Kesten Measures, Wigner Semicircle Law and Beyond

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1 Independence and Central Limit Theorem

Classical Central Limit Theorem

$\{X_n\}$: **a sequence of independent random variables,** $\mathbf{E}(X_i) = 0$, $\mathbf{E}(X_i^2) = 1$

$$\frac{X_1 + X_2 + \cdots + X_N}{\sqrt{N}} \longrightarrow \textbf{Gaussian measure}$$

which means, for example,

$$\lim_{N\to\infty} \mathbf{E}\left[\left(\frac{X_1 + X_2 + \cdots + X_N}{\sqrt{N}}\right)^m\right] = \int_{-\infty}^{+\infty} x^m \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, \qquad m = 1, 2, \ldots .$$

Free Central Limit Theorem

$\{a_n\}$: **a sequence of free independent random variables in** $(\mathcal{A}, \varphi)$

$a_i = a_i^*$, $\varphi(a_i) = 0$, $\varphi(a_i^2) = 1$

$$\frac{a_1 + a_2 + \cdots + a_N}{\sqrt{N}} \longrightarrow \textbf{Wigner semicircle law}$$

which means that

$$\lim_{N\to\infty} \varphi\left[\left(\frac{X_1 + X_2 + \cdots + X_N}{\sqrt{N}}\right)^m\right] = \int_{-2}^{+2} x^m \frac{1}{2\pi} \sqrt{4 - x^2}\, dx, \qquad m = 1, 2, \ldots .$$

Various Concepts of Independence

$(\mathcal{A}, \varphi)$: **Algebraic probability space**

independence = a rule of computing mixed moments: $\varphi(a_{i_1} \dots a_{i_n})$

independence	commutative	free	Boolean	monotone
$\varphi(aba) =$	$\varphi(a^2)\varphi(b)$	$\varphi(a^2)\varphi(b)$	$\varphi(a)^2\varphi(b)$	$\varphi(a^2)\varphi(b)$
$\varphi(bab) =$	$\varphi(a)\varphi(b^2)$	$\varphi(a)\varphi(b^2)$	$\varphi(a)\varphi(b)^2$	$\varphi(a)\varphi(b)^2$
$\varphi(abab) =$	$\varphi(a^2)\varphi(b^2)$	$\varphi(a)^2\varphi(b^2) + \varphi(a^2)\varphi(b)^2 - \varphi(a)^2\varphi(b)^2$	$\varphi(a)^2\varphi(b)^2$	$\varphi(a^2)\varphi(b)^2$
CLM	Gaussian	Wigner	Bernoulli	arcsine

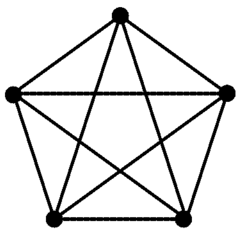
CLM (=central limit measure) is defined as

$$\lim_{N\to\infty} \varphi\left[\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} a_n\right)^m\right] = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots,$$

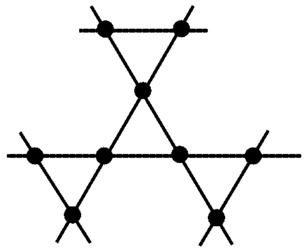
where $\{a_i\}$ **is a sequence of {commutative/free/Boolean/monotone} independent random variables such that** $a_i = a_i^*$, $\varphi(a_i) = 0$, $\varphi(a_i^2) = 1$.

2 Spectral Analysis on Graphs

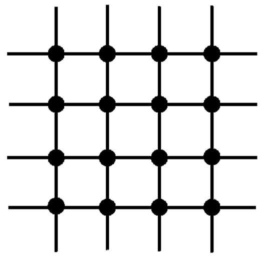
Definition A *graph* is a pair $\mathcal{G} = (V, E)$, where V is the set of *vertices* and E the set of *edges*. We write $x \sim y$ (adjacent) if they are connected by an edge.



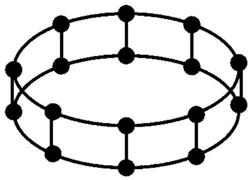
4-regular (drg)
finite



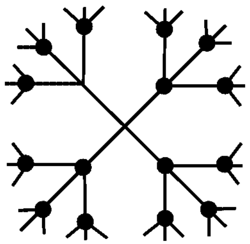
4-regular (drg)
infinite



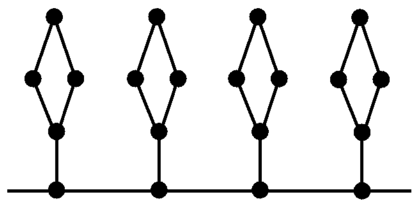
4-regular
2-dim lattice



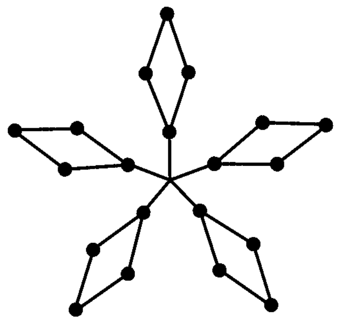
3-regular
ladder graph



4-regular (drg)
homogeneous tree



comb graph



star graph

Definition The ***adjacency matrix*** A of a graph $\mathcal{G} = (V, E)$ is defined by

$$A_{xy} = \begin{cases} 1, & x \sim y, \\ 0, & \text{otherwise}. \end{cases}$$

Question 1 Given a graph $\mathcal{G} = (V, E)$ and a certain state $\langle \cdot \rangle$, find a probability measure μ on R (spectral distribution in $\langle \cdot \rangle$) satisfying

$$\langle A^m \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx) \qquad m = 1, 2, \ldots .$$

