Quantum Stochastic Gradients and Quantum Hitsuda-Skorokhod Integrals

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1 Motivation of Our Work

1.1 Classical and quantum stochastic integrals of Itô type

⋆ Classical stochastic integral (Itô):

$$X_t = \int_0^t F\,dB_s \quad \text{for an adapted integrand}$$

⋆ Quantum stochastic integral (Hudson-Parthasarathy):

$$\Xi_t = \int_0^t E dA_s + \int_0^t F dA_s^* + \int_0^t G d\Lambda_s \quad \text{for adapted integrands}$$

- taking the actions on exponential vectors (operator symbols)
- and using a pararell arguments as in the case of classical Itô integrals
- $\{A_t\}, \{A_t^*\}, \{\Lambda_t\}$ appear in *quantum decompositions* of Brownian motion and Poisson process:

$$B_t = A_t + A_t^* = \int_0^t a_s\,ds + \int_0^t a_s^* ds$$

$$P_t = \Lambda_t + \sqrt{\lambda}(A_t + A_t^*) + \lambda = \int_0^t (a_s^* + \sqrt{\lambda})(a_s + \sqrt{\lambda}) ds$$

⋆ *How about generalizations to non-adapted integrands?*

1.2 Classical stochastic gradient and its adjoint action

$$H = L^2(\mathbf{R}, dt) \quad \text{with norm } \|f\|_0^2 = \langle \bar{f}, f \rangle = \int_{\mathbf{R}} |f(t)|^2 dt$$

$$\Gamma(H) = (\text{Boson Fock space over } H) = \left\{ \phi = (f_n)\,;\, f_n \in H^{\widehat{\otimes} n},\ \|\phi\|_0^2 = \sum_{n=0}^{\infty} n!\,|f_n|_0^2 < \infty \right\}$$

<u>Definition</u> The *classical stochastic gradient* is defined by

$$\nabla\phi(t) = ((n+1)f_{n+1}(t,\cdot))_{n=0}^{\infty} \qquad \text{for a suitable } \phi = (f_n) \in \Gamma(H).$$

The domain of ∇ is taken, for example, to be

$$\mathbf{D} = \left\{ \phi = (f_n)\,;\, \|\phi\|_{\mathbf{D}}^2 = \sum_{n=0}^{\infty} (n+1)n!\,|f_n|_0^2 < \infty \right\} \quad \text{[Malliavin,Nualart, Kuo,...]}$$

$$\mathcal{G}^* = \operatorname*{ind\,lim}_{p\to-\infty} \mathcal{G}_p\,, \quad \mathcal{G}_p = \left\{ \phi = (f_n)\,;\, \|\phi\|_p^2 = \sum_{n=0}^{\infty} n! e^{2pn} |f_n|_0^2 < \infty \right\} \quad \text{[Belavkin, Øksendal,...]}$$

Then, we obtain continuous linear maps:

$$\nabla : \mathbf{D} \to L^2(\mathbf{R}, \Gamma(H)) \quad \text{or} \quad \mathcal{G}^* \to L^2(\mathbf{R}, \mathcal{G}^*)$$

$$\delta = \nabla^* : L^2(\mathbf{R}, \Gamma(H)) \to \mathbf{D}^* \quad \text{or} \quad L^2(\mathbf{R}, \mathcal{G}) \to \mathcal{G}$$

The adjoint map $\delta = \nabla^*$ (also called the divergence operator) defines (non-adapted) stochastic integrals which generalize the Itô integrals (Hitsuda-Skorokhod, Zakai-Nualart-Pardoux), see e.g., Malliavin: "Stochastic Analysis."

