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GENERALIZED WHITE NOISE OPERATOR FIELDS AND QUANTUM WHITE NOISE DERIVATIVES

by

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Abstract. — Regarding a Fock space operator Ξ as a function of quantum white noise $\Xi = \Xi(a_t, a_t^* \,;\, t \in T)$, we introduce its quantum white noise derivatives (qwn-derivatives) as a kind of functional derivatives with respect to a_t and a_t^*. We prove that every white noise operator is differentiable and the qwn-derivatives form a generalized white noise operator field on T. We show a relation between qwn-derivatives and generalized quantum stochastic integrals. We obtain a condition under which qwn-derivatives are defined pointwisely.

***Résumé* (Champs d'opérateurs de bruit blanc généralisé et dérivées de bruit blanc quantique)**
Considérant un opérateur Ξ sur l'espace de Fock comme une fonction d'un bruit blanc quantique $\Xi = \Xi(a_t, a_t^*; t \in T)$, nous introduisons ses dérivées de bruit blanc quantique qui sont analogues à des dérivés fonctionnelles par rapport à a_t et a_t^*. Nous montrons que tout opérateur de bruit blanc est différentiable et que ses dérivées de bruit blanc quantique constituent un champ d'opérateurs de bruit blanc sur T. Nous établissons une relation entre les dérivées de bruit blanc quantique et des intégrales stochastiques quantiques généralisées. Nous obtenons une condition pour que les dérivées de bruit blanc quantique soient définies en tout point.

1. Introduction

This paper is a continuation of the preceding work [6], where the new idea of quantum white noise derivative (qwn-derivative for brevity) of a Fock space operator was introduced. A Fock space of interest is of the form $\Gamma(H)$, where $H = L^2(T, \mathrm{d}t)$ is the Hilbert space of L^2-functions on a topological space T equipped with a σ-finite

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Borel measure dt. Many questions in quantum stochastic analysis, quantum physics and infinite dimensional harmonic analysis are concerned with (often unbounded) operators on Fock space $\Gamma(H)$ and a key role is played by the quantum white noise, i.e., the pair of annihilation and creation operators $\{a_t, a_t^*\,;\, t \in T\}$ though some difficulty is caused by their singularity. This difficulty may be overcome by adopting a suitable Gelfand triple. In this paper, for simplicity and avoiding cumbersome notation, we adopt the Hida-Kubo-Takenaka space:

$$(E) \subset \Gamma(H) \subset (E)^*,$$

see Section 2 for details. We then concentrate our attention to the class $\mathcal{L}((E),(E)^*)$ of continuous linear operators from (E) into $(E)^*$. Such an operator is called a *white noise operator*. This restriction, nevertheless, covers a wide class of Fock space operators including bounded operators on $\Gamma(H)$ as well as pointwisely defined annihilation and creation operators. A systematic study of white noise operators was started in [12] and has developed into *quantum white noise theory*, see [4] and references cited therein.

The idea of quantum white noise derivative is naive. Recall that every white noise operator admits a Fock expansion, i.e., every $\Xi \in \mathcal{L}((E),(E)^*)$ is decomposed into an infinite series:

$$\Xi = \sum_{\ell,m=0}^{\infty} \Xi_{\ell,m}(\kappa_{\ell,m}), \tag{1.1}$$

where the integral kernel operator $\Xi_{\ell,m}(\kappa_{\ell,m})$ is expressed in a formal integral (because $\kappa_{\ell,m}$ may be a distribution):

$$\begin{aligned}\Xi_{\ell,m}(\kappa_{\ell,m}) = \int_{T^{\ell+m}} & \kappa_{\ell,m}(s_1,\dots,s_\ell,t_1,\dots,t_m) \\ & \times a_{s_1}^* \dots a_{s_\ell}^* a_{t_1} \dots a_{t_m}\, ds_1 \dots ds_\ell dt_1 \dots dt_m.\end{aligned} \tag{1.2}$$

Accepting (1.2) as a "polynomial" in a_t and a_t^*, and hence (1.1) as a "function" of them: $\Xi = \Xi(a_t, a_t^*\,;\, t \in T)$, we naturally come to a kind of functional derivatives:

$$D_t^- \Xi = \frac{\delta \Xi}{\delta a_t}, \quad D_t^+ \Xi = \frac{\delta \Xi}{\delta a_t^*}. \tag{1.3}$$

These are respectively called the *annihilation-* and *creation-derivatives* of Ξ, and both together the *quantum white noise derivatives* (*qwn-derivatives* for brevity). However, the above intuitive definition does not work due to singularity brought by a_t and a_t^*.

In the preceding paper [6], taking $\zeta \in H = L^2(T)$, we defined the annihilation- and creation-derivatives $D_\zeta^\pm \Xi$ for an *admissible* white noise operator Ξ and proved that $D_\zeta^\pm \Xi$ is again an admissible white noise operator. In comparison with (1.3) one may write down the definition as

$$D_\zeta^- \Xi = \frac{\delta \Xi}{\delta a(\zeta)}, \quad D_\zeta^+ \Xi = \frac{\delta \Xi}{\delta a^*(\zeta)},$$

where $a(\zeta)$ and $a^*(\zeta)$ are smeared operators:

$$a(\zeta) = \int_T \zeta(t) a_t \, \mathrm{d}t, \quad a^*(\zeta) = \int_T \zeta(t) a_t^* \, \mathrm{d}t.$$

In fact, the derivatives $D_\zeta^\pm \Xi$ for an admissible white noise operator Ξ were defined by means of the Gross derivative for an admissible white noise function.

In this paper, for an *arbitrary* white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ we define its qwn-derivatives without using the Gross derivative. In general, $t \mapsto D_t^\pm \Xi$ does not make sense but is given a meaning as a *generalized white noise operator field* on T (Theorem 4.6 and corollaries). Moreover, we extend the qwn-derivatives for generalized white noise operator fields (Corollary 4.13). A generalized integral kernel operator has been discussed as a generalization of quantum stochastic integral [**13**, **14**]. We find an interesting relation between the qwn-differential operators and such stochastic integrals (Corollary 4.15). Finally, we discuss a condition under which the qwn-derivative is defined pointwisely (Theorem 5.4).

This paper is organized as follows: In Section 2 we prepare some basic notation of the Hida-Kubo-Takenaka space. In Section 3 we review white noise operators, in particular, integral kernel operators, Fock expansion and operator symbols. In Section 4 we introduce generalized white noise operator fields and study the qwn-derivatives in detail. In Section 5 we discuss pointwise qwn-derivatives.

The qwn-derivatives are considered as a quantum counterpart of "classical" stochastic derivatives studied extensively by many authors, see e.g., Gross [**1**], Hida [**2**], Krée [**7**], Kuo [**8**], Malliavin [**9**], Nualart [**10**], Strook [**16**], and references cited therein. We expect that the qwn-derivatives have applications in quantum stochastic integrals, particularly in representation theory of quantum martingales [**3**], [**15**]. Further study is now in progress.

2. Preliminaries

2.1. Underlying Gelfand Triple. — Let T be a topological space equipped with a σ-finite Borel measure $\mathrm{d}t$ (we do not use a specific symbol for this measure). Let $H = L^2(T, \mathrm{d}t)$ be the (complex) Hilbert space of L^2-functions on T and the norm is denoted by $|\cdot|_0$. Let A be a selfadjoint operator (densely defined) in H satisfying conditions (A1)–(A4) below.

