi_{n+1} + O(N^{-1}), \\ \frac{A^-}{\sqrt{2N}}\,\Phi_n &= \sqrt{n}\;\Phi_{n-1} + O(N^{-3/2}). \end{aligned} \tag{5.4}$$

For this situation we say that $\Gamma(\mathbf{Z}^N)$ is asymptotically invariant under the quantum components of A. Formally we set

$$\lim_{N\to\infty} \frac{A_N^\pm}{\sqrt{2N}} = B^\pm.$$

The actions of $B^\pm$ are clear from (5.4). They are the actions of the annihilation and creation operators of the Boson Fock space, i.e., an interacting Fock space associated with a Jacobi sequence $\{\omega_n = n\}$. Thus (5.4) means that the normalized quantum components converge to the annihilation and creation operators of the Boson Fock space. The convergence is justified in the sense of stochastic convergence (Definition 4.1).

Proposition 5.1 *Let A_N be the adjacency matrix of an N-dimensional integer lattice $\mathbf{Z}^N$, $N \ge 1$, and regard it as an algebraic random variable equipped with the vacuum state at a fixed origin $o \in \mathbf{Z}^N$. Then, in the sense of stochastic convergence we have*

$$\lim_{N\to\infty} \frac{A_N^\pm}{\sqrt{2N}} = B_{\mathrm{Boson}}^\pm,$$

where $B_{\mathrm{Boson}}^\pm$ is the annihilation and creation operators of the Boson Fock space equipped with the vacuum state. As a classical reduction we have

$$\lim_{N\to\infty} \left\langle \left(\frac{A_N}{\sqrt{2N}}\right)^m \right\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^m e^{-x^2/2} dx, \qquad m = 1, 2, \dots . \tag{5.5}$$

where the probability distribution in the right hand side is the standard Gaussian distribution.

Remark 5.2 Proposition 5.1 is a consequence of the classical ("commutative" in our context) central limit theorem. In fact, the adjacency matrix of $\mathbf{Z}^N$ is decomposed into a sum of commutative independent random variables, which become classical independent random variables through the Fourier transform.

6 Growing Regular Graphs

Suggested by Propositions 4.2, 5.1 and many other examples, we may abstract natural conditions for a growing regular graph in order that the asymptotic spectral distribution in the vacuum state (as well as in the deformed vacuum states) is derived along with a kind of quantum central limit theorem.

Before going into the growing regular graphs, we prepare some notations. Let $\mathcal{G} = (V, E)$ be a graph. Given a fixed origin $o \in V$, consider the stratification:

$$V = \bigcup_{n=0}^{\infty} V_n$$

and define as before

$$\omega_\epsilon(x) = \{y \in V\,;\, y \sim x,\ \partial(o,y) = \partial(o,x) + \epsilon\}, \qquad x \in V, \quad \epsilon \in \{+,-,\circ\}.$$

Some statistics of $|\omega_\epsilon(x)|$ plays a crucial role. We set

$$M(\omega_\epsilon|V_n) = \frac{1}{|V_n|}\sum_{x\in V_n}|\omega_\epsilon(x)|$$

$$\Sigma^2(\omega_\epsilon|V_n) = \frac{1}{|V_n|}\sum_{x\in V_n}\bigl\{|\omega_\epsilon(x)| - M(\omega_\epsilon|V_n)\bigr\}^2,$$

$$L(\omega_\epsilon|V_n) = \max\{|\omega_\epsilon(x)|\,;\, x \in V_n\}.$$

$M(\omega_\epsilon|V_n)$ is the mean value of $|\omega_\epsilon(x)|$ when x runs over V_n, and $\Sigma^2(\omega_\epsilon|V_n)$ its variance.

Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph, where the growing parameter ν runs over an infinite directed set. The degree of $\mathcal{G}^{(\nu)}$ is denoted by $\kappa(\nu)$. For each graph $\mathcal{G}^{(\nu)}$ we fix an origin $o_\nu \in V^{(\nu)}$ and consider as usual the stratification:

$$V^{(\nu)} = \bigcup_{n=0}^{\infty} V_n^{(\nu)}, \qquad V_n^{(\nu)} = \{y \in V^{(\nu)};\, \partial(o,y) = n\}. \tag{6.1}$$

($V_n^{(\nu)} = \emptyset$ may occur.) Then, for $n = 0, 1, 2, \dots$ we define a unit vector in $\ell^2(V^{(\nu)})$ by

$$\Phi_n^{(\nu)} = |V_n^{(\nu)}|^{-1/2}\sum_{x\in V_n^{(\nu)}}\delta_x. \tag{6.2}$$

Let $\Gamma(\mathcal{G}^{(\nu)})$ denote the linear span of $\{\Phi_0^{(\nu)}, \Phi_1^{(\nu)}, \dots\}$. Let A_ν denote the adjacency matrix of $\mathcal{G}^{(\nu)}$. According to the stratification (6.1) we have a *quantum decomposition*:

$$A_\nu = A_\nu^+ + A_\nu^- + A_\nu^\circ, \tag{6.3}$$

We do not assume that $\Gamma(\mathcal{G}^{(\nu)})$ is invariant under the actions of quantum components A_ν^ϵ, but we need asymptotic invariance. This requirement is fulfilled by natural conditions on how the graphs grow.