1.3 Non-adapted quantum stochastic integrals

Belavkin (1991), Lindsay (1993) defined quantum stochastic integrals $\delta^+(\Xi), \delta^-(\Xi), \delta^0(\Xi)$ by

$$\delta^+(\Xi)\phi = \delta(\Xi\phi), \qquad \delta^-(\Xi)\phi = \int_{\mathbf{R}} \Xi(t)(\nabla\phi(t))dt, \qquad \delta^0(\Xi\phi) = \delta(\Xi\nabla\phi)$$

- taking a suitable vector ϕ to apply the classical δ

Our Aim

(1) Introduce *quantum stochastic gradients* acting on "quantum random variables," as the classical stochastic gradient acts on "random variables."

(2) Define non-adapted quantum stochastic integrals directly by the adjoint actions of quantum stochastic gradients.

(3) Show some applications (regularity properties of quantum stochastic integrals, quantum white noise derivatives, representations of quantum martingales,...)

2 Classical Stochastic Gradient Revisited

2.1 White noise triple

$(E) \subset \Gamma(H) \subset (E)^*$: originally by Kubo-Takenaka (1980), later many generalizations.

$$E_p = \text{completion of } \mathcal{S}(\mathbf{R}) \text{ with respect to } |f|_p = |A^p f|_0, \quad A = 1 + t^2 - \frac{d^2}{dt^2},$$

$$(E) = \operatorname{proj}\lim_{p\to\infty} \Gamma(E_p)$$

$\star$ (E) is a nuclear Fréchet space.

2.2 Quantum white noise

$$a_t : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, n\xi(t)\xi^{\otimes(n-1)}, 0, 0, \dots) \quad \text{annihilation process}$$

$$a_t^* : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, 0, \xi^{\otimes n}\widehat{\otimes}\delta_t, 0, \dots) \quad \text{creation process}$$

<u>Fundamental Lemma</u> The map $t \mapsto a_t$ is an $\mathcal{L}((E),(E))$-valued rapidly decreasing function, i.e., belong to $\mathcal{S}(\mathbf{R}) \otimes \mathcal{L}((E),(E)) \cong \mathcal{L}((E), \mathcal{S}(\mathbf{R}) \otimes (E)) \cong \mathcal{S}(\mathbf{R}, \mathcal{L}((E),(E)))$.

<u>Definition</u> Define a continuous linear map $\nabla : (E) \to \mathcal{S}(\mathbf{R}) \otimes (E) \cong \mathcal{S}(\mathbf{R}, (E))$ by

$$\nabla\phi(t) = a_t\phi, \qquad \phi \in (E), \quad t \in \mathbf{R}.$$

2.3 Several domains of classical stochastic gradient

According to the inclusion relations:

$$(E) \subset \mathcal{G} \subset \mathbf{D} \subset \Gamma(H) \subset \mathbf{D}^* \subset \mathcal{G}^* \subset (E)^*$$

we can define the *classical stochastic gradients* as continuous linear maps:

$$\begin{array}{ccccccccc}
(E) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathbf{D} & \longrightarrow & \Gamma(H) & \longrightarrow & \mathcal{G}^* \\
\nabla\big\downarrow & & \nabla\big\downarrow & & \nabla\big\downarrow & & \nabla\big\downarrow & & \nabla\big\downarrow \\
\mathcal{S}(\mathbf{R}, (E)) & \longrightarrow & L^2(\mathbf{R}, \mathcal{G}) & \longrightarrow & L^2(\mathbf{R}, \Gamma(H)) & \longrightarrow & L^2(\mathbf{R}, \mathbf{D}^*) & \longrightarrow & L^2(\mathbf{R}, \mathcal{G}^*)
\end{array}$$

where $L^2(\mathbf{R}, \mathcal{G}) \stackrel{\text{def}}{=} \operatorname{proj\,lim}_{p\to\infty} L^2(\mathbf{R}, \mathcal{G}_p) \cong \operatorname{proj\,lim}_{p\to\infty} L^2(\mathbf{R}) \otimes \mathcal{G}_p$

$$L^2(\mathbf{R}, \mathcal{G}^*) \stackrel{\text{def}}{=} \operatorname{ind\,lim}_{p\to\infty} L^2(\mathbf{R}, \mathcal{G}_{-p}) \cong \operatorname{ind\,lim}_{p\to\infty} L^2(\mathbf{R}) \otimes \mathcal{G}_{-p}$$