(A1) $\inf \mathrm{Spec}(A) > 1$ *and* A^{-1} *is of Hilbert-Schmidt type.*

Then there exist a sequence

$$1 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots, \quad \| A^{-1} \|_{\mathrm{HS}}^2 = \sum_{j=0}^{\infty} \lambda_j^{-2} < \infty, \tag{2.4}$$

and an orthonormal basis $\{e_j\}_{j=0}^{\infty}$ of H such that $Ae_j = \lambda_j e_j$. For $p \in \mathbb{R}$ we define

$$|\xi|_p^2 = |A^p \xi|_0^2 = \sum_{j=0}^{\infty} \lambda_j^{2p} \left| \langle \xi, e_j \rangle \right|^2, \quad \xi \in H.$$

Now let $p \ge 0$. Setting $E_p = \{\xi \in H\,;\, |\xi|_p < \infty\}$ and defining E_{-p} to be the completion of H with respect to $|\,.\,|_{-p}$, we obtain a chain of Hilbert spaces $\{E_p\,;\, p \in \mathbb{R}\}$. Define their limit spaces:

$$E = \mathcal{S}_A(T) = \operatorname*{proj\,lim}_{p\to\infty} E_p, \quad E^* = \mathcal{S}_A^*(T) = \operatorname*{ind\,lim}_{p\to\infty} E_{-p},$$

which are mutually dual spaces. Note also that $\mathcal{S}_A(T)$ becomes a countably Hilbert nuclear space. Identifying H with its dual space, we obtain a complex Gelfand triple:

$$E = \mathcal{S}_A(T) \subset H = L^2(T, \mathrm{d}t) \subset E^* = \mathcal{S}_A^*(T). \tag{2.5}$$

As usual, we understand that $\mathcal{S}_A(T)$ and $\mathcal{S}_A^*(T)$ are spaces of test functions and generalized functions (or distributions) on T, respectively. We denote by $\langle .,. \rangle$ the canonical $\mathbb{C}$-bilinear form on $E^* \times E$, which is characterized by $\langle e_i, e_j \rangle = \delta_{ij}$. The same symbol is used for the canonical $\mathbb{C}$-bilinear form on H.

For white noise theory $\mathcal{S}_A^*(T)$ must contain delta functions. But this is not automatic and we need further assumptions:

(A2) *For each function $\xi \in \mathcal{S}_A(T)$ there exists a unique continuous function $\widetilde{\xi}$ on T such that $\xi(t) = \widetilde{\xi}(t)$ for a.e. $t \in T$.*

Thus $\mathcal{S}_A(T)$ is regarded as a space of continuous functions on T and we do not use the exclusive symbol $\widetilde{\xi}$. The uniqueness in (A2) is equivalent to that a continuous function a.e. vanishing on T is identically zero.

(A3) *For each $t \in T$ the evaluation map $\delta_t : \xi \mapsto \xi(t)$, $\xi \in \mathcal{S}_A(T)$, is a continuous linear functional, i.e., $\delta_t \in \mathcal{S}_A^*(T)$.*

(A4) *The map $t \mapsto \delta_t \in \mathcal{S}_A^*(T)$, $t \in T$, is continuous with respect to the strong dual topology of $\mathcal{S}_A^*(T)$.*

It is verified easily that

$$\| A^{-p} \|_{\mathrm{HS}}^2 = \int_T |\delta_t|_{-p}^2 \mathrm{d}t < \infty, \quad p \ge 1. \tag{2.6}$$

See [12, Chapter 1] for relevant discussion.

Remark 2.1. — The space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions is obtained by taking $H = L^2(\mathbb{R}, \mathrm{d}t)$ together with the selfadjoint operator $A = 1 + t^2 - \mathrm{d}^2/\mathrm{d}t^2$, where $\mathrm{d}t$ is Lebesgue measure. This choice is suitable for stochastic processes for $\mathbb{R}$ plays a role of the time axis. On the other hand, within our general framework T can be a manifold (space-time), a discrete space or even a finite set.

2.2. Hida-Kubo-Takenaka space. — Let E_p be the Hilbert space defined in Section 2.1. The (Boson) Fock space over E_p is defined by

$$\Gamma(E_p) = \Big\{\phi = (f_n)_{n=0}^{\infty}\,;\, f_n \in E_p^{\widehat{\otimes} n},\ \|\,\phi\,\|_p^2 = \sum_{n=0}^{\infty} n!|\,f_n\,|_p^2 < \infty\Big\}.$$

(The weight factor $n!$ is for convention.) Having obtained a chain of Fock spaces $\{\Gamma(E_p)\,;\, p \in \mathbb{R}\}$, we set

$$(E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p), \quad (E)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p}).$$

Then we obtain a complex Gelfand triple:

$$(E) \subset \Gamma(H) \subset (E)^*,$$

which is referred to as the *Hida-Kubo-Takenaka space.* It is known that (E) is a countably Hilbert nuclear space.

By definition the topology of (E) is defined by the norms

$$\|\,\phi\,\|_p^2 = \sum_{n=0}^{\infty} n!\,|\,f_n\,|_p^2, \quad \phi = (f_n),\ p \in \mathbb{R}.$$

On the other hand, for each $\Phi \in (E)^*$ there exists $p \geq 0$ such that $\Phi \in \Gamma(E_{-p})$. In this case, we have

$$\|\,\Phi\,\|_{-p}^2 \equiv \sum_{n=0}^{\infty} n!\,|\,F_n\,|_{-p}^2 < \infty, \quad \Phi = (F_n).$$

The canonical $\mathbb{C}$-bilinear form on $(E)^* \times (E)$ takes the form:

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \quad \Phi = (F_n) \in (E)^*, \quad \phi = (f_n) \in (E).$$

Here we recall two important elements of $(E)^*$.

(a) *White noise.* By assumption (A3),

$$W_t = (0, \delta_t, 0, \dots), \quad t \in T,$$

belongs to $(E)^*$. The family $\{W_t\,;\, t \in T\} \subset (E)^*$ is called a *white noise field* on T or a *white noise process* when T is a time interval.

(b) *Exponential vector.* An *exponential vector* (or also called a *coherent vector*) associated with $x \in E^*$ is defined by

$$\phi_x = \Big(1, x, \frac{x^{\otimes 2}}{2!}, \dots, \frac{x^{\otimes n}}{n!}, \dots\Big).$$

Obviously, $\phi_x \in (E)^*$. Moreover, ϕ_ξ belongs to (E) (resp. $\Gamma(E_p)$) if and only if ξ belongs to E (resp. E_p). In particular, ϕ_0 is called the *vacuum vector.*

Remark 2.2. — Let $E_{\mathbb{R}} \subset H_{\mathbb{R}} \subset E_{\mathbb{R}}^*$ be the real Gelfand triple obtained by taking the real parts of (2.5), i.e., for example, $E_{\mathbb{R}}$ is the closed real subspace of E spanned by $\{e_0, e_1, \dots\}$. Let μ be the standard Gaussian measure on $E_{\mathbb{R}}^*$. Since $L^2(E_{\mathbb{R}}^*, \mu)$ is unitarily isomorphic to $\Gamma(H)$ by the celebrated Wiener-Itô decomposition theorem, we may regard (E) as a subspace of $L^2(E^*, \mu)$. In this sense an element of (E) is called a *test white noise function*, and hence an element of $(E)^*$ is called a *generalized white noise function.*

3. White noise operators

3.1. Integral kernel operators and Fock expansion. — A continuous operator from (E) into $(E)^*$ is called a *white noise operator.* The space of white noise operators is denoted by $\mathcal{L}((E), (E)^*)$ and is equipped with the bounded convergence topology. It is noted that $\mathcal{L}((E), (E))$ is a subspace of $\mathcal{L}((E), (E)^*)$.