Question 2 Given a growing family of graphs $\mathcal{G}_\nu = (V^{(\nu)}, E^{(\nu)})$ and a certain state $\langle \cdot \rangle$, find a probability measure μ on R (spectral distribution in $\langle \cdot \rangle$) satisfying

$$\langle A_\nu^m \rangle \approx \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \ldots, \qquad \text{as } \nu \to \infty$$

States under consideration

(1) $\langle X \rangle = \langle \delta_o, X\delta_o \rangle$: ***vacuum state***, where $o \in V$ is a fixed origin.

(2) $\langle X \rangle = \langle T_q\delta_o, X\delta_o \rangle$: ***deformed vacuum state***, where T_q is a certain positive operator.

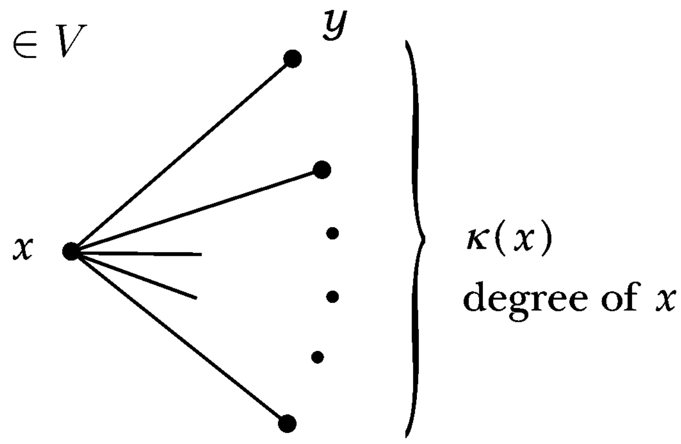
Assumptions on $\mathcal{G} = (V, E)$

(a) A graph is ***connected***,

i.e., any pair of distinct vertices are connected by a walk.

(b) A graph is ***locally finite***, i.e.,

$$\kappa(x) = |\{y \in V\,;\, y \sim x\}| < \infty \qquad \text{for any } x \in V$$



♦ A is a selfadjoint operator acting on $\ell^2(V)$:

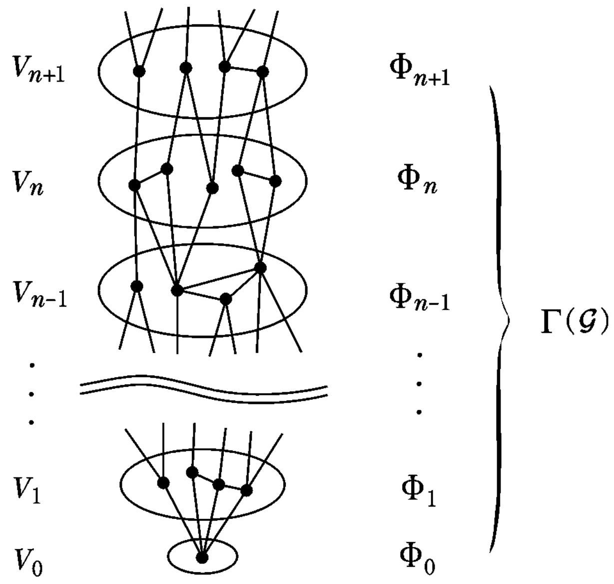
$$Af(x) = \sum_{y \in V} A_{xy} f(y) = \sum_{y \sim x} f(y) \qquad \text{or equivalently,} \quad A_{xy} = \langle \delta_x, A\delta_y \rangle$$

♦ A is bounded if and only if $\mathcal{G}$ is uniformly locally finite, i.e.,

$$\sup_{x \in V} \kappa(x) < \infty$$

3 Quantum Decomposition of Adjacency Matrix A

Fix an origin $o \in V$ of $\mathcal{G} = (V, E)$.

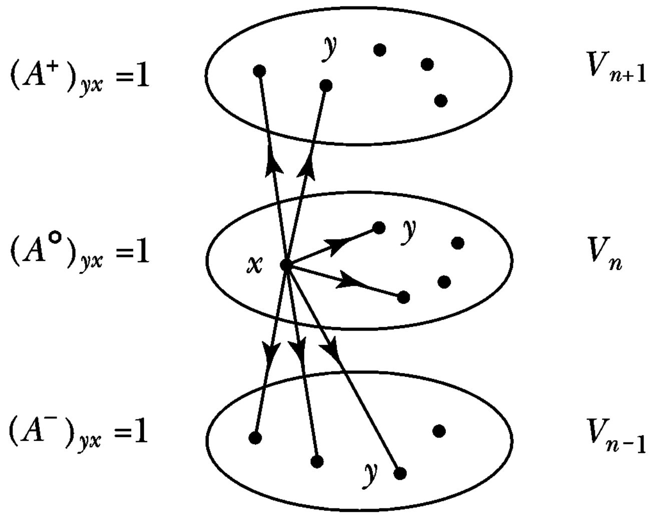


Stratification (Distance Partition)

$$V = \bigcup_{n=0}^{\infty} V_n, \qquad V_n = \{x \in V\,;\, \partial(o,x) = n\}$$