We consider the following conditions on a growing regular graph $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$.

(A1) $\lim_\nu \kappa(\nu) = \infty$;

(A2) for each $n = 1, 2, \dots$ there exists a limit

$$\omega_n = \lim_\nu M(\omega_-|V_n^{(\nu)}) < \infty. \tag{6.4}$$

Moreover,

$$\lim_\nu \Sigma^2(\omega_-|V_n^{(\nu)}) = 0, \tag{6.5}$$

$$\sup_\nu L(\omega_-|V_n^{(\nu)}) < \infty; \tag{6.6}$$

(A3) for each $n = 0, 1, 2, \dots$ there exists a limit

$$\alpha_{n+1} = \lim_\nu M\left(\frac{\omega_\circ}{\sqrt{\kappa(\nu)}} \,\middle|\, V_n^{(\nu)} \right) = \lim_\nu \frac{M(\omega_\circ | V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty. \tag{6.7}$$

Moreover,

$$\lim_\nu \Sigma^2\left(\frac{\omega_\circ}{\sqrt{\kappa(\nu)}} \,\middle|\, V_n^{(\nu)} \right) = \lim_\nu \frac{\Sigma^2(\omega_\circ | V_n^{(\nu)})}{\kappa(\nu)} = 0, \tag{6.8}$$

$$\sup_\nu \frac{L(\omega_\circ | V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty. \tag{6.9}$$

Remark 6.1 Condition (A2) for $n = 1$ and (A3) for $n = 0$ are always satisfied. Note also that $\omega_1 = 1$ and $\alpha_1 = 0$.

Remark 6.2 At a first glance the above conditions seem to be incomplete because there is no statement for the case of $V_n^{(\nu)} = \emptyset$. However, as is proved in Proposition 6.3 below, for each $n \geq 1$ we have $V_n^{(\nu)} = \emptyset$ for all large ν.

The meaning of (A1) is clear. It follows from condition (A2) that, in each stratum most of the vertices have the same number of downward edges independently of the growth and the statistical fluctuation (variance and range) of that number is controlled. Condition (A3) gives a similar restriction for level edges. The number of level edges may increase as the graph grows, but the growth rate is bounded by $\sqrt{\kappa(\nu)}$. Roughly speaking, as the graph grows keeping conditions (A1)–(A3), most of the new edges connect new vertices lying in a upper stratum with new and old ones in a lower stratum.

Proposition 6.3 *Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph satisfying conditions (A1)–(A3). Then, $(\{\omega_n\}, \{\alpha_n\})$ defined in these conditions is a Jacobi coefficient of infinite type.*

Proof. It is sufficient to show that $\omega_n \geq 1$ for all $n \geq 1$. For each ν we define

$$N(\nu) = \sup\{n \geq 1\,;\, V_n^{(\nu)} \neq \emptyset\}.$$

Note that $1 \leq N(\nu) \leq \infty$. We shall prove that $\lim_\nu N(\nu) = \infty$. Suppose otherwise. Then there exist an integer $N \geq 1$ and $\nu_1 < \nu_2 < \cdots \to \infty$ such that $N(\nu_i) \leq N$. Since $N(\nu_i)$ is an integer, we may assume $N(\nu_i) = N$ (by taking a subsequence and another N). Consider a vertex $x \in V_N^{(\nu_i)}$. Since x has no upward edge, we have

$$\kappa(\nu_i) = |\omega_-(x)| + |\omega_\circ(x)| \leq L(\omega_- | V_N^{(\nu_i)}) + L(\omega_\circ | V_N^{(\nu_i)}) = O(\sqrt{\kappa(\nu_i)}\,), \tag{6.10}$$

where the last estimate follows from (6.6) and (6.9). Then (6.10) causes contradiction against (A1). We have thus proved that $\lim_\nu N(\nu) = \infty$. In other words, for $n \geq 1$ there exists $\nu_0 = \nu_0(n)$ such that $V_n^{(\nu)} \neq \emptyset$ for all $\nu \geq \nu_0$, and hence $M(\omega_- | V_n^{(\nu)}) \geq 1$ because $|\omega_-(x)| \geq 1$ for all $x \in V_n^{(\nu)}$. Consequently, $\omega_n \geq 1$ for all $n \geq 1$. ∎