For each $t \in T$ there exists a continuous operator on (E) uniquely specified by

$$a_t : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \longmapsto \big(0, \dots, 0, n\xi(t)\xi^{\otimes(n-1)}, 0, \dots\big), \quad \xi \in E.$$

This is called an *annihilation operator at a point t.* By duality, the *creation operator at a point t* is defined by $a_t^* \in \mathcal{L}((E)^*, (E)^*)$. Note that $\mathcal{L}((E)^*, (E)^*)$ is regarded as a subspace of $\mathcal{L}((E), (E)^*)$ in an obvious manner. The pair of annihilation and creation operators $\{a_t, a_t^* \,;\, t \in T\}$ is called a *quantum white noise (field)* on T or a *quantum white noise process* when T is a time interval.

Note that a normal-ordered composition of annihilation and creation operators is always well defined, i.e., for any pair of integers $\ell \geq 0$, $m \geq 0$, we have a map

$$T^{\ell+m} \ni (s_1, \dots, s_\ell, t_1, \dots, t_m) \longmapsto a_{s_1}^* \cdots a_{s_\ell}^* a_{t_1} \cdots a_{t_m} \in \mathcal{L}\big((E), (E)^*\big). \tag{3.7}$$

As a consequence (see also below) we know that (3.7) is an $\mathcal{L}((E), (E)^*)$-valued test function on $T^{\ell+m}$, and the canonical pairing with any $\kappa_{\ell,m} \in (E^{\otimes(\ell+m)})^*$ defines a white noise operator. This operator is called an *integral kernel operator* with kernel distribution $\kappa_{\ell,m}$ and is often written in a formal integral form as in (1.2).

To be more precise, given $\kappa_{\ell,m} \in (E^{\otimes(\ell+m)})^*$ we take $K_{\ell,m} \in \mathcal{L}(E^{\otimes m}, (E^{\otimes \ell})^*)$ uniquely specified through the kernel theorem as

$$\langle \kappa_{\ell,m}, \eta^{\otimes \ell} \otimes \xi^{\otimes m} \rangle = \langle K_{\ell,m} \xi^{\otimes m}, \eta^{\otimes \ell} \rangle, \quad \xi, \eta \in E.$$

We define

$$K^0_{\ell,m} = S_{n+\ell} \circ (I_n \otimes K_{\ell,m}),$$

where $I_n : E^{\otimes n} \to E^{\otimes n}$ is the identity and $S_{n+\ell} : E^{\otimes(n+\ell)} \to E^{\widehat{\otimes}(n+\ell)}$ the symmetrizing operator. The symbol $K^0_{\ell,m}$ is used commonly for all $n \geq 0$. Then an *integral kernel operator* $\Xi_{\ell,m}(K_{\ell,m})$ is defined by the action $\phi = (f_n) \mapsto (g_n)$ given by

$$g_n = 0, \quad 0 \leq n < \ell; \quad g_{n+\ell} = \frac{(n+m)!}{n!} K^0_{\ell,m} f_{n+m}, \quad n \geq 0.$$

It is proved by norm estimates that $\Xi_{\ell,m}(K_{\ell,m}) \in \mathcal{L}((E), (E)^*)$. We also write $\Xi_{\ell,m}(\kappa_{\ell,m})$ for $\Xi_{\ell,m}(K_{\ell,m})$.

A role of integral kernel operators is shown in the following

***Theorem 3.1* (see [11], [12]).** — *A white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ is decomposed uniquely into a sum of integral kernel operators:*

$$\Xi = \sum_{\ell,m=0}^{\infty} \Xi_{\ell,m}(\kappa_{\ell,m}), \tag{3.8}$$

where the right hand side converges in $\mathcal{L}((E), (E)^*)$.

The above (3.8) is called the *Fock expansion* of Ξ. Convergence of Fock expansion has been considerably discussed for various classes of white noise operators, see e.g., [4].

3.2. Characterization theorem for operator symbols. — The *symbol* of a white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ is defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\!\langle \Xi \phi_\xi, \phi_\eta \rangle\!\rangle, \quad \xi, \eta \in E.$$

A white noise operator is uniquely specified by the symbol since $\{\phi_\xi \,;\, \xi \in E\}$ spans a dense subspace of (E). Moreover, we have an analytic characterization of symbols.

***Theorem 3.2* (see [11], [12]).** — *A $\mathbb{C}$-valued function Θ on $E \times E$ is the symbol of a white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ if and only if*

(i) *Θ is Gâteaux entire;*

(ii) *there exist $C \geq 0$, $K \geq 0$ and $p \geq 0$ such that*

$$\big|\Theta(\xi, \eta)\big| \leq C \exp K\big(|\,\xi\,|_p^2 + |\,\eta\,|_p^2\big), \quad \xi, \eta \in E.$$

For an integral kernel operator $\Xi = \Xi_{\ell,m}(\kappa_{\ell,m}) = \Xi_{\ell,m}(K_{\ell,m})$, where $\kappa_{\ell,m} \in (E^{\otimes(\ell+m)})^*$ and $K_{\ell,m} \in \mathcal{L}(E^{\otimes m}, (E^{\otimes \ell})^*)$ are in correspondence, we have

$$\widehat{\Xi}(\xi,\eta) = \left\langle K_{\ell,m}\xi^{\otimes m}, \eta^{\otimes \ell}\right\rangle e^{\langle \xi,\eta\rangle} = \left\langle \kappa_{\ell,m}, \eta^{\otimes \ell} \otimes \xi^{\otimes m}\right\rangle e^{\langle \xi,\eta\rangle}. \tag{3.9}$$

In particular, for the quantum white noise we have

$$\widehat{a_t}(\xi,\eta) = \xi(t)\, e^{\langle \xi,\eta\rangle}, \quad \widehat{a_t^*}(\xi,\eta) = \eta(t)\, e^{\langle \xi,\eta\rangle},$$

which follow directly from $a_t\phi_\xi = \xi(t)\phi_\xi$ for $\xi \in E$.

The characterization theorem for operator symbols has been considerably studied for various classes of white noise operators, see e.g., [5] and references cited therein.

Remark 3.3. — To avoid the factor $e^{\langle \xi,\eta\rangle}$ in (3.9) the *Wick symbol* of a white noise operator Ξ is defined by

$$\widetilde{\Xi}(\xi,\eta) = \widehat{\Xi}(\xi,\eta)\, e^{-\langle \xi,\eta\rangle} = \langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle\, e^{-\langle \xi,\eta\rangle}, \quad \xi,\eta \in E.$$

The statement in Theorem 3.2 remains valid for the Wick symbol.