Hilbert space $\Gamma(\mathcal{G}) \subset \ell^2(V)$

$$\Gamma(\mathcal{G}) = \sum_{n=0}^{\infty} \oplus \mathbf{C}\Phi_n, \qquad \Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} \delta_x$$



Quantum decomposition

$$A = A^+ + A^- + A^\circ$$

$$(A^+)^* = A^-, \qquad (A^\circ)^* = A^\circ$$

(Case 1) $\Gamma(\mathcal{G})$ **is invariant under** A^ϵ

(Case 2) $\Gamma(\mathcal{G})$ **is asymptotically invariant under** A^ϵ

(Case 3) $\Gamma(\mathcal{G})$ **is not invariant under** A^ϵ

Theorem **1** *If* $\Gamma(\mathcal{G})$ *is invariant under* A^+, A^-, A°, *there exists a pair of sequences* $\{\alpha_n\}$ *and* $\{\omega_n\}$ *such that*

$$A^+\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1}, \qquad A^-\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1}, \qquad A^\circ\Phi_n = \alpha_{n+1}\Phi_n.$$

In particular, $(\Gamma(\mathcal{G}), A^+, A^-)$ *is an interacting Fock space with Jacobi sequence* $\{\omega_n\}$.

Theorem **2** *Notations and Assumptions being as above, there exists a probability measure* μ *on* $\mathbf{R}$ *such that*

$$\langle \Phi_0, A^m\Phi_0\rangle = \langle \Phi_0, (A^+ + A^- + A^\circ)^m\Phi_0\rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots.$$

Moreover, the Stieltjes transform $G_\mu(z)$ *of* μ *satisfies*

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z-\alpha_1} \, \underset{-}{} \, \frac{\omega_1}{z-\alpha_2} \, \underset{-}{} \, \frac{\omega_2}{z-\alpha_3} \, \underset{-}{} \, \frac{\omega_3}{z-\alpha_4} \, \underset{-}{} \cdots .$$

Remark μ **is the orthogonalizing measure for the polynomials defined by**

$$P_0(x) = 1, \quad P_1(x) = x - \alpha_1, \quad xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x).$$

$(\{\omega_n\}, \{\alpha_n\})$ **is called the *Szegö-Jacobi parameter of*** μ.

Continued fraction

$$G_\mu(z) = \frac{1}{z-\alpha_1 -} \frac{\omega_1}{z-\alpha_2 -} \frac{\omega_2}{z-\alpha_3 -} \frac{\omega_3}{z-\alpha_4 -} \cdots$$

$$= \cfrac{1}{z-\alpha_1 - \cfrac{\omega_1}{z-\alpha_2 - \cfrac{\omega_2}{z-\alpha_3 - \cfrac{\omega_3}{z-\alpha_4 - \cdots}}}}.$$

Stieltjes transform

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \int_{-\infty}^{+\infty} \frac{dF(x)}{z-x} \qquad z \in \mathbf{C}, \quad \operatorname{Im} z \neq 0,$$

$F(x) = \mu((-\infty, x])$ **distribution function (right-continuous)**

Stieltjes inversion formula

$$\frac{1}{2}\{F(t)+F(t-0)\} - \frac{1}{2}\{F(s)+F(s-0)\} = -\frac{1}{\pi}\lim_{y\to+0}\int_s^t \operatorname{Im} G_\mu(x+iy)dx, \qquad s<t$$

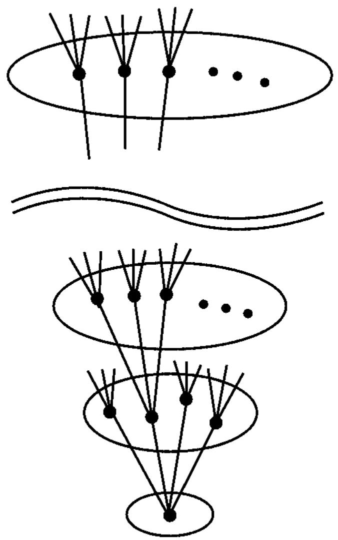
$\rho(x) = -\frac{1}{\pi}\lim_{y\to+0} \operatorname{Im} G_\mu(x+iy)$ **absolutely continuous part of** $\mu(dx)$