Some part of conditions (A1)–(A3) are rephrased in slightly different forms. Here we only prove the following

Proposition 6.4 *In the conditions (A1)–(A3), we may replace* (6.4), (6.5) *with a single condition: for each $n = 1, 2, \dots$ there exists a constant number ω_n independent of ν such that*

$$\lim_\nu \frac{|\{x \in V_n^{(\nu)}\,;\, |\omega_-(x)| = \omega_n\}|}{|V_n^{(\nu)}|} = 1. \tag{6.11}$$

Proof. Throughout the proof $n = 1, 2, \dots$ is fixed arbitrarily. We first prove that (6.11) implies (6.4) and (6.5). Divide $V_n^{(\nu)}$ into two parts:

$$U_{\mathrm{reg}}^{(\nu)} = \{x \in V_n^{(\nu)}\,;\, |\omega_-(x)| = \omega_n\}, \qquad U_{\mathrm{sing}}^{(\nu)} = \{x \in V_n^{(\nu)}\,;\, |\omega_-(x)| \neq \omega_n\},$$

where the index n is omitted for simplicity. The average of $|\omega_-(x)|$ is given by

$$\begin{aligned} M(\omega_-|V_n^{(\nu)}) &= \frac{1}{|V_n^{(\nu)}|}\Bigg(\sum_{x\in U_{\mathrm{reg}}^{(\nu)}} |\omega_-(x)| + \sum_{x\in U_{\mathrm{sing}}^{(\nu)}} |\omega_-(x)|\Bigg) \\ &= \frac{|U_{\mathrm{reg}}^{(\nu)}|}{|V_n^{(\nu)}|}\,\omega_n + \frac{1}{|V_n^{(\nu)}|}\sum_{x\in U_{\mathrm{sing}}^{(\nu)}} |\omega_-(x)|. \end{aligned}$$

In view of (6.6) we set

$$L_n = \sup_\nu L(\omega_-|V_n^{(\nu)}) < \infty.$$

Then $|\omega_-(x)| \le L_n$ for $x \in V_n^{(\nu)}$ and we obtain

$$|M(\omega_-|V_n^{(\nu)}) - \omega_n| \le \Bigg(1 - \frac{|U_{\mathrm{reg}}^{(\nu)}|}{|V_n^{(\nu)}|}\Bigg)\omega_n + \frac{|U_{\mathrm{sing}}^{(\nu)}|}{|V_n^{(\nu)}|}\,L_n \le \frac{|U_{\mathrm{sing}}^{(\nu)}|}{|V_n^{(\nu)}|}\,(\omega_n + L_n).$$

Since

$$\lim_\nu \frac{|U_{\mathrm{sing}}^{(\nu)}|}{|V_n^{(\nu)}|} = 0, \tag{6.12}$$

by (6.11), we obtain

$$\lim_\nu M(\omega_-|V_n^{(\nu)}) = \omega_n, \tag{6.13}$$

which proves (6.4). We next consider the variance. By Minkowski's inequality, we obtain

$$\begin{aligned} &\Sigma(\omega_-|V_n^{(\nu)}) \\ &= \Bigg\{\frac{1}{|V_n^{(\nu)}|}\sum_{x\in V_n^{(\nu)}} (|\omega_-(x)| - M(\omega_-|V_n^{(\nu)}))^2\Bigg\}^{1/2} \\ &\le \Bigg\{\frac{1}{|V_n^{(\nu)}|}\sum_{x\in V_n^{(\nu)}} (|\omega_-(x)| - \omega_n)^2\Bigg\}^{1/2} + \Bigg\{\frac{1}{|V_n^{(\nu)}|}\sum_{x\in V_n^{(\nu)}} (\omega_n - M(\omega_-|V_n^{(\nu)}))^2\Bigg\}^{1/2} \\ &= \Bigg\{\frac{1}{|V_n^{(\nu)}|}\sum_{x\in U_{\mathrm{sing}}^{(\nu)}} (|\omega_-(x)| - \omega_n)^2\Bigg\}^{1/2} + \big|\omega_n - M(\omega_-|V_n^{(\nu)})\big|. \end{aligned}$$

Since $||\omega_-(x)| - \omega_n| \le |\omega_-(x)| + \omega_n \le L_n + \omega_n$ for $x \in V_n^{(\nu)}$, we have

$$\Sigma(\omega_-|V_n^{(\nu)}) \le \left(\frac{|U_{\mathrm{sing}}^{(\nu)}|}{|V_n^{(\nu)}|} \right)^{1/2} (L_n + \omega_n) + \big|\omega_n - M(\omega_-|V_n^{(\nu)})\big|$$

and hence (6.5) follows by (6.12) and (6.13).