Norm estimates

$$\|\nabla\phi\|_{L^2(\mathbf{R},\Gamma(H))} \le \|\phi\|_{\mathbf{D}}, \qquad \phi \in \mathbf{D},$$

$$\|\nabla\phi\|_{L^2(\mathbf{R},\mathbf{D}^*)} \le \|\phi\|_{\Gamma(H)}, \qquad \phi \in \Gamma(H),$$

$$\|\nabla\phi\|^2_{L^2(\mathbf{R},\mathcal{G}_p)} = \int_{\mathbf{R}} \|\nabla\phi(t)\|_p^2\, dt \le K(p,r)\|\phi\|^2_{p+r}, \quad K(p,r) = \sup_n n e^{-2rn-2p}, \quad r > 0.$$

2.4 The adjoint actions of ∇ (Hitsuda–Skorohod integrals)

Taking the adjoint maps in the above diagram,

$$
\begin{array}{ccccccccc}
L^2(\mathbf{R},\mathcal{G}) & \longrightarrow & L^2(\mathbf{R},\mathbf{D}) & \longrightarrow & L^2(\mathbf{R},\Gamma(H)) & \longrightarrow & L^2(\mathbf{R},\mathcal{G}^*) & \longrightarrow & \mathcal{S}'(\mathbf{R},(E)^*) \\
{\scriptstyle \delta=\nabla^*}\big\downarrow & & {\scriptstyle \delta}\big\downarrow & & {\scriptstyle \delta}\big\downarrow & & {\scriptstyle \delta}\big\downarrow & & {\scriptstyle \delta}\big\downarrow \\
\mathcal{G} & \longrightarrow & \Gamma(H) & \longrightarrow & \mathbf{D}^* & \longrightarrow & \mathcal{G}^* & \longrightarrow & (E)^*
\end{array}
$$

Definition We call $\delta(\Psi)$ the *Hitsuda–Skorohod integral*.

By definition it holds that

$$\langle\langle \delta(\Psi), \phi \rangle\rangle = \int_{\mathbf{R}} \langle\langle \Psi(t), \nabla\phi(t) \rangle\rangle \, dt, \qquad \text{for a suitable pair } \Psi \text{ and } \phi.$$

• The adjoint action of the classical stochastic gradient defines a (non-adapted) stochastic integrals which generalize the Itô integral.

⋆ *This idea can be applied to quantum stochastic integrals!*

We first introduce three kinds of quantum stochastic gradients:

creation gradient ∇^+ annihilation gradient ∇^- conservation gradient ∇^0

3 Quantum Stochastic Gradients

3.1 White noise operators

We have already introduced the inclusion relations:

$$(E) \subset \mathcal{G} \subset \mathbf{D} \subset \Gamma(H) \subset \mathbf{D}^* \subset \mathcal{G}^* \subset (E)^*$$

- classical random variables $\Longleftrightarrow$ vectors in these spaces
- quantum random variables $\Longleftrightarrow$ operators between these spaces

<u>Definition</u> A continuous linear operator from (E) into $(E)^*$ is called a *white noise operator*. The space of white noise operators is denoted by $\mathcal{L}((E),(E)^*)$ and is equipped with the bounded convergence topology.