4 Kesten Measures

4.1 Homogeneous Trees

T_κ: **homogeneous tree of degree** $\kappa \geq 2$

(T_{2N} **Cayley graph of the free group on** N **generators)**



Stratification of T_4

Quantum decomposition $A = A^+ + A^- + A^\circ$

$$A^+\Phi_0 = \sqrt{\kappa}\,\Phi_1, \quad A^+\Phi_n = \sqrt{\kappa - 1}\,\Phi_{n+1} \quad \text{for } n \geq 1$$

$$A^-\Phi_0 = 0, \quad A^-\Phi_1 = \sqrt{\kappa}\,\Phi_0, \quad A^-\Phi_n = \sqrt{\kappa - 1}\,\Phi_{n-1} \quad \text{for } n \geq 2$$

$$A^\circ = 0.$$

Szegö-Jacobi parameter

$$\omega_1 = \kappa, \quad \omega_2 = \omega_3 = \cdots = \kappa - 1, \qquad \alpha_n \equiv 0$$

Stieltjes transform

$$\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z - x} = G_\mu(z) = \frac{1}{z -}\,\frac{\omega_1}{z -}\,\frac{\omega_2}{z -}\,\frac{\omega_3}{z -}\,\frac{\omega_4}{z -}\cdots$$

$$= \frac{1}{z -}\,\frac{\kappa}{z -}\,\frac{\kappa - 1}{z} - \frac{\kappa - 1}{z} - \frac{\kappa - 1}{z} - \cdots$$

Definition **The probability measure** $\mu = \mu_{a,b}$ **determined by**

$$\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z-}\,\frac{a}{z-}\,\frac{b}{z-}\,\frac{b}{z-}\,\frac{b}{z-}\cdots \qquad a>0, \quad b>0,$$

is called the *Kesten measure* with parameter a, b**. The density is given by**

$$\mu_{a,b}(dx) = \begin{cases} \rho_{a,b}(x)dx, & \textbf{if } 0 < a \le 2b, \\ \rho_{a,b}(x)dx + \dfrac{a-2b}{2(a-b)}(\delta_{-a/\sqrt{a-b}} + \delta_{a/\sqrt{a-b}}), & \textbf{if } 0 < 2b < a, \end{cases}$$

where

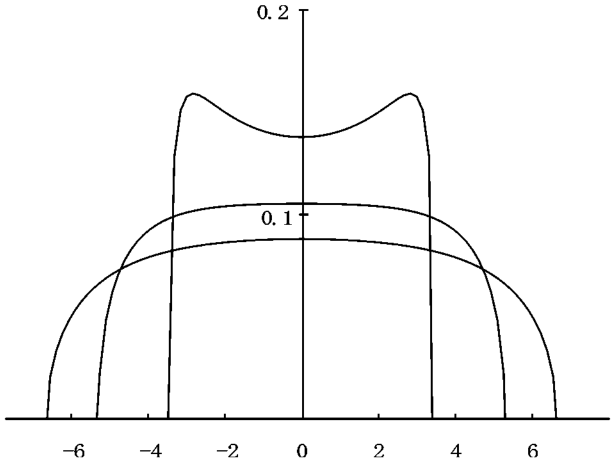
$$\rho_{a,b}(x) = \frac{a}{2\pi}\,\frac{\sqrt{4b-x^2}}{a^2-(a-b)x^2}, \qquad |x| \le 2\sqrt{b}.$$

For a homogeneous tree T_κ $(\kappa \ge 2)$

Setting $a = \kappa$**,** $b = \kappa - 1$**, we obtain**

$$\langle A_\kappa^m \rangle = \int_{-2\sqrt{\kappa-1}}^{+2\sqrt{\kappa-1}} x^m \frac{\kappa}{2\pi}\,\frac{\sqrt{4(\kappa-1)-x^2}}{\kappa^2-x^2}\,dx$$

$$m = 1, 2, \dots$$



Kesten measures (for T_4, T_8, T_{12}**)**

4.2 Asymptotics: T_κ as $\kappa \to \infty$

Actions of normalized quantum components $\langle A_\kappa \rangle = 0$, $\langle A_\kappa^2 \rangle = \kappa$.

$$\frac{A^+}{\sqrt{\kappa}}\Phi_0 = \Phi_1, \quad \frac{A^+}{\sqrt{\kappa}}\Phi_n = \sqrt{\frac{\kappa-1}{\kappa}}\,\Phi_{n+1} \quad \textbf{for } n \geq 1$$

$$\frac{A^-}{\sqrt{\kappa}}\Phi_0 = 0, \quad \frac{A^-}{\sqrt{\kappa}}\Phi_1 = \Phi_0, \quad \frac{A^-}{\sqrt{\kappa}}\Phi_n = \sqrt{\frac{\kappa-1}{\kappa}}\,\Phi_{n-1} \quad \textbf{for } n \geq 2$$

$$\frac{A^\circ}{\sqrt{\kappa}} = 0.$$

Szegö-Jacobi parameter in the limit as $\kappa \to \infty$

$$\omega_n = \lim_{\kappa\to\infty} \omega_n(\kappa) = 1 \quad \textbf{for all } n \geq 1; \qquad \alpha_n = 0 \quad \textbf{for all } n \geq 1$$