We next show that (6.11) is derived from (6.4) and (6.5). By (6.4), for any $\epsilon > 0$ there exists ν_0 such that

$$|M(\omega_-|V_n^{(\nu)}) - \omega_n| < \epsilon, \quad \nu \ge \nu_0.$$

If $x \in V_n^{(\nu)}$ satisfies $||\omega_-(x)| - \omega_n| \ge 2\epsilon$, we have

$$\big||\omega_-(x)| - M(\omega_-|V_n^{(\nu)})\big| \ge ||\omega_-(x)| - \omega_n| - |\omega_n - M(\omega_-|V_n^{(\nu)})| \ge \epsilon.$$

Hence

$$\frac{|\{x \in V_n^{(\nu)}\,;\, ||\omega_-(x)| - \omega_n| \ge 2\epsilon\}|}{|V_n^{(\nu)}|} \le \frac{|\{x \in V_n^{(\nu)}\,;\, \big||\omega_-(x)| - M(\omega_-|V_n^{(\nu)})\big| \ge \epsilon\}|}{|V_n^{(\nu)}|}.$$

By Chebyshev's inequality and (6.5) we have

$$\frac{|\{x \in V_n^{(\nu)}\,;\, ||\omega_-(x)| - \omega_n| \ge 2\epsilon\}|}{|V_n^{(\nu)}|} \le \frac{\Sigma^2(\omega_-|V_n^{(\nu)})}{\epsilon^2} \to 0, \qquad \nu \to \infty. \tag{6.14}$$

We prove that ω_n is an integer. Suppose otherwise. Then, since $|\omega_-(x)|$ is always an integer, we can choose a sufficiently small $\epsilon > 0$ such that

$$V_n^{(\nu)} = \{x \in V_n^{(\nu)}\,;\, ||\omega_-(x)| - \omega_n| \ge 2\epsilon\}.$$

But this contradicts (6.14) and hence ω_n is an integer. Since $|\omega_-(x)|$ and ω_n are all integers, we may choose a sufficiently small $\epsilon > 0$ such that

$$\frac{|\{x \in V_n^{(\nu)}\,;\, |\omega_-(x)| \ne \omega_n\}|}{|V_n^{(\nu)}|} = \frac{|\{x \in V_n^{(\nu)}\,;\, ||\omega_-(x)| - \omega_n| \ge 2\epsilon\}|}{|V_n^{(\nu)}|}. \tag{6.15}$$

As is shown in (6.14), the right hand side of (6.15) tends to 0 as $\nu \to \infty$. Therefore

$$\lim_{\nu} \frac{|\{x \in V_n^{(\nu)}\,;\, \omega_-(x) \ne \omega_n\}|}{|V_n^{(\nu)}|} = 0$$

and (6.11) follows. ∎

During the above proof we have observed the following

Proposition 6.5 *Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph satisfying conditions (A1)–(A3). Then, the Jacobi sequence $\{\omega_n\}$ defined therein consists of positive integers.*

7 Quantum Central Limit Theorem in Vacuum State

Theorem 7.1 *Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing graph satisfying conditions (A1)–(A3) and A_ν its adjacency matrix. Let $(\Gamma, \{\Psi_n\}, B^+, B^-)$ be the interacting Fock space associated with $\{\omega_n\}$ and B° the diagonal operator associated with $\{\alpha_n\}$, where $\{\omega_n\}$ and $\{\alpha_n\}$ are given in conditions (A1)–(A3). Then we have*

$$\lim_\nu \frac{A_\nu^\epsilon}{\sqrt{\kappa(\nu)}} = B^\epsilon, \qquad \epsilon \in \{+, -, \circ\},$$

in the sense of stochastic convergence with respect to the vacuum states in both sides.

As a classical reduction, we obtain immediately the following

Theorem 7.2 *Notations and assumptions being the same as in Theorem 7.1, let μ be a probability distribution of which the Jacobi coefficient is $(\{\omega_n\}, \{\alpha_n\})$. Then μ is the asymptotic spectral distribution in the vacuum state, i.e.,*

$$\lim_\nu \left\langle \left(\frac{A_\nu}{\sqrt{\kappa(\nu)}} \right)^m \right\rangle_o = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 0, 1, 2, \dots .$$

We shall sketch the proof of Theorem 7.1. For the time being, let us consider a single regular graph $\mathcal{G} = (V, E)$ with degree κ. As usual, fix an origin $o \in V$ and consider the stratification.