- $\mathcal{L}((E),(E)^*)$ covers a wide class of Fock space operators, e.g., $\mathcal{L}(\mathcal{X},\mathcal{Y})$.
- The nuclear kernel theorem (since (E) is a nuclear space) claims the canonical isomorphism:

  $$\mathcal{K} : \mathcal{L}((E),(E)^*) \xrightarrow{\cong} (E)^* \otimes (E)^* \quad \text{defined by} \quad \langle\langle \Xi\phi, \psi \rangle\rangle = \langle\langle \mathcal{K}\Xi, \psi \otimes \phi \rangle\rangle, \quad \phi, \psi \in (E).$$

- This $\mathcal{K}$ induces many isomorphisms such as

  $$\mathcal{L}((E),(E)) \cong (E) \otimes (E)^*, \qquad \mathcal{L}((E)^*,\mathcal{X}) \cong \mathcal{X} \otimes (E), \quad \text{etc.}$$

  where $\otimes$ is the completed π-tensor product.

3.2 Creation gradient ∇^+

Classical stochastic gradient $\nabla : \mathbf{D} \to L^2(\mathbf{R}, \Gamma(H))$, i.e., (random variables) $\to$ (stochastic processes)
Quantum stochastic gradient ∇^ϵ : (quantum random variables) $\to$ (quantum stochastic processes)

Consider ∇^+ acting on $\mathcal{L}((E), \mathbf{D})$

$$\mathcal{L}((E), \mathbf{D}) \xrightarrow{\cong} \mathbf{D} \otimes (E)^* \xrightarrow{\nabla \otimes I} L^2(\mathbf{R}, \Gamma(H)) \otimes (E)^*$$

The last space becomes

$$\begin{aligned} L^2(\mathbf{R}, \Gamma(H)) \otimes (E)^* &\cong \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbf{R}, \Gamma(H)) \otimes \Gamma(E_{-p}) \\ &\cong \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbf{R}, \Gamma(H) \otimes \Gamma(E_{-p})) \stackrel{\text{def}}{=} L^2(\mathbf{R}, \Gamma(H) \otimes (E)^*) \\ &\cong \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbf{R}, \mathcal{L}_2(\Gamma(E_p), \Gamma(H))) \stackrel{\text{def}}{=} L^2(\mathbf{R}, \mathcal{L}((E), \Gamma(H))) \end{aligned}$$

Thus, the *creation gradient* is defined:

$$\begin{aligned} \nabla^+ : \mathcal{L}((E), \mathbf{D}) &\to L^2(\mathbf{R}, \Gamma(H) \otimes (E)^*) \cong L^2(\mathbf{R}, \mathcal{L}((E), \Gamma(H))) \\ \mathcal{L}_2(\Gamma(E_p), \mathbf{D}) &\to L^2(\mathbf{R}, \Gamma(H) \otimes \Gamma(E_{-p})) \cong L^2(\mathbf{R}, \mathcal{L}_2(\Gamma(E_p), \Gamma(H))) \end{aligned}$$

Norm estimate

$$\|\nabla^+ \Xi\|^2_{L^2(\mathbf{R}, \mathcal{L}_2(\Gamma(E_p), \Gamma(H)))} = \int_{\mathbf{R}} \|\nabla^+ \Xi(t)\|^2_{\mathcal{L}_2(\Gamma(E_p), \Gamma(H))} \, dt \le \|\Xi\|^2_{\mathcal{L}_2(\Gamma(E_p), \mathbf{D})}.$$

In particular,

Theorem 3.1 (HS criterion) *For any* $\Xi \in \mathcal{L}_2(\Gamma(H), \mathbf{D})$ *we have*

$$\int_{\mathbf{R}} \|\nabla^+\Xi(t)\|^2_{\mathcal{L}_2(\Gamma(H),\Gamma(H))} dt \le \|\Xi\|^2_{\mathcal{L}_2(\Gamma(H),\mathbf{D})}.$$

Therefore, $\nabla^+\Xi(t)$ *is a Hilbert–Schmidt operator on* $\Gamma(H)$ *for a.e.* $t \in \mathbf{R}$*.*