***Theorem* 3** *For the adjacency matrix A_κ of the homogeneous tree T_κ,*

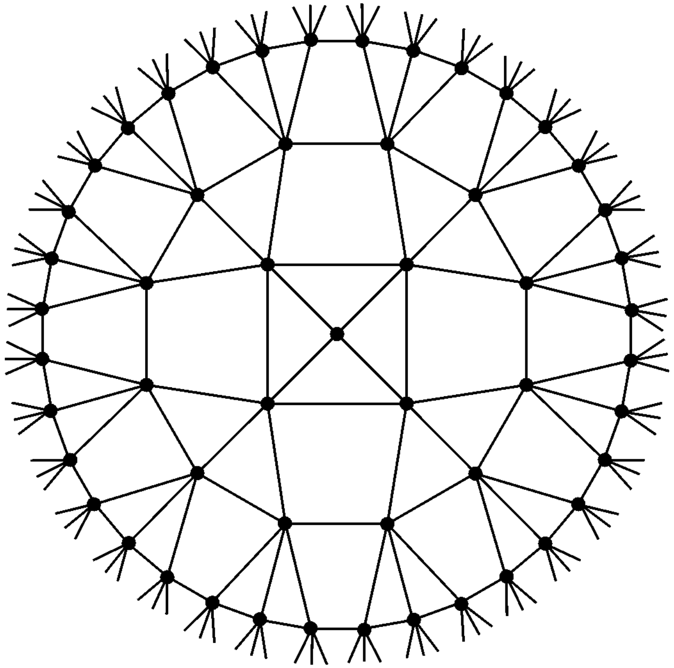
$$\left\langle \left(\frac{A_\kappa}{\sqrt{\kappa}}\right)^m \right\rangle = \langle (B^+_{\text{free}} + B^-_{\text{free}})^m \rangle = \int_{-2}^{+2} x^m \frac{1}{2\pi}\sqrt{4-x^2}\,dx, \qquad m = 1, 2, \dots.$$

Remark Let $(\pi, \ell^2(F_N))$ be the regular representation of $F_N = \langle g_1, \dots, g_N \rangle$.

$$\frac{A_{2N}}{\sqrt{2N}} = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} \frac{\pi(g_k) + \pi(g_k^{-1})}{\sqrt{2}}$$ is a sum of *free independent* random variables.

4.3 Spidernet $S(a, b, c)$ where $a \geq 1$, $b \geq 2$, $b - 1 \geq c \geq 1$

also called a *semi-regular graph* [Urakawa (2003)]



$S(4, 6, 3)$

$$\kappa(x) = \begin{cases} a & x = o \text{ (origin)} \\ b & x \neq o \end{cases}$$

$$\begin{cases} \omega_-(x) = 1 \\ \omega_+(x) = c \\ \omega_\circ(x) = b - 1 - c \end{cases} \qquad \textbf{for } x \neq o.$$

Actions of quantum components of A

$$A^+\Phi_0 = \sqrt{a}\,\Phi_1, \qquad A^+\Phi_n = \sqrt{c}\,\Phi_{n+1}, \quad n \geq 1,$$

$$A^-\Phi_0 = 0, \qquad A^-\Phi_1 = \sqrt{a}\,\Phi_0, \qquad A^-\Phi_n = \sqrt{c}\,\Phi_{n-1}, \quad n \geq 2,$$

$$A^\circ\Phi_0 = 0, \qquad A^\circ\Phi_n = (b - 1 - c)\Phi_n, \quad n \geq 1.$$

Szegö-Jacobi parameter

$$\omega_1 = a, \qquad \omega_2 = \omega_3 = \cdots = c,$$
$$\alpha_1 = 0, \qquad \alpha_2 = \alpha_3 = \cdots = b - 1 - c.$$

Asymptotics: $S(a,b,c)$ as $a \to \infty$

Noting $\langle A \rangle = 0$ and $\langle A^2 \rangle = a$,

$$\frac{A^+}{\sqrt{a}}\Phi_0 = \Phi_1, \qquad \frac{A^+}{\sqrt{a}}\Phi_n = \sqrt{\frac{c}{a}}\,\Phi_{n+1}, \quad n \geq 1,$$
$$\frac{A^-}{\sqrt{a}}\Phi_0 = 0, \qquad \frac{A^-}{\sqrt{a}}\Phi_1 = \Phi_0, \qquad \frac{A^-}{\sqrt{a}}\Phi_n = \sqrt{\frac{c}{a}}\,\Phi_{n-1}, \quad n \geq 2,$$
$$\frac{A^\circ}{\sqrt{a}}\Phi_0 = 0, \qquad \frac{A^\circ}{\sqrt{a}}\Phi_n = \frac{b-1-c}{\sqrt{a}}\Phi_n, \quad n \geq 1.$$

Letting $a \to \infty$ with

$$\frac{c}{a} \longrightarrow p, \qquad \frac{b-1-c}{\sqrt{a}} \longrightarrow q,$$

Szegö-Jacobi parameter in the limit

$$\omega_1 = 1, \quad \omega_2 = \omega_3 = \cdots = p, \quad 0 \leq p < \infty$$
$$\alpha_1 = 0, \quad \alpha_2 = \alpha_3 = \cdots = q, \quad 0 \leq q < \infty$$

Definition **Let** $a > 0$, $p \geq 0$, $q \geq 0$. **The probability measure** $\mu = \mu_{a,p,q}$ **determined by**

$$\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z-} \frac{a}{z-q-} \frac{p}{z-q-} \frac{p}{z-q-} \frac{p}{z-q-} \cdots$$

is called the *deformed Kesten measure*.