Lemma 7.3 *For any $n = 0, 1, 2, \dots$ with $V_n \neq \emptyset$ we have*

$$M(\omega_-|V_{n+1})|V_{n+1}| = \kappa|V_n| \left(1 - \frac{M(\omega_-|V_n)}{\kappa} - \frac{M(\omega_\circ|V_n)}{\kappa} \right). \tag{7.1}$$

Proof. Suppose first that $V_{n+1} \neq \emptyset$. Since $|\omega_+(x)| + |\omega_-(x)| + |\omega_\circ(x)| = \kappa$ for all $x \in V$, we have

$$\begin{aligned} \kappa|V_n| &= \sum_{x \in V_n} |\omega_+(x)| + \sum_{x \in V_n} |\omega_-(x)| + \sum_{x \in V_n} |\omega_\circ(x)| \\ &= \sum_{y \in V_{n+1}} |\omega_-(y)| + \sum_{x \in V_n} |\omega_-(x)| + \sum_{x \in V_n} |\omega_\circ(x)| \\ &= M(\omega_-|V_{n+1})|V_{n+1}| + M(\omega_-|V_n)|V_n| + M(\omega_\circ|V_n)|V_n|. \end{aligned} \tag{7.2}$$

This proves (7.1). If $V_{n+1} = \emptyset$, then the first term in (7.2) is zero and (7.1) remains true understanding the left hand side is zero. ∎

Lemma 7.4 *Let $n = 1, 2, \dots$. If $V_n \neq \emptyset$, then $M(\omega_-|V_k) \geq 1$ for $k = 1, 2, \dots, n$ and*

$$|V_n| = \frac{\kappa^n}{\prod_{k=1}^n M(\omega_-|V_k)} + O(\kappa^{n-1}),$$

where $O(\kappa^{n-1})$ is a polynomial in κ of degree $(n-1)$.

Proof. An immediate consequence by repeated application of (7.1). ∎

The explicit actions of the quantum components A^ϵ are given in (2.5)–(2.7). Inserting the mean values $M(\omega_\epsilon|V_n)$ therein, we obtain

$$A^+\Phi_n = M(\omega_-|V_{n+1})\left(\frac{|V_{n+1}|}{|V_n|}\right)^{1/2}\Phi_{n+1} + \frac{1}{\sqrt{|V_n|}}\sum_{y\in V_{n+1}}(|\omega_-(y)| - M(\omega_-|V_{n+1}))\delta_y,$$
$$A^-\Phi_n = M(\omega_+|V_{n-1})\left(\frac{|V_{n-1}|}{|V_n|}\right)^{1/2}\Phi_{n-1} + \frac{1}{\sqrt{|V_n|}}\sum_{y\in V_{n-1}}(|\omega_+(y)| - M(\omega_+|V_{n-1}))\delta_y,$$
$$A^\circ\Phi_n = M(\omega_\circ|V_n)\Phi_n + \frac{1}{\sqrt{|V_n|}}\sum_{y\in V_n}(|\omega_\circ(y)| - M(\omega_\circ|V_n))\delta_y,$$

for $n = 0, 1, 2, \dots$, understanding that $A^-\Phi_0 = 0$ for the second formula. It is convenient to unify the above three formulae. We set

$$\gamma_n^+ = M(\omega_-|V_n)\left(\frac{|V_n|}{\kappa|V_{n-1}|}\right)^{1/2}, \quad n = 1, 2, \dots, \tag{7.3}$$
$$\gamma_n^- = M(\omega_+|V_n)\left(\frac{|V_n|}{\kappa|V_{n+1}|}\right)^{1/2}, \quad n = 0, 1, 2, \dots, \tag{7.4}$$
$$\gamma_n^\circ = \frac{M(\omega_\circ|V_n)}{\sqrt{\kappa}}, \quad n = 0, 1, 2, \dots, \tag{7.5}$$

and

$$S_n^+ = \frac{1}{\sqrt{\kappa|V_{n-1}|}}\sum_{y\in V_n}(|\omega_-(y)| - M(\omega_-|V_n))\delta_y, \quad n = 1, 2, \dots,$$
$$S_n^- = \frac{1}{\sqrt{\kappa|V_{n+1}|}}\sum_{y\in V_n}(|\omega_+(y)| - M(\omega_+|V_n))\delta_y, \quad n = 0, 1, 2, \dots,$$
$$S_n^\circ = \frac{1}{\sqrt{\kappa|V_n|}}\sum_{y\in V_n}(|\omega_\circ(y)| - M(\omega_\circ|V_n))\delta_y, \quad n = 0, 1, 2, \dots.$$

We tacitly set

$$\gamma_{-1}^-\Phi_{-1} = S_{-1}^- = 0.$$

With these notations we have

$$\frac{A^\epsilon}{\sqrt{\kappa}}\Phi_n = \gamma_{n+\epsilon}^\epsilon\Phi_{n+\epsilon} + S_{n+\epsilon}^\epsilon, \qquad \epsilon\in\{+,-,\circ\}, \quad n = 0, 1, 2, \dots. \tag{7.6}$$