The above discussion on the creation gradient is summarized into the following diagram:

$$\begin{array}{ccccc}
\mathcal{L}((E)^*, \mathbf{D}) & \longrightarrow & \mathcal{L}_2(\Gamma(E_p), \mathbf{D}) & \longrightarrow & \mathcal{L}((E), \mathbf{D}) \\
{\scriptstyle \nabla^+}\big\downarrow & & {\scriptstyle \nabla^+}\big\downarrow & & {\scriptstyle \nabla^+}\big\downarrow \\
L^2(\mathbf{R}, \mathcal{L}((E)^*, \Gamma(H))) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}_2(\Gamma(E_p), \Gamma(H))) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}((E), \Gamma(H)))
\end{array}$$

Different domains:

$$\begin{array}{ccccc}
\mathcal{L}((E)^*, \Gamma(H)) & \longrightarrow & \mathcal{L}_2(\Gamma(E_p), \Gamma(H)) & \longrightarrow & \mathcal{L}((E), \Gamma(H)) \\
{\scriptstyle \nabla^+}\big\downarrow & & {\scriptstyle \nabla^+}\big\downarrow & & {\scriptstyle \nabla^+}\big\downarrow \\
L^2(\mathbf{R}, \mathcal{L}((E)^*, \mathbf{D}^*)) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}_2(\Gamma(E_p), \mathbf{D}^*)) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}((E), \mathbf{D}^*))
\end{array}$$

where p runs over $\mathbf{R}$ and

$$\mathcal{L}((E), \Gamma(H)) \cong \operatorname*{ind\,lim}_{p\to\infty} \mathcal{L}_2(\Gamma(E_p), \Gamma(H)), \quad \mathcal{L}((E)^*, \Gamma(H)) \cong \operatorname*{proj\,lim}_{p\to\infty} \mathcal{L}_2(\Gamma(E_{-p}), \Gamma(H)).$$

3.3 Annihilation gradient ∇^-

The *annihilation gradient* is defined by

$$\nabla^- : \mathcal{L}(\mathbf{D}^*, (E)^*) \xrightarrow{\cong} (E)^* \otimes \mathbf{D} \xrightarrow{I\otimes\nabla} (E)^* \otimes L^2(\mathbf{R}, \Gamma(H))$$
$$\xrightarrow{\cong} L^2(\mathbf{R}, (E)^* \otimes \Gamma(H)) \xrightarrow{\cong} L^2(\mathbf{R}, \mathcal{L}(\Gamma(H), (E)^*)),$$

$$\text{where } L^2(\mathbf{R}, (E)^* \otimes \Gamma(H)) \stackrel{\text{def}}{=} \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbf{R}, \Gamma(E_{-p}) \otimes \Gamma(H)).$$

$$\begin{array}{ccccc}
\mathcal{L}(\mathbf{D}^*, (E)) & \longrightarrow & \mathcal{L}_2(\mathbf{D}^*, \Gamma(E_p)) & \longrightarrow & \mathcal{L}(\mathbf{D}^*, (E)^*) \\
{\scriptstyle\nabla^-}\downarrow & & {\scriptstyle\nabla^-}\downarrow & & {\scriptstyle\nabla^-}\downarrow \\
L^2(\mathbf{R}, \mathcal{L}(\Gamma(H), (E))) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}_2(\Gamma(H), \Gamma(E_p))) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}(\Gamma(H), (E)^*))
\end{array}$$

where

$$\mathcal{L}(\mathbf{D}^*, (E)) \cong \operatorname*{proj\,lim}_{p\to\infty} \mathcal{L}_2(\mathbf{D}^*, \Gamma(E_p)), \quad \mathcal{L}(\mathbf{D}^*, (E)^*) \cong \operatorname*{ind\,lim}_{p\to\infty} \mathcal{L}_2(\mathbf{D}^*, \Gamma(E_{-p})).$$