***Theorem* 4** *Let $A = A(a,b,c)$ be the adjacency matrix of a spidernet $S(a,b,c)$. Then we have*

$$\langle A^m \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx) \quad m = 1, 2, \ldots,$$

where $\mu = \mu_{a,c,b-1-c}$ is the deformed Kesten measure.

***Theorem* 5** *Let $A = A(a,b,c)$ be the adjacency matrix of a spidernet $S(a,b,c)$. Then in the limit as $a \to \infty$, $c/a \to p$, $(b-c)/\sqrt{a} \to q$ we have*

$$\lim \left\langle \left(\frac{A_{a,b,c}}{\sqrt{a}} \right)^m \right\rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \ldots,$$

where $\mu = \mu_{1,p,q}$ is the deformed Kesten measure.

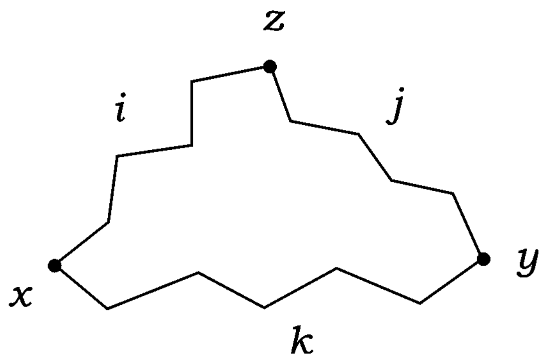
Remark **(1)** $\mu_{a,p,0}$ **is the Kesten measure with parameter** (a, p).
(2) $\mu_{1,1,0}$ **is the Wigner semicircle law.**

5 Distance Regular Graphs

Definition **A graph $\mathcal{G} = (V, E)$ is called *distance regular* if the intersection numbers:**

$$p^k_{i,j} = |\{z \in V \,;\, \partial(x,z) = i,\ \partial(y,z) = j\}|,$$

is constant for all pairs x, y such that $\partial(x,y) = k$.



Theorem **6** *Let $\mathcal{G} = (V, E)$ be a distance-regular graph with a fixed origin $o \in V$. Then $\Gamma(\mathcal{G})$ is invariant under the quantum components A^ϵ. Moreover,*

$$A^+\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1}, \qquad A^-\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1}, \qquad A^\circ\Phi_n = \alpha_{n+1}\Phi_n,$$

holds with the coefficients given by

$$\omega_n = p^n_{1,n-1}p^{n-1}_{1,n} \qquad \alpha_n = p^{n-1}_{1,n-1}, \qquad n = 1, 2, \ldots.$$

Examples (a) *Homogeneous tree T_κ*

$$p_{1,1}^0 = \kappa, \quad p_{n,n}^0 = \kappa(\kappa-1)^{n-1}, \quad p_{1,n}^{n-1} = \kappa - 1, \quad p_{1,n}^n = 0.$$

Kesten measure (Kesten 1959)

Wigner semicircle law in the limit as $\kappa \to \infty$ (Voiculescu 1980s)

(b) *Hamming graphs $H(d,N)$* $V = \{1, 2, \dots, N\}^d$

$$p_{1,1}^0 = d(N-1), \quad p_{n,n}^0 = \binom{d}{n}(N-1)^n, \quad p_{1,n}^{n-1} = (d-n+1)(N-1), \quad p_{1,n}^n = n(N-2).$$

Gaussian measure or Poisson measure in the limit as $d, N \to \infty$

"Semiclassical" method (Hora 1998)

Quantum decomposition (Hashimoto-Obata-Tabei 2001)

(c) *Johnson graphs $J(v,d)$* $V = \{x \subset \{1, 2, \dots, v\}\,;\, |x| = d\}$ $(2d \le v)$

$$p_{1,1}^0 = d(v-d), \quad p_{n,n}^0 = \binom{d}{n}\binom{v-d}{n}, \quad p_{1,n}^{n-1} = (d-n+1)(v-d-n+1), \quad p_{1,n}^n = n(v-2n).$$

Exponential distribution or geometric distribution in the limit as $v, d \to \infty$

"Semiclassical" method (Hora 1998)

Quantum decomposition (Hashimoto-Hora-Obata 2003)

(New Example) ***Odd graph*** O_k $(k \geq 2)$

Vertex set

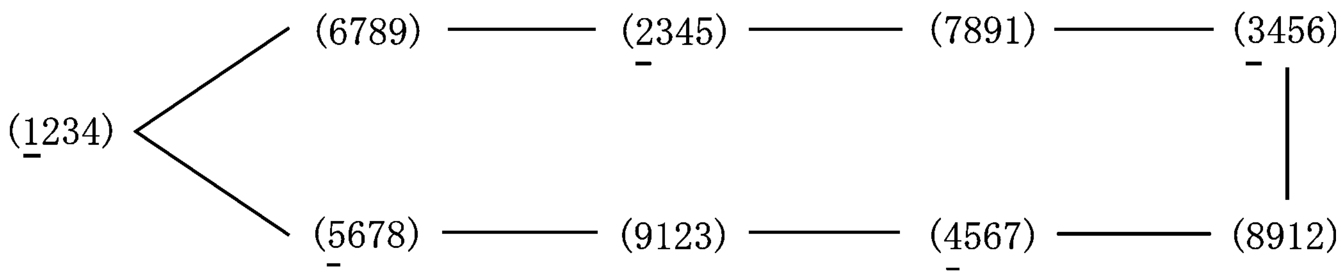
$$V = \{x \subset \Omega\,;\ |x| = k-1\}, \qquad \Omega = \{1, 2, \dots, 2k-1\}.$$