Then its repeated action is expressible in a concise form:

$$\frac{A^{\epsilon_m}}{\sqrt{\kappa}}\cdots\frac{A^{\epsilon_1}}{\sqrt{\kappa}}\Phi_n = \gamma_{n+\epsilon_1}^{\epsilon_1}\gamma_{n+\epsilon_1+\epsilon_2}^{\epsilon_2}\cdots\gamma_{n+\epsilon_1+\dots+\epsilon_m}^{\epsilon_m}\Phi_{n+\epsilon_1+\dots+\epsilon_m}$$
$$+ \sum_{k=1}^m \underbrace{\gamma_{n+\epsilon_1}^{\epsilon_1}\cdots\gamma_{n+\epsilon_1+\dots+\epsilon_{k-1}}^{\epsilon_{k-1}}}_{(k-1)\text{ times}}\underbrace{\frac{A^{\epsilon_m}}{\sqrt{\kappa}}\cdots\frac{A^{\epsilon_{k+1}}}{\sqrt{\kappa}}}_{(m-k)\text{ times}}S_{n+\epsilon_1+\dots+\epsilon_k}^{\epsilon_k}. \tag{7.7}$$

By observing the up-down actions of A^ϵ we see immediately that

$$\frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} \Phi_n = 0$$

unless

$$n+\epsilon_1 \ge 0, \quad n+\epsilon_1+\epsilon_2 \ge 0, \quad \dots, \quad n+\epsilon_1+\epsilon_2+\cdots+\epsilon_m \ge 0. \tag{7.8}$$

We need to estimate the error term of (7.7). Let us recall

$$L(\omega_-|V_k) = \max\{|\omega_-(x)|\,;\, x\in V_k\}, \qquad L(\omega_\circ|V_k) = \max\{|\omega_\circ(x)|\,;\, x\in V_k\}.$$

Then, for $n, q = 1, 2, \dots$ and $q = 1, 2, \dots$ we define $M^-_{n,q}$ by

$$M^-_{n,q} = \max\left\{ \prod_{j=1}^q L(\omega_-|V_{k_j})\,;\, 1 \le k_1, k_2, \dots, k_q \le n \right\}, \tag{7.9}$$

and set $M^-_{n,0} = 1$. Similarly, (taking condition (A3) in mind) we set

$$M^\circ_{n,q} = \max\left\{ \prod_{j=1}^q \frac{L(\omega_\circ|V_{k_j})}{\sqrt{\kappa}}\,;\, 1 \le k_1, k_2, \dots, k_q \le n \right\}, \tag{7.10}$$

and set $M^\circ_{n,0} = 1$.

Lemma 7.5 *Let $\epsilon_1, \dots, \epsilon_m \in \{+, -, \circ\}$, $m \ge 1$, be given arbitrarily. Let p, q and r be the numbers of $+$, $-$ and $\circ$ in $\{\epsilon_1, \dots, \epsilon_m\}$, respectively. Then for any $n \ge 1$ with $n+p-q \ge 0$ we have*

$$\left| \left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S^+_n \right\rangle \right| \le \Sigma(\omega_-|V_n) M^-_{n+p,q} M^\circ_{n+p,r} \frac{\kappa^{p+\frac{r-m-1}{2}} |V_n|}{\sqrt{|V_{n+p-q}||V_{n-1}|}}, \tag{7.11}$$

$$\left| \left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S^-_n \right\rangle \right|$$

$$\le \{\Sigma(\omega_-|V_n) + \Sigma(\omega_\circ|V_n)\} M^-_{n+p,q} M^\circ_{n+p,r} \frac{\kappa^{p+\frac{r-m-1}{2}} |V_n|}{\sqrt{|V_{n+p-q}||V_{n+1}|}}, \tag{7.12}$$

$$\left| \left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S^\circ_n \right\rangle \right| \le \Sigma(\omega_\circ|V_n) M^-_{n+p,q} M^\circ_{n+p,r} \frac{\kappa^{p+\frac{r-m-1}{2}} \sqrt{|V_n|}}{\sqrt{|V_{n+p-q}|}}. \tag{7.13}$$

Proof. We only show the outline for (7.11). It is sufficient to prove the assertion under (7.8), since otherwise the left hand side of (7.11) vanishes. Note first

$$\begin{aligned} \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S^+_n &= \frac{1}{(\kappa|V_{n-1}|)^{1/2}} \sum_{y\in V_n} (|\omega_-(y)| - M(\omega_-|V_n)) \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} \delta_y \\ &= \frac{\kappa^{-m/2}}{(\kappa|V_{n-1}|)^{1/2}} \sum_{y\in V_n} (|\omega_-(y)| - M(\omega_-|V_n)) A^{\epsilon_m} \cdots A^{\epsilon_1} \delta_y. \end{aligned} \tag{7.14}$$