Different domains:

$$\begin{array}{ccccc}
\mathcal{L}(\Gamma(H), (E)) & \longrightarrow & \mathcal{L}_2(\Gamma(H), \Gamma(E_p)) & \longrightarrow & \mathcal{L}(\Gamma(H), (E)^*) \\
{\scriptstyle\nabla^-}\downarrow & & {\scriptstyle\nabla^-}\downarrow & & {\scriptstyle\nabla^-}\downarrow \\
L^2(\mathbf{R}, \mathcal{L}(\mathbf{D}, (E))) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}_2(\mathbf{D}, \Gamma(E_p))) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}(\mathbf{D}, (E)^*))
\end{array}$$

Norm estimate, e.g.,

$$\|\nabla^- \Xi\|^2_{L^2(\mathbf{R},\mathcal{L}_2(\Gamma(H),\Gamma(E_p)))} = \int_{\mathbf{R}} \|\nabla^- \Xi(t)\|^2_{\mathcal{L}_2(\Gamma(H),\Gamma(E_p))} dt \le \|\Xi\|^2_{\mathcal{L}_2(\mathbf{D}^*,\Gamma(E_p))}.$$

Theorem 3.2 (HS criterion) *For any* $\Xi \in \mathcal{L}_2(\mathbf{D}^*, \Gamma(H))$ *we have*

$$\int_{\mathbf{R}} \|\nabla^- \Xi(t)\|^2_{\mathcal{L}_2(\Gamma(H),\Gamma(H))} dt \le \|\Xi\|^2_{\mathcal{L}_2(\mathbf{D}^*,\Gamma(H))}.$$

In particular, $\nabla^- \Xi(t)$ *is a Hilbert–Schmidt operator on* $\Gamma(H)$ *for a.e.* $t \in \mathbf{R}$.

Theorem 3.3 (Relation between annihilation and creation gradients) *Let* Ξ *be a member of one of the domains of the creation gradient in the above diagrams. Then,*

$$\nabla^- \Xi(t) = (\nabla^+ \Xi^*(t))^* \quad \textit{for a.e. } t \in \mathbf{R}.$$

Theorem 3.4 (Annihilation and creation gradients as densities) *Let* Ξ *be a member of one of the domains of the creation gradient in the above diagrams. Then, for* $\zeta \in H = L^2(\mathbf{R})$ *the compositions* $a(\zeta)\Xi$ *and* $\Xi a(\zeta)^*$ *are well defined (as continuous operators) and admit the integral expressions:*

$$a(\zeta)\Xi = \int_{\mathbf{R}} \zeta(t) \nabla^+ \Xi(t)\, dt, \qquad \Xi a^*(\zeta) = \int_{\mathbf{R}} \zeta(t) \nabla^- \Xi(t)\, dt.$$

3.4 Conservation gradient ∇^0

creation gradiaent $\nabla^+ : \mathcal{L}((E), \mathbf{D}) \xrightarrow{\cong} \mathbf{D} \otimes (E)^* \xrightarrow{\nabla \otimes I} L^2(\mathbf{R}, \Gamma(H)) \otimes (E)^* \to \cdots$

annihilation gradient $\nabla^- : \mathcal{L}(\mathbf{D}^*, (E)^*) \xrightarrow{\cong} (E)^* \otimes \mathbf{D} \xrightarrow{I \otimes \nabla} (E)^* \otimes L^2(\mathbf{R}, \Gamma(H)) \to \cdots$

The *conservation gradient* will be defined as

$$\nabla^0 : \mathcal{L}((E)^*, \mathbf{D}) \xrightarrow{\cong} \mathbf{D} \otimes (E) \xrightarrow{\nabla \oslash \nabla} L^2(\mathbf{R}, \Gamma(H) \otimes (E)) \xrightarrow{\cong} L^2(\mathbf{R}, \mathcal{L}((E)^*, \Gamma(H))),$$

where $\nabla \oslash \nabla$ is the "diagonalized" tensor product.

$$[(\nabla \oslash \nabla)\phi \otimes \psi](t) = \nabla\phi(t) \otimes \nabla\psi(t) \qquad \text{for suitable } \phi, \psi$$

Lemma $\nabla \oslash \nabla : \mathbf{D} \otimes (E) \to L^2(\mathbf{R}, \Gamma(H) \otimes (E))$ is a continuous linear map.