Adjacency matrix A **defined by**

$$A_{xy} = \begin{cases} 1, & \textbf{if } x \cap y = \emptyset, \\ 0, & \textbf{otherwise.} \end{cases}$$

Some geometric data

$$\deg O_k = k, \qquad \operatorname{diam} O_k = k-1$$



a cycle of O_5

Intersection numbers

$$p^n_{1,n-1} = \begin{cases} \dfrac{n}{2} \\ \dfrac{n+1}{2} \end{cases} \qquad p^n_{1,n+1} = \begin{cases} k - \dfrac{n}{2}, & n\text{: even} \\ k - \dfrac{n+1}{2}, & n\text{: odd} \end{cases} \qquad p^n_{1,n} = \begin{cases} 0, & 1 \le n \le k-2 \\ \dfrac{k+1}{2}, & n = k-1,\ k\text{: odd} \\ \dfrac{k}{2}, & n = k-1,\ k\text{: even} \end{cases}$$

Limit as $k \to \infty$

$$\omega_{2n+1} = \lim_{k\to\infty} \frac{p^{2n+1}_{1,2n}\, p^{2n}_{1,2n+1}}{k} = n+1, \quad \omega_{2n} = \lim_{k\to\infty} \frac{p^{2n}_{1,2n-1}\, p^{2n-1}_{1,2n}}{k} = n, \quad \alpha_n = \lim_{k\to\infty} \frac{p^{n-1}_{1,n-1}}{\sqrt{k}} = 0.$$

Theorem **7 (Igarashi-Obata 2004)** *For the adjacency matricx A_k of the odd graph O_k we have*

$$\lim_{k\to\infty} \left\langle \left(\frac{A_k}{\sqrt{k}} \right)^m \right\rangle = \langle \Phi_0, (B^+ + B^-)^m \Phi_0 \rangle, \qquad m = 1, 2, \dots,$$

where $B^\pm$ are the annihilation and creation operators on the interacting Fock space associated with the Jacobi parameter $\{\omega_n\} = \{1, 1, 2, 2, 3, 3, \dots\}$.

Stieltjes transform

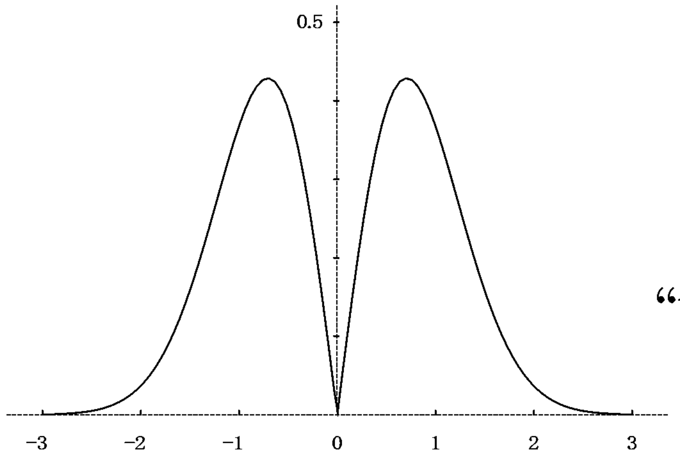
$$\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = G(z) = \frac{1}{z-}\,\frac{1}{z-}\,\frac{1}{z-}\,\frac{2}{z-}\,\frac{2}{z-}\,\frac{3}{z-}\,\frac{3}{z-}\,\frac{4}{z-}\cdots$$

$$= \frac{z}{z^2-1-}\,\frac{1^2}{z^2-3-}\,\frac{2^2}{z^2-5-}\,\frac{3^2}{z^2-7-}\cdots-\frac{(n-1)^2}{z^2-(2n-1)-}\cdots$$

$$= \int_{-\infty}^{+\infty} \frac{|x|e^{-x^2}}{z-x}\,dx$$

Theorem 8 (Igarashi-Obata, 2004) *For the adjacency matrix A_k of the odd graph O_k we have*