We use a new notation. For $y, z \in V$ and $\epsilon \in \{+, -, \circ\}$ we write $y \xrightarrow{\epsilon} z$ if $z \sim y$ and $\partial(z, o) = \partial(y, o) + \epsilon$. For $y, z \in V$ we put

$$\begin{aligned} &w(y; \epsilon_1, \dots, \epsilon_m; z) \\ &\qquad = |\{(z_1, \dots, z_{m-1}) \in V^{m-1}\,;\, y \xrightarrow{\epsilon_1} z_1 \xrightarrow{\epsilon_2} z_2 \cdots \xrightarrow{\epsilon_{m-1}} z_{m-1} \xrightarrow{\epsilon_m} z\}|. \end{aligned}$$

This counts the walks from y to z along edges with directions $\epsilon_1, \dots, \epsilon_m$. Then (7.14) becomes

$$\frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S_n^+ = \frac{\kappa^{-m/2}}{(\kappa |V_{n-1}|)^{1/2}} \sum_{y \in V_n} \sum_{z \in V_{n+p-q}} (|\omega_-(y)| - M(\omega_-|V_n)) w(y; \epsilon_1, \dots, \epsilon_m; z) \delta_z.$$

Therefore,

$$\begin{aligned} &\left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S_n^+ \right\rangle \\ &= \frac{1}{|V_{n+p-q}|^{1/2}} \frac{\kappa^{-m/2}}{(\kappa |V_{n-1}|)^{1/2}} \sum_{y \in V_n} \sum_{z \in V_{n+p-q}} (|\omega_-(y)| - M(\omega_-|V_n)) w(y; \epsilon_1, \dots, \epsilon_m; z). \end{aligned} \tag{7.15}$$

Let $y \in V_n$ be fixed. Then

$$\sum_{z \in V_{n+p-q}} w(y; \epsilon_1, \dots, \epsilon_m; z) \tag{7.16}$$

coincides with the number of walks from y to a certain point in V_{n+p-q} along m edges with directions $\epsilon_1, \dots, \epsilon_m$ in order. Consider an intermediate point $\xi \in V_k$ in such a walk. The number of edges from ξ with $-$ direction is bounded by $L(\omega_-|V_k)$, with $\circ$ direction by $L(\omega_\circ|V_k)$, and with $+$ direction by κ. Given $(\epsilon_1, \dots, \epsilon_m)$, $+$, $-$ and $\circ$ directions appear p, q and r times, respectively, and the intermediate point ξ lie in $V_0 \cup V_1 \cup \dots \cup V_{n+p}$. Hence by (7.9) and (7.10) we obtain

$$\sum_{z \in V_{n+p-q}} w(y; \epsilon_1, \dots, \epsilon_m; z) \le \kappa^{p+\frac{r}{2}} M^-_{n+p,q} M^\circ_{n+p,r},$$

where the right hand side is independent of $y \in V_n$. Now we come to an estimate of (7.15). In fact,

$$\begin{aligned} &\left| \left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S_n^+ \right\rangle \right| \\ &\qquad \le \frac{\kappa^{p+\frac{r}{2}} M^-_{n+p,q} M^\circ_{n+p,r}}{|V_{n+p-q}|^{1/2}} \frac{\kappa^{-m/2}}{(\kappa |V_{n-1}|)^{1/2}} \sum_{y \in V_n} ||\omega_-(y)| - M(\omega_-|V_n)| \\ &\qquad \le \frac{\kappa^{p+\frac{r}{2}-\frac{m}{2}-\frac{1}{2}} M^-_{n+p,q} M^\circ_{n+p,r}}{|V_{n+p-q}|^{1/2} |V_{n-1}|^{1/2}} \left(\sum_{y \in V_n} ||\omega_-(y)| - M(\omega_-|V_n)|^2 \right)^{1/2} |V_n|^{1/2} \\ &\qquad = \Sigma(\omega_-|V_n) M^-_{n+p,q} M^\circ_{n+p,r} \frac{\kappa^{p+\frac{r-m-1}{2}} |V_n|}{|V_{n+p-q}|^{1/2} |V_{n-1}|^{1/2}}. \end{aligned}$$

This proves inequality (7.11). ∎

We go back to the proof of Theorem 7.1. Let $\mathcal{G}_\nu = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph as stated therein. We consider a general matrix element:

$$\left\langle \Phi_j^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle. \tag{7.17}$$