Proof (1) For any $p \geq 0$ and $q > 0$ with $p + q > 5/12$ there exists a constant $C(p,q) > 0$ such that

$$\sup_{t \in \mathbf{R}} \|\nabla\psi(t)\|_p^2 \leq C(p,q)\, \|\psi\|_{p+q}^2, \qquad \psi \in (E).$$

(2) Then one can show easily that

$$\int_{\mathbf{R}} \|[(\nabla \oslash \nabla)\phi \otimes \psi](t)\|^2_{\Gamma(H)\otimes\Gamma(E_p)} dt \leq C(p,q)\|\phi\|^2_{\mathbf{D}}\|\psi\|^2_{p+q}.$$

Therefore, $\nabla \oslash \nabla : \mathbf{D} \otimes_\pi \Gamma(E_{p+q}) \to L^2(\mathbf{R}, \Gamma(H) \otimes \Gamma(E_p))$ is continuous.

(3) Combine a continuous injection $\mathbf{D} \otimes (E) = \mathbf{D} \otimes_\pi (E) \to \mathbf{D} \otimes_\pi \Gamma(E_{p+q})$.

Remarks (1) $5/12$ arises from $\sup_{t\in\mathbf{R}} |\,\delta_t\,|_{-r} < \infty$ for $r > 5/12$.

(2) $L^2(\mathbf{R}, \Gamma(H) \otimes (E)) \overset{\text{def}}{=} \operatorname{proj\,lim}_{p\to\infty} L^2(\mathbf{R}, \Gamma(H) \otimes \Gamma(E_p)) \cong (L^2(\mathbf{R}) \otimes \Gamma(H)) \otimes (E)$.

With a different domain,

$$\begin{array}{ccc} \mathcal{L}((E)^*, \mathbf{D}) & \longrightarrow & \mathcal{L}((E)^*, \Gamma(H)) \\ \nabla^0 \big\downarrow & & \big\downarrow \nabla^0 \\ L^2(\mathbf{R}, \mathcal{L}((E)^*, \Gamma(H))) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}((E)^*, \mathbf{D}^*)). \end{array}$$

We call ∇^0 the *conservation gradient*.

4 Hitsuda–Skorokhod Quantum Stochastic Integrals

4.1 Creation integral

The *creation integral* δ^+ is by definition the adjoint map of the creation gradient ∇^+.

$$\begin{array}{ccccc} L^2(\mathbf{R}, \mathcal{L}((E)^*, \Gamma(H))) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}_2(\Gamma(E_p), \Gamma(H))) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}((E), \Gamma(H))) \\ \delta^+ \downarrow & & \delta^+ \downarrow & & \delta^+ \downarrow \\ \mathcal{L}((E)^*, \mathbf{D}^*) & \longrightarrow & \mathcal{L}_2(\Gamma(E_p), \mathbf{D}^*) & \longrightarrow & \mathcal{L}((E), \mathbf{D}^*) \end{array}$$

Different domains:

$$\begin{array}{ccccc} L^2(\mathbf{R}, \mathcal{L}((E)^*, \mathbf{D})) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}_2(\Gamma(E_p), \mathbf{D})) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}((E), \mathbf{D})) \\ \delta^+ \downarrow & & \delta^+ \downarrow & & \delta^+ \downarrow \\ \mathcal{L}((E)^*, \Gamma(H)) & \longrightarrow & \mathcal{L}_2(\Gamma(E_p), \Gamma(H)) & \longrightarrow & \mathcal{L}((E), \Gamma(H)). \end{array}$$