4. Quantum white noise derivatives

4.1. Generalized white noise operator fields

Definition 4.1. — A continuous linear map $L : E \to \mathcal{L}((E),(E)^*)$ is called a *generalized white noise operator field* on T. A generalized white noise operator field on a time interval is called a *generalized quantum stochastic process* [13].

Obviously, if L is a generalized white noise operator field on T, so is the adjoint L^* defined by $L^*(\zeta) = L(\zeta)^*$.

Remark 4.2. — The symbol L^* might be also used for the adjoint of

$$L : E \longrightarrow \mathcal{L}\big((E),(E)^*\big) \cong \big((E)\otimes(E)\big)^*,$$

i.e., $L^* : (E)\otimes(E) \to E^*$, but the confusion will not occur by context.

For $f \in E^*$ we define white noise operators:

$$a(f) = \Xi_{0,1}(f) = \int_T f(t)a_t\,dt, \quad a^*(f) = \Xi_{1,0}(f) = \int_T f(t)a_t^*\,dt,$$

which are called respectively the annihilation and creation operators associated with f. Then, both $\zeta \mapsto a(\zeta)$ and $\zeta \mapsto a^*(\zeta)$ are generalized white noise operator fields.

An important example of a generalized white noise operator field is obtained from an integral kernel operator. We recall notation. For $\kappa \in (E^{\otimes n})^*$ and $\zeta \in E$ we define their right and left contractions $\kappa * \zeta$, $\zeta * \kappa \in (E^{\otimes (n-1)})^*$ by

$$\langle \kappa * \zeta, \eta_1 \otimes \cdots \otimes \eta_{n-1} \rangle = \langle \kappa, \eta_1 \otimes \cdots \otimes \eta_{n-1} \otimes \zeta \rangle,$$
$$\langle \zeta * \kappa, \eta_1 \otimes \cdots \otimes \eta_{n-1} \rangle = \langle \kappa, \zeta \otimes \eta_1 \otimes \cdots \otimes \eta_{n-1} \rangle,$$

where $\eta_1, \dots, \eta_{n-1} \in E$.

Proposition 4.3. — *Let $\kappa_{\ell,m} \in (E^{\otimes(\ell+m)})^*$. Then $\zeta \mapsto \Xi_{\ell,m-1}(\kappa_{\ell,m} * \zeta)$ defines a generalized white noise operator field on T $(m \geq 1)$. Similarly, so does $\zeta \mapsto \Xi_{\ell-1,m}(\zeta * \kappa_{\ell,m})$ $(\ell \geq 1)$.*

Proof. — By using a standard estimate of an integral kernel operator, see e.g., [12, Theorem 4.3.2], we see that for any $\phi \in (E)$,

$$\begin{aligned} \big\| \Xi_{\ell,m-1}(\kappa_{\ell,m} * \zeta)\phi \big\|_{-p} &\leq C | \kappa_{\ell,m} * \zeta |_{-p} \cdot \| \phi \|_p \\ &\leq C | \kappa_{\ell,m} |_{-p} | \zeta |_p \cdot \| \phi \|_p, \end{aligned} \tag{4.10}$$

where $p > 0$ is chosen as $| \kappa_{\ell,m} |_{-p} < \infty$ and C is a constant depending on ℓ, m, p. Then the continuity of $\zeta \mapsto \Xi_{\ell,m-1}(\kappa_{\ell,m} * \zeta) \in \mathcal{L}((E),(E)^*)$ is a direct consequence of the above estimate. For $\Xi_{\ell-1,m}(\zeta * \kappa_{\ell,m})$ the argument is similar. □

4.2. Quantum white noise derivatives. — As is checked by direct norm estimate, for $f \in E^*$ and $\phi \in (E)$ we have

$$\| a(f)\phi \|_p \leq C_q | f |_{-(p+q)} \| \phi \|_{p+q}, \tag{4.11}$$

$$\| a^*(f)\phi \|_p \leq C_q | f |_p \| \phi \|_{p+q}, \tag{4.12}$$

where $p \in \mathbb{R}$, $q > 0$ and $C_q = \sup_{n \geq 0} \sqrt{n+1}\, \lambda_0^{-qn}$ with λ_0 being a constant defined in (2.4). It then follows from (4.11) and (4.12) that $a(f) \in \mathcal{L}((E),(E))$ and $a^*(f) \in \mathcal{L}((E)^*,(E)^*)$ for all $f \in E^*$ (that is, the integral kernel operator $\Xi_{1,0}(f)$ extends to a continuous operator from $(E)^*$ into itself). Moreover, (4.11) and (4.12) imply the following

Lemma 4.4. — *If $\zeta \in E$, then $a(\zeta)$ extends to a continuous linear operator from $(E)^*$ into itself (denoted by the same symbol) and $a^*(\zeta)$ (restricted to (E)) is a continuous linear operator from (E) into itself.*

Thus, for any white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ and $\zeta \in E$ the commutators

$$\big[a(\zeta), \Xi\big] = a(\zeta)\Xi - \Xi a(\zeta), \quad -\big[a^*(\zeta), \Xi\big] = \Xi a^*(\zeta) - a^*(\zeta)\Xi,$$

are well-defined white noise operators, i.e., belongs to $\mathcal{L}((E),(E)^*)$. We define

$$D_\zeta^+ \Xi = \big[a(\zeta), \Xi\big], \quad D_\zeta^- \Xi = -\big[a^*(\zeta), \Xi\big].$$

Definition 4.5. — $D_\zeta^+\Xi$ and $D_\zeta^-\Xi$ are respectively called the *creation derivative* and *annihilation derivative* of Ξ, and both together the *quantum white noise derivatives* (*qwn-derivatives* for brevity) of Ξ.

Clearly, $D_\zeta^\pm$ becomes a linear map from $\mathcal{L}((E),(E)^*)$ into itself. Moreover, we have

Theorem 4.6. — *The map*

$$E \times \mathcal{L}\big((E),(E)^*\big) \longrightarrow \mathcal{L}\big((E),(E)^*\big), \quad (\zeta,\Xi) \longmapsto D_\zeta^\pm \Xi$$

is continuous bilinear.