Of course, this is zero unless (7.8) is fulfilled. So we assume (7.8) and $j = n + p - q$, where p, q, r are the numbers of $+, -, \circ$ appearing in $\{\epsilon_1, \ldots, \epsilon_m\}$. Using (7.7), one obtains

$$\begin{aligned} &\left\langle \Phi_j^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle \\ &= \gamma_{n+\epsilon_1}^{\epsilon_1} \gamma_{n+\epsilon_1+\epsilon_2}^{\epsilon_2} \cdots \gamma_{n+\epsilon_1+\cdots+\epsilon_m}^{\epsilon_m} \\ &\qquad + \sum_{k=1}^{m} \gamma_{n+\epsilon_1}^{\epsilon_1} \cdots \gamma_{n+\epsilon_1+\cdots+\epsilon_{k-1}}^{\epsilon_{k-1}} \left\langle \Phi_j^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_{k+1}}}{\sqrt{\kappa(\nu)}} S_{n+\epsilon_1+\cdots+\epsilon_k}^{\epsilon_k} \right\rangle. \end{aligned} \tag{7.18}$$

We shall prove that the second term vanishes in the limit. The coefficients γ_n^ϵ depends on ν. Explicit expressions of γ_n^ϵ being given in (7.3)–(7.5), with the help of Lemma 7.3 and conditions (A1)–(A3) we come to

$$\lim_\nu \gamma_n^+ = \lim_\nu \sqrt{M(\omega_-|V_n)} = \sqrt{\omega_n}, \tag{7.19}$$

$$\lim_\nu \gamma_n^- = \lim_\nu \{\kappa - M(\omega_-|V_n) - M(\omega_\circ|V_n)\} \frac{\sqrt{M(\omega_-|V_{n+1})}}{\kappa} = \sqrt{\omega_{n+1}}, \tag{7.20}$$

$$\lim_\nu \gamma_n^\circ = \alpha_{n+1}. \tag{7.21}$$

Therefore, in order to prove that the second term of (7.18) vanishes in the limit it is sufficient to show that

$$\lim_\nu \left\langle \Phi_j^{(\nu)}, \frac{A^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A^{\epsilon_{k+1}}}{\sqrt{\kappa(\nu)}} S_{n+\epsilon_1+\cdots+\epsilon_k}^{\epsilon_k} \right\rangle = 0. \tag{7.22}$$

This follows by Lemma 7.5. In fact, for $\epsilon_k = +$ we use (7.11), where we see that

$$M_{n+p,q}^- M_{n+p,r}^\circ \frac{\kappa^{p+\frac{r-m-1}{2}} |V_n|}{\sqrt{|V_{n+p-q}||V_{n-1}|}}$$

stays bounded in the limit. Hence the condition $\lim_\nu \Sigma(\omega_-|V_n) = 0$ works to obtain (7.22). The arguments for $\epsilon_k = -, \circ$ are similar. Thus, only the first term of (7.18) contributes to the limit and we come to

$$\lim_\nu \left\langle \Phi_j^{(\nu)}, \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle = \lim_\nu \gamma_{n+\epsilon_1}^{\epsilon_1} \gamma_{n+\epsilon_1+\epsilon_2}^{\epsilon_2} \cdots \gamma_{n+\epsilon_1+\cdots+\epsilon_m}^{\epsilon_m}. \tag{7.23}$$

The right hand side is readily known from (7.19)–(7.21) and equal to

$$\langle \Psi_j, B^{\epsilon_m} \ldots B^{\epsilon_1} \Psi_n \rangle. \tag{7.24}$$

by the definition of an interacting Fock space $\Gamma_{\{\omega_n\}} = (\Gamma, B^+, B^-)$. Unless (7.8) and $j = n + p - q$ are fulfilled, (7.17) is zero and hence so is its limit. In that case, obviously, (7.24) is zero. Consequently,

$$\lim_{\nu} \left\langle \Phi_j^{(\nu)}, \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle = \langle \Psi_j, B^{\epsilon_m} \cdots B^{\epsilon_1} \Psi_n \rangle$$

holds for any choice of $\epsilon_1, \dots \epsilon_m \in \{+, -, \circ\}$, $m = 1, 2, \dots,$ and $j, n = 0, 1, 2, \dots$. This being slightly more than necessary, the proof of Theorem 7.1 is now complete.

Theorem 7.1 generalizes the main result in Hora–Obata [15]. Some concrete examples are found in Hashimoto [5], Hashimoto–Hora–Obata [6], Hashimoto–Obata–Tabei [7], Hora [9, 10], Hora–Obata [16]. Further study, in particular, concering deformed vacuum states is now in progress and will appear elsewhere.

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