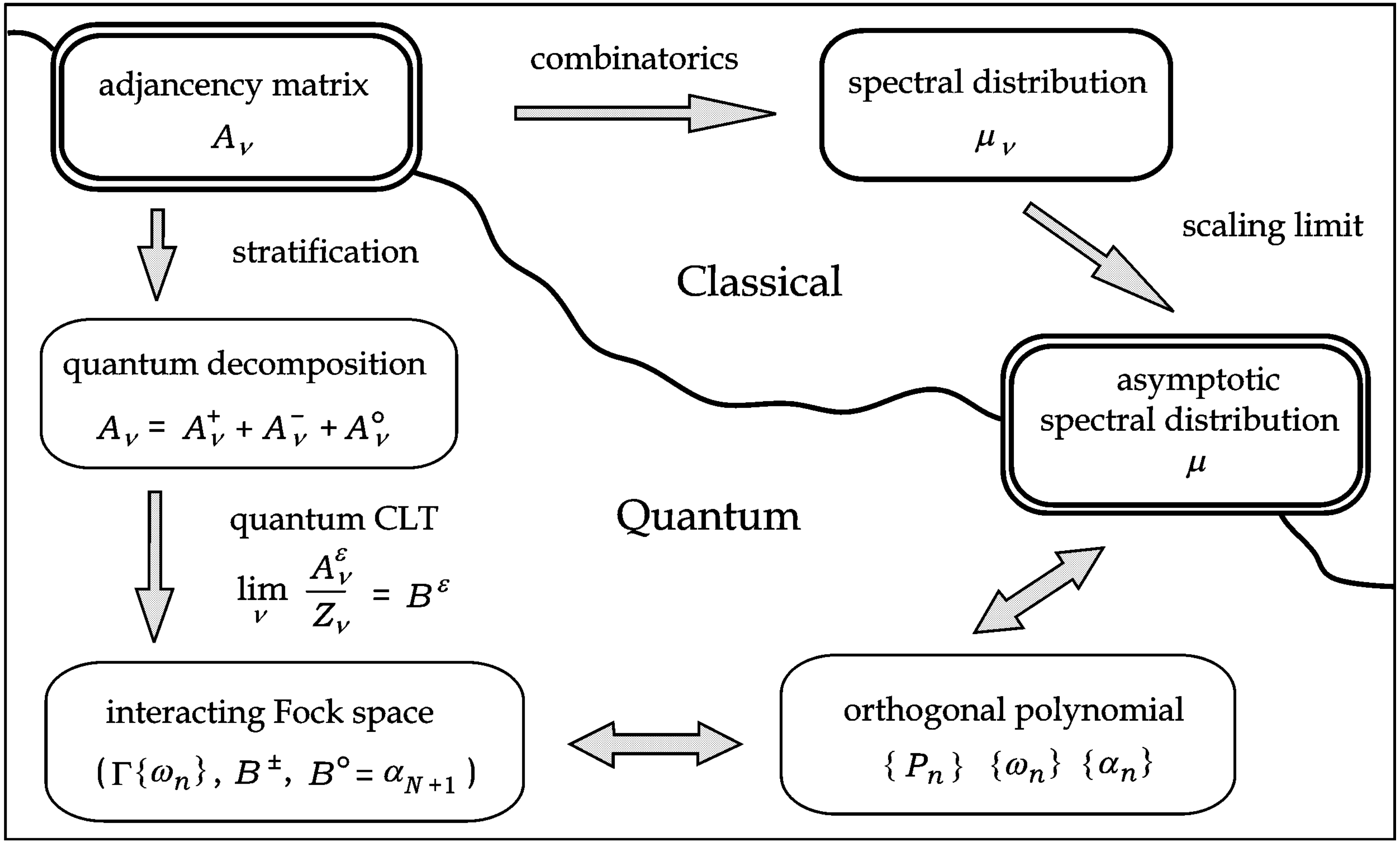
$$\lim_{k\to\infty} \left\langle \left(\frac{A_k}{\sqrt{k}}\right)^m \right\rangle = \int_{-\infty}^{+\infty} x^m |x| e^{-x^2} dx, \qquad m = 1, 2, \ldots.$$



"two-sided" Rayleigh distribution

Overview of Our Method

Given $\{\mathcal{G}_\nu = (V_\nu, E_\nu)\}$, **we wish to find** μ **such that** $\langle A_\nu^m \rangle \approx \int_{-\infty}^{+\infty} x^m \mu(dx)$



SUMMARY

1. **Free CLT was revisited from the viewpoint of** ***Asymptotic Spectral Analysis*** **on homogeneous trees.**

2. **Method of** ***Quantum Decomposition*** **of the adjacency matrix** $A = A^+ + A^- + A^\circ$ **was discussed.**

3. ***Deformed Kesten measures*****, generalizing Kesten measures and Wigner semicircle law, emerged from spidernets.**

4. **Another direction of generalization of Free CLT was discussed from the viewpoint of** ***Asymptotic Spectral Analysis*** **on distance regular graphs.**

5. **A new example (odd graphs) was shown.**

A. Hora and N. Obata: "Quantum Probability and Spectral Analysis on Graphs,"
A monograph in preparation

5.1 A New Aspect: Graph Products $\Longleftrightarrow$ Independence Concepts

$$\mathcal{G} = \text{“product” of } \mathcal{G}_1 \text{ and } \mathcal{G}_2$$

$$\Updownarrow$$

$$A = \widetilde{A^{(1)}} + \widetilde{A^{(2)}} = \text{sum of “independent” random variables}$$

independence	monotone	Boolean	commutative	free
CLM	arcsine	Bernoulli	Gaussian	Wigner
graph product	comb	star	direct	free
examples	comb graph	star graph	integer lattice	homogeneous tree