Proof. — For $\Xi \in \mathcal{L}((E),(E)^*)$ we denote by $\|\Xi\|_{-p}$ the norm of the corresponding element in $((E)\otimes(E))^*$ through the isomorphism:

$$\mathcal{L}\big((E),(E)^*\big) \cong \big((E)\otimes(E)\big)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p})\otimes\Gamma(E_{-p}).$$

Then, for $\phi,\psi \in (E)$ we have

$$\begin{aligned}\big|\langle\!\langle (D_\zeta^-\Xi)\phi,\psi\rangle\!\rangle\big| &= \big|\langle\!\langle \Xi a^*(\zeta)\phi,\psi\rangle\!\rangle - \langle\!\langle a^*(\zeta)\Xi\phi,\psi\rangle\!\rangle\big| \\ &\le \big|\langle\!\langle \Xi,\psi\otimes a^*(\zeta)\phi\rangle\!\rangle\big| + \big|\langle\!\langle \Xi,a(\zeta)\psi\otimes\phi\rangle\!\rangle\big| \\ &\le \|\Xi\|_{-p}\cdot\|\psi\|_p\cdot\big\|a^*(\zeta)\phi\big\|_p + \|\Xi\|_{-p}\cdot\|a(\zeta)\psi\|_p\cdot\|\phi\|_p.\end{aligned}$$

By (4.11) and (4.12) we obtain

$$\begin{aligned}\big|\langle\!\langle (D_\zeta^-\Xi)\phi,\psi\rangle\!\rangle\big| \le \|\Xi\|_{-p}\big(&C_q|\zeta|_p\cdot\|\phi\|_{p+q}\cdot\|\psi\|_p \\ &+ C_q|\zeta|_{-(p+q)}\cdot\|\phi\|_p\cdot\|\psi\|_{p+q}\big).\end{aligned}$$

Using $|\zeta|_{-(p+q)} \le \lambda_0^{-(2p+q)}|\zeta|_p$ and setting $C = C_q + \lambda_0^{-(2p+q)}C_q$, we come to

$$\big|\langle\!\langle (D_\zeta^-\Xi)\phi,\psi\rangle\!\rangle\big| \le C\|\Xi\|_{-p}\cdot|\zeta|_p\cdot\|\phi\|_{p+q}\cdot\|\psi\|_{p+q}. \tag{4.13}$$

This proves that $(\zeta,\Xi)\mapsto D_\zeta^-\Xi$ is a continuous bilinear map from $E\times\mathcal{L}((E),(E)^*)$ into $\mathcal{L}((E),(E)^*)$. The proof for D^+ is similar. □

The following results are noteworthy, though immediate from Theorem 4.6.

Corollary 4.7. — *For each $\zeta \in E$, the qwn-differential operator $D_\zeta^\pm$ is a continuous operator from $\mathcal{L}((E),(E)^*)$ into itself.*

Corollary 4.8. — *Let $\Xi \in \mathcal{L}((E),(E)^*)$. Then*

$$E \ni \zeta \longmapsto D_\zeta^\pm\Xi \in \mathcal{L}\big((E),(E)^*\big)$$

is a generalized white noise operator field on T.

We define

$$D^\pm : \mathcal{L}\big((E),(E)^*\big) \longrightarrow \mathcal{L}\big(E,\mathcal{L}\big((E),(E)^*\big)\big), \quad D^\pm\Xi(\zeta) = D_\zeta^\pm\Xi,.$$

4.3. Symbol and Fock expansion. — The symbols of the qwn-derivatives $D_\zeta^\pm \Xi$ are expressed in a concise form by using the Wick product (see e.g., [8] for definition). The proof is straightforward from definition and is omitted.

Proposition 4.9. — *Let $\Xi \in \mathcal{L}((E),(E)^*)$ and $\zeta \in E$. Then for $\xi, \eta \in E$ we have*

$$\widehat{D_\zeta^- \Xi}(\xi,\eta) = \langle\!\langle a(\zeta)(\Xi^* \phi_\eta \diamond \phi_{-\eta}), \phi_\xi \rangle\!\rangle \, \mathrm{e}^{\langle \xi,\eta\rangle},$$

$$\widehat{D_\zeta^+ \Xi}(\xi,\eta) = \langle\!\langle a(\zeta)(\Xi \phi_\xi \diamond \phi_{-\xi}), \phi_\eta \rangle\!\rangle \, \mathrm{e}^{\langle \xi,\eta\rangle},$$

where $\diamond$ is the Wick product.

We next discuss the qwn-derivatives of $\Xi \in \mathcal{L}((E),(E)^*)$ in terms of Fock expansion. Let us start with the following

Proposition 4.10. — *The qwn-derivatives of an integral kernel operator $\Xi_{\ell,m}(\kappa_{\ell,m})$, $\kappa_{\ell,m} \in (E^{\otimes(\ell+m)})^*$ are given by*

$$\begin{cases} D_\zeta^- \Xi_{\ell,m}(\kappa_{\ell,m}) = m \Xi_{\ell,m-1}(\kappa_{\ell,m} * \zeta), \\ D_\zeta^+ \Xi_{\ell,m}(\kappa_{\ell,m}) = \ell \Xi_{\ell-1,m}(\zeta * \kappa_{\ell,m}), \end{cases} \tag{4.14}$$

where $\zeta \in E$. (The right hand sides are understood to be zero for $m = 0$ and $\ell = 0$, respectively.)

Proof. — Note first that for any $p \in \mathbb{R}$,

$$\left\| \frac{\phi_{\xi+\theta\zeta} - \phi_\xi}{\theta} - a^*(\zeta)\phi_\xi \right\|_p \le |\theta| \exp \tfrac{1}{2}\big(|\,\xi\,|_p + |\,\zeta\,|_p\big)^2, \quad \xi, \zeta \in E, \ 0 < |\theta| \le 1,$$

which is verified by direct calculation. We then have

$$\langle\!\langle \Xi_{\ell,m}(\kappa_{\ell,m}) a^*(\zeta)\phi_\xi, \phi_\eta \rangle\!\rangle = \frac{\mathrm{d}}{\mathrm{d}\theta} \langle\!\langle \Xi_{\ell,m}(\kappa_{\ell,m}) \phi_{\xi+\theta\zeta}, \phi_\eta \rangle\!\rangle \Big|_{\theta=0}.$$

The right hand side is then computed with the help of (3.9) and becomes

$$= \big\langle \kappa_{\ell,m}, \eta^{\otimes \ell} \otimes m \xi^{\otimes(m-1)} \otimes \zeta \big\rangle \mathrm{e}^{\langle \xi,\eta\rangle} + \langle \zeta, \eta\rangle \big\langle \kappa_{\ell,m}, \eta^{\otimes\ell} \otimes \xi^{\otimes m} \big\rangle \mathrm{e}^{\langle\xi,\eta\rangle}. \tag{4.15}$$

On the other hand,

$$\langle\!\langle a^*(\zeta) \Xi_{\ell,m}(\kappa_{\ell,m}) \phi_\xi, \phi_\eta \rangle\!\rangle = \langle \zeta,\eta\rangle \big\langle \kappa_{\ell,m}, \eta^{\otimes\ell} \otimes \xi^{\otimes m} \big\rangle \mathrm{e}^{\langle\xi,\eta\rangle}. \tag{4.16}$$