Norm estimates:

$$\|\delta^+(\Xi)\|^2_{\mathcal{L}_2(\Gamma(E_p), \mathbf{D}^*)} \le \int_{\mathbf{R}} \|\Xi(t)\|^2_{\mathcal{L}_2(\Gamma(E_p), \Gamma(H))} \, dt$$

$$\|\delta^+(\Xi)\|^2_{\mathcal{L}_2(\Gamma(E_p), \Gamma(H))} \le \int_{\mathbf{R}} \|\Xi(t)\|^2_{\mathcal{L}_2(\Gamma(E_p), \mathbf{D})} \, dt$$

Theorem 4.1 (HS criterion) *For any* $\Xi \in L^2(\mathbf{R}, \mathcal{L}_2(\Gamma(H), \mathbf{D}))$ *the creation integral* $\delta^+(\Xi)$ *is a Hilbert–Schmidt operator on* $\Gamma(H)$.

Theorem 4.2 *Let* Ξ *be a member of one of the domains of the creation integral in the above diagrams. Then it holds that*

$$\langle\langle \delta^+(\Xi)\phi, \psi \rangle\rangle = \int_{\mathbf{R}} \langle\langle \Xi(t)\phi, \nabla\psi(t) \rangle\rangle \, dt$$

for a suitable pair ϕ, ψ. *Therefore, denoting* $(\Xi\phi)(t) = \Xi(t)\phi$ *we have*

$$\delta^+(\Xi)\phi = \delta\left(\Xi\phi\right), \qquad \phi \in (E). \tag{BL1}$$

Remark (BL1) means that our $\delta^+(\Xi)$ coincides with the non-adapted quantum stochastic integrals defined by Belavkin (1991) and Lindsay (1993) when Ξ is in the common domain.

- We have some criteria for boundedness of $\delta^+(\Xi)$, for example,

Theorem 4.3 (Boundedness criterion. cf. Theorem 4.1) *For any* $\Xi \in L^2(\mathbf{R}, \mathcal{L}(\Gamma(H), \mathbf{D}))$ *the creation integral* $\delta^+(\Xi)$ *is a bounded operator on* $\Gamma(H)$.

4.2 Annihilation Integral

The *annihilation integral* δ^- is defined to be the adjoint map of the annihilation gradient ∇^-:

$$\begin{array}{ccccc} L^2(\mathbf{R}, \mathcal{L}(\Gamma(H), (E))) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}_2(\Gamma(H), \Gamma(E_p))) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}(\Gamma(H), (E)^*)) \\ \delta^- \big\downarrow & & \delta^- \big\downarrow & & \delta^- \big\downarrow \\ \mathcal{L}(\mathbf{D}, (E)) & \longrightarrow & \mathcal{L}_2(\mathbf{D}, \Gamma(E_p)) & \longrightarrow & \mathcal{L}(\mathbf{D}, (E)^*) \end{array}$$

Different domains:

$$\begin{array}{ccccc} L^2(\mathbf{R}, \mathcal{L}(\mathbf{D}^*, (E))) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}_2(\mathbf{D}^*, \Gamma(E_p))) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}(\mathbf{D}^*, (E)^*)) \\ \delta^- \big\downarrow & & \delta^- \big\downarrow & & \delta^- \big\downarrow \\ \mathcal{L}(\Gamma(H), (E)) & \longrightarrow & \mathcal{L}_2(\Gamma(H), \Gamma(E_p)) & \longrightarrow & \mathcal{L}(\Gamma(H), (E)^*), \end{array}$$

Norm estimates:

$$\|\delta^-(\Xi)\|^2_{\mathcal{L}_2(\mathbf{D},\Gamma(E_p))} \leq \int_{\mathbf{R}} \|\Xi(t)\|^2_{\mathcal{L}_2(\Gamma(H),\Gamma(E_p))}\, dt$$

$$\|\delta^-(\Xi)\|^2_{\mathcal{L}_2(\Gamma(H),\Gamma(E_p))} \leq \int_{\mathbf{R}} \|\Xi(t)\|^2_{\mathcal{L}_2(\mathbf