From (4.15) and (4.16) we see that

$$\begin{aligned} \langle\!\langle (D_\zeta^- \Xi_{\ell,m}(\kappa_{\ell,m}))\phi_\xi, \phi_\eta \rangle\!\rangle &= \langle\!\langle \Xi_{\ell,m}(\kappa_{\ell,m}) a^*(\zeta)\phi_\xi, \phi_\eta \rangle\!\rangle - \langle\!\langle a^*(\zeta)\Xi_{\ell,m}(\kappa_{\ell,m})\phi_\xi, \phi_\eta \rangle\!\rangle \\ &= \big\langle \kappa_{\ell,m}, \eta^{\otimes\ell} \otimes m\xi^{\otimes(m-1)} \otimes \zeta \big\rangle \mathrm{e}^{\langle\xi,\eta\rangle} \\ &= m \big\langle \kappa_{\ell,m} * \zeta, \eta^{\otimes\ell} \otimes \xi^{\otimes(m-1)} \big\rangle \mathrm{e}^{\langle\xi,\eta\rangle} \\ &= m \langle\!\langle \Xi_{\ell,m-1}(\kappa_{\ell,m} * \zeta)\phi_\xi, \phi_\eta \rangle\!\rangle, \end{aligned}$$

which shows the first equality in (4.14). The second one is proved similarly. □

Theorem 4.11. — *Let $\Xi \in \mathcal{L}((E),(E)^*)$ with Fock expansion*

$$\Xi = \sum_{\ell,m=0}^{\infty} \Xi_{\ell,m}(\kappa_{\ell,m}).$$

Then,

$$D_\zeta^- \Xi = \sum_{\ell=0}^{\infty}\sum_{m=1}^{\infty} m\Xi_{\ell,m-1}(\kappa_{\ell,m} * \zeta), \quad D_\zeta^+ \Xi = \sum_{\ell=1}^{\infty}\sum_{m=0}^{\infty} \ell\Xi_{\ell-1,m}(\zeta * \kappa_{\ell,m}),$$

where the right hand sides converge in $\mathcal{L}((E),(E)^*)$.

Proof. — By virtue of Proposition 4.10 one needs only to check the convergence, which is verified by using standard estimates of integral kernel operators (see [**12**, Section 4.3]). □

4.4. QWN-derivatives of generalized white noise operator fields

Lemma 4.12. — *Let $L \in \mathcal{L}(E, \mathcal{L}((E),(E)^*))$ be a generalized white noise operator field. Then, $(\zeta,\xi) \mapsto D_\zeta^\pm(L(\xi))$ is a continuous bilinear map from $E \times E$ into $\mathcal{L}((E),(E)^*)$.*

Proof. — Inequality (4.13) in the proof of Theorem 4.6 being applied to $\Xi = L(\xi)$, we obtain

$$\big|\langle\!\langle (D_\zeta^- L(\xi))\phi, \psi\rangle\!\rangle\big| \le C \| L(\xi) \|_{-p} \cdot |\zeta|_p \cdot \|\phi\|_{p+q} \cdot \|\psi\|_{p+q}. \tag{4.17}$$

Since $L : E \to \mathcal{L}((E),(E)^*) \cong ((E)\otimes(E))^*$ is continuous, there exists $p \ge 0$ and $K \ge 0$ such that $\| L(\xi) \|_{-p} \le K|\xi|_p$. Hence (4.17) becomes

$$\big|\langle\!\langle (D_\zeta^- L(\xi))\phi, \psi\rangle\!\rangle\big| \le CK|\xi|_p \cdot |\zeta|_p \cdot \|\phi\|_{p+q} \cdot \|\psi\|_{p+q}, \tag{4.18}$$

which proves the continuity of $(\zeta,\xi) \mapsto D_\zeta^-(L(\xi))$. The rest is similar. □

For $L \in \mathcal{L}(E, \mathcal{L}((E),(E)^*))$ the qwn-derivative $D_\zeta^\pm L$, $\zeta \in E$, is defined by

$$(D_\zeta^\pm L)(\xi) = D_\zeta^\pm(L(\xi)), \quad \xi \in E. \tag{4.19}$$

Then $D_\zeta^\pm L \in \mathcal{L}(E, \mathcal{L}((E),(E)^*))$ by Lemma 4.12. Moreover, (4.18) yields

Corollary 4.13. — *For each $\zeta \in E$, the qwn-differential operator $D_\zeta^\pm$ is a continuous operator from $\mathcal{L}(E, \mathcal{L}((E),(E)^*))$ into itself.*

4.5. Generalized integral kernel operators. — Let

$$L \in \mathcal{L}(E, \mathcal{L}((E), (E)^*))$$

be a generalized white noise operator field. Then by the characterization theorem for operator symbols (Theorem 3.2) one may prove without difficulty that there exists a unique $\Xi_1 \in \mathcal{L}((E), (E)^*)$ satisfying

$$\langle\!\langle \Xi_1 \phi_\xi, \phi_\eta \rangle\!\rangle = \langle\!\langle L(\xi)\phi_\xi, \phi_\eta \rangle\!\rangle, \quad \xi, \eta \in E. \tag{4.20}$$

In a similar manner there exists $\Xi_2 \in \mathcal{L}((E), (E)^*)$ such that

$$\langle\!\langle \Xi_2 \phi_\xi, \phi_\eta \rangle\!\rangle = \langle\!\langle L(\eta)\phi_\xi, \phi_\eta \rangle\!\rangle, \quad \xi, \eta \in E. \tag{4.21}$$

The white noise operators defined in (4.20) and (4.21) are written in formal integral forms:

$$\Xi_1 = Q^-(L) = \int_T L_t a_t \, dt, \quad \Xi_2 = Q^+(L) = \int_T a_t^* L_t \, dt, \tag{4.22}$$

respectively and called generalized integral kernel operators [14]. These are considered as generalizations of quantum stochastic integrals, see also [13].

A generalized integral kernel operator emerges naturally in an integral kernel operator $\Xi_{\ell,m}(\kappa_{\ell,m})$. Assume $m \geq 1$. Then $L : \zeta \mapsto \Xi_{\ell,m-1}(\kappa_{\ell,m} * \zeta)$ is a generalized white noise operator field by Proposition 4.3. We write

$$\int_T L_t a_t \, dt = \int_T \Xi_{\ell,m-1}(\kappa_{\ell,m} * \delta_t) a_t \, dt.$$

Similarly, if $\ell \geq 1$, a generalized integral kernel operator

$$\int_T a_t^* \Xi_{\ell-1,m}(\delta_t * \kappa_{\ell,m}) \, dt$$

is defined.

Proposition 4.14. — *For $\kappa \in (E^{\otimes(\ell+m)})^*$ it holds that*

$$\Xi_{\ell,m}(\kappa_{\ell,m}) = \int_T \Xi_{\ell,m-1}(\kappa_{\ell,m} * \delta_t) a_t \, dt, \quad m \geq 1, \tag{4.23}$$

$$\Xi_{\ell,m}(\kappa_{\ell,m}) = \int_T a_t^* \Xi_{\ell-1,m}(\delta_t * \kappa_{\ell,m}) \, dt, \quad \ell \geq 1, \tag{4.24}$$

where the right hand sides are generalized integral kernel operators.

Proof. — We set $L(\zeta) = \Xi_{\ell,m-1}(\kappa_{\ell,m} * \zeta)$ and

$$\Xi = \int_T L_t a_t \, dt = \int_T \Xi_{\ell,m-1}(\kappa_{\ell,m} * \delta_t) a_t \, dt.$$

By definition we have

$$\begin{aligned}\langle\!\langle \Xi\phi_\xi, \phi_\eta\rangle\!\rangle &= \big\langle\!\big\langle L(\xi)\phi_\xi, \phi_\eta\big\rangle\!\big\rangle = \big\langle\!\big\langle \Xi_{\ell,m-1}(\kappa_{\ell,m} * \xi)\phi_\xi, \phi_\eta\big\rangle\!\big\rangle \\ &= \big\langle \kappa_{\ell,m} * \xi, \eta^{\otimes \ell} \otimes \xi^{\otimes(m-1)}\big\rangle \mathrm{e}^{\langle \xi,\eta\rangle} = \big\langle \kappa_{\ell,m}, \eta^{\otimes\ell} \otimes \xi^{\otimes m}\big\rangle \mathrm{e}^{\langle\xi,\eta\rangle} \\ &= \langle\!\langle \Xi_{\ell,m}(\kappa_{\ell,m})\phi_\xi, \phi_\eta\rangle\!\rangle,\end{aligned}$$

which proves (4.23). The proof of (4.24) is similar. □

Corollary 4.15. — *For $\kappa_{\ell,m} \in (E^{\otimes(\ell+m)})^*$ it holds that*

$$Q^- D^- \Xi_{\ell,m}(\kappa_{\ell,m}) = m\Xi_{\ell,m}(\kappa_{\ell,m}), \quad Q^+D^+\Xi_{\ell,m}(\kappa_{\ell,m}) = \ell\,\Xi_{\ell,m}(\kappa_{\ell,m}). \tag{4.25}$$

Moreover,

$$[N, \Xi] = (Q^+D^+ - Q^-D^-)\Xi, \quad \Xi \in \mathcal{L}\big((E), (E)^*\big), \tag{4.26}$$

where N is the number operator.

Proof. — Formulae (4.25) follow by simple combination of Propositions 4.10 and 4.14. From $[N, \Xi_{\ell,m}(\kappa_{\ell,m})] = (\ell - m)\,\Xi_{\ell,m}(\kappa_{\ell,m})$, which is easily seen and well known, we obtain (4.26). □

Theorem 4.16. — *For $\zeta \in E$ and $L \in \mathcal{L}(E, \mathcal{L}((E),(E)^*))$ it holds that*

$$D_\zeta^- \int_T L_t a_t \,\mathrm{d}t = \int_T (D_\zeta^- L)_t\, a_t \,\mathrm{d}t + L(\zeta), \tag{4.27}$$

$$D_\zeta^+ \int_T a_t^* L_t \,\mathrm{d}t = \int_T a_t^* (D_\zeta^+ L)_t \,\mathrm{d}t + L(\zeta). \tag{4.28}$$

Proof. — We show only (4.27) for the proof of (4.28) is similar. Set

$$\Xi = \int_T L_t a_t \,\mathrm{d}t.$$

Then we have

$$\begin{aligned}\big\langle\!\big\langle \Xi a^*(\zeta)\phi_\xi, \phi_\eta\big\rangle\!\big\rangle &= \frac{\mathrm{d}}{\mathrm{d}\theta}\langle\!\langle \Xi\phi_{\xi+\theta\zeta}, \phi_\eta\rangle\!\rangle\Big|_{\theta=0} = \frac{\mathrm{d}}{\mathrm{d}\theta}\big\langle\!\big\langle L(\xi+\theta\zeta)\phi_{\xi+\theta\zeta}, \phi_\eta\big\rangle\!\big\rangle\Big|_{\theta=0} \\ &= \big\langle\!\big\langle L(\xi)a^*(\zeta)\phi_\xi, \phi_\eta\big\rangle\!\big\rangle + \big\langle\!\big\langle L(\zeta)\phi_\xi, \phi_\eta\big\rangle\!\big\rangle \end{aligned}\tag{4.29}$$

and

$$\begin{aligned}\big\langle\!\big\langle a^*(\zeta)\Xi\phi_\xi, \phi_\eta\big\rangle\!\big\rangle &= \langle \zeta, \eta\rangle \langle\!\langle \Xi\phi_\xi, \phi_\eta\rangle\!\rangle = \langle\zeta,\eta\rangle\langle\!\langle L(\xi)\phi_\xi, \phi_\eta\rangle\!\rangle \\ &= \big\langle\!\big\langle L(\xi)\phi_\xi, a(\zeta)\phi_\eta\big\rangle\!\big\rangle = \big\langle\!\big\langle a^*(\zeta)L(\xi)\phi_\xi, \phi_\eta\big\rangle\!\big\rangle.\end{aligned}\tag{4.30}$$

Then from (4.29) and (4.30) we see that

$$\begin{aligned}\big\langle\!\big\langle (D_\zeta^-\Xi)\phi_\xi, \phi_\eta\big\rangle\!\big\rangle &= \big\langle\!\big\langle [\Xi, a^*(\zeta)]\phi_\xi, \phi_\eta\big\rangle\!\big\rangle \\ &= \big\langle\!\big\langle [L(\xi), a^*(\zeta)]\phi_\xi, \phi_\eta\big\rangle\!\big\rangle + \big\langle\!\big\langle L(\zeta)\phi_\xi, \phi_\eta\big\rangle\!\big\rangle \\ &= \big\langle\!\big\langle (D_\zeta^- L(\xi))\phi_\xi, \phi_\eta\big\rangle\!\big\rangle + \big\langle\!\big\langle L(\zeta)\phi_\xi, \phi_\eta\big\rangle\!\big\rangle.\end{aligned}$$

Therefore, by (4.19) we have

$$\langle\!\langle (D_\zeta^- \Xi)\phi_\xi, \phi_\eta \rangle\!\rangle = \langle\!\langle (D_\zeta^- L)(\xi)\phi_\xi, \phi_\eta \rangle\!\rangle + \langle\!\langle L(\zeta)\phi_\xi, \phi_\eta \rangle\!\rangle$$

which proves (4.27). □

5. Pointwise quantum white noise derivatives

Corollary 4.8 says that for an arbitrary white noise operator

$$\Xi \in \mathcal{L}\big((E), (E)^*\big)$$

the qwn-derivatives $D_t^{\pm}\Xi$ are defined as $\mathcal{L}((E), (E)^*)$-valued distributions in $t \in T$. We shall discuss when $D_t^{\pm}\Xi$ has a pointwise meaning.

Definition 5.1. — A generalized white noise operator field

$$L \in \mathcal{L}\big(E, \mathcal{L}\big((E), (E)^*\big)\big)$$

is called L^2-*smooth* if it admits a continuous extension to $H = L^2(T)$.

Definition 5.2. — A map $t \mapsto L_t \in \mathcal{L}((E), (E)^*)$ is called an $\mathcal{L}((E), (E)^*)$-*valued* L^2-*function on* T if there exists a continuous norm $\|.\|_+$ on $(E) \otimes (E)$ such that

$$\int_T \| L_t \|_-^2 \, \mathrm{d}t < \infty,$$

where $\|.\|_-$ is the dual norm on $((E)\otimes(E))^* \cong \mathcal{L}((E), (E)^*)$ derived from $\|.\|_+$ through $(E) \otimes (E) \subset \Gamma(H) \otimes \Gamma(H) \subset ((E) \otimes (E))^*$.

Lemma 5.3. — *Let $t \mapsto L_t \in \mathcal{L}((E), (E)^*)$ be an $\mathcal{L}((E), (E)^*)$-valued L^2-function on T. Then for any $\phi, \psi \in (E)$ the function $t \mapsto \langle\!\langle L_t\phi, \psi \rangle\!\rangle$ belongs to $H = L^2(T)$. Moreover, there exists an L^2-smooth generalized white noise operator field Ξ such that*

$$\langle\!\langle \Xi(\zeta)\phi, \psi \rangle\!\rangle = \int_T \zeta(t) \langle\!\langle L_t\phi, \psi \rangle\!\rangle \mathrm{d}t, \quad \zeta \in H, \quad \phi, \psi \in (E). \tag{5.31}$$

Proof. — Note first that

$$\int_T \big|\langle\!\langle L_t\phi, \psi \rangle\!\rangle\big|^2 \mathrm{d}t = \int_T \big|\langle\!\langle L_t, \psi \otimes \phi \rangle\!\rangle\big|^2 \mathrm{d}t \le \| \psi \otimes \phi \|_+^2 \int_T \| L_t \|_-^2 \, \mathrm{d}t.$$

It is then obvious that $t \mapsto \langle\!\langle L_t\phi, \psi \rangle\!\rangle$ belongs to $H = L^2(T)$. Moreover, since we have

$$\Big| \int_T \zeta(t) \langle\!\langle L_t\phi, \psi \rangle\!\rangle \mathrm{d}t \Big| \le \| \phi \otimes \psi \|_+ |\, \zeta \,|_0 \Big(\int_T \| L_t \|_-^2 \, \mathrm{d}t \Big)^{1/2}, \tag{5.32}$$

there exists $\Xi = \Xi(\zeta) \in \mathcal{L}((E), (E)^*)$ such that

$$\int_T \zeta(t) \langle\!\langle L_t\phi, \psi \rangle\!\rangle \mathrm{d}t = \langle\!\langle \Xi(\zeta)\phi, \psi \rangle\!\rangle, \quad \phi, \psi \in (E).$$

In view of (5.32) again, we see that $\Xi : H \to \mathcal{L}((E), (E)^*)$ is continuous. Namely, Ξ is an L^2-smooth generalized white noise operator field. □

For $p,q\in\mathbb{R}$ we define a new norm of $\kappa\in(E^{\otimes(\ell+m)})^*$ by

$$|\,\kappa\,|^2_{\ell,m;p,q}=\sum_{\boldsymbol{i},\boldsymbol{j}}\big|\langle\kappa,e(\boldsymbol{i})\otimes e(\boldsymbol{j})\rangle\big|^2\cdot|\,e(\boldsymbol{i})\,|_p^2\cdot|\,e(\boldsymbol{j})\,|_q^2,$$

where $e(\boldsymbol{i})=e_{i_1}\otimes\cdots\otimes e_{i_\ell}$ and $e(\boldsymbol{j})=e_{j_1}\otimes\cdots\otimes e_{j_m}$, see Section 2.1.

Theorem 5.4. — *For $\kappa_{\ell,m}\in(E^{\otimes\ell})^*\otimes E_1^{\otimes m}$, $m\geq 1$, there exists an $\mathcal{L}((E),(E)^*)$-valued L^2-function $M:T\to\mathcal{L}((E),(E)^*)$ such that*

$$\langle\!\langle(D_\zeta^-\Xi_{\ell,m}(\kappa_{\ell,m}))\phi,\psi\rangle\!\rangle=m\int_T\zeta(t)\langle\!\langle M_t\phi,\psi\rangle\!\rangle\,\mathrm{d}t,\quad\phi,\psi\in(E).\tag{5.33}$$

Similarly, for $\kappa_{\ell,m}\in E_1^{\otimes\ell}\otimes(E^{\otimes m})^*$, $\ell\geq 1$, there exists an $\mathcal{L}((E),(E)^*)$-valued L^2-function $L:T\to\mathcal{L}((E),(E)^*)$ such that

$$\langle\!\langle(D_\zeta^+\Xi_{\ell,m}(\kappa_{\ell,m}))\phi,\psi\rangle\!\rangle=\ell\int_T\zeta(t)\langle\!\langle L_t\phi,\psi\rangle\!\rangle\,\mathrm{d}t,\quad\phi,\psi\in(E).\tag{5.34}$$

Proof. — We show only (5.33) for the proof of (5.34) is similar. By assumption (A4) the map $t\mapsto\delta_t\in E^*$ is continuous, however, it is not known whether there is $p\geq 0$ such that $|\,\delta_t\,|_{-p}<\infty$ for all $t\in T$. Nevertheless, from (2.6) we see that $|\,\delta_t\,|_{-1}<\infty$ for a.e. $t\in T$. For such a $t\in T$ we define $\kappa_{\ell,m}*\delta_t$ by

$$\langle\kappa_{\ell,m}*\delta_t,f\otimes g\rangle=\langle\kappa_{\ell,m},f\otimes g\otimes\delta_t\rangle,\quad f\in E^{\otimes\ell},\quad g\in E_{-1}^{\otimes(m-1)}.$$

In fact, taking $q>0$ such that $|\,\kappa_{\ell,m}\,|_{\ell,m;-q,1}<\infty$, we obtain

$$\big|\langle\kappa_{\ell,m}*\delta_t,f\otimes g\rangle\big|\leq|\,\kappa_{\ell,m}\,|_{\ell,m;-q,1}\cdot|\,\delta_t\,|_{-1}\cdot|\,f\,|_q\cdot|\,g\,|_{-1},$$

which means that $\kappa_{\ell,m}*\delta_t\in E_{-q}^{\otimes\ell}\otimes E_1^{\otimes(m-1)}$. On the other hand, by similar argument as in (4.10) we obtain

$$\begin{aligned}\big\|\,\Xi_{\ell,m-1}(\kappa_{\ell,m}*\delta_t)\phi\,\big\|_{-q}&\leq C|\,\kappa_{\ell,m}*\delta_t\,|_{-q}\cdot\|\,\phi\,\|_q\\&\leq C|\,\kappa_{\ell,m}\,|_{\ell+m-1,1;-q,1}\cdot|\,\delta_t\,|_{-1}\cdot\|\,\phi\,\|_q,\quad\phi\in(E).\end{aligned}\tag{5.35}$$

Define $M_t=\Xi_{\ell,m-1}(\kappa_{\ell,m}*\delta_t)$ if $|\,\delta_t\,|_{-1}<\infty$ and and $M_t=0$ otherwise. Then we see easily from (5.35) that $M:T\to\mathcal{L}((E),(E)^*)$ is an $\mathcal{L}((E),(E)^*)$-valued L^2-function. Moreover, (5.33) follows by applying (5.31). □

The *pointwise qwn-derivatives* of $\Xi_{\ell,m}(\kappa_{\ell,m})$ are defined by

$$D_t^-\Xi_{\ell,m}(\kappa_{\ell,m})=mM_t,\quad D_t^+\Xi_{\ell,m}(\kappa_{\ell,m})=\ell\,L_t,$$

where L,M are the $\mathcal{L}((E),(E)^*)$-valued L^2-functions introduced in Theorem 5.4. In this case we also write

$$D_t^-\Xi_{\ell,m}(\kappa_{\ell,m})=m\,\Xi_{\ell,m-1}(\kappa_{\ell,m}*\delta_t),\quad D_t^+\Xi_{\ell,m}(\kappa_{\ell,m})=\ell\,\Xi_{\ell,m-1}(\kappa_{\ell,m}*\delta_t).$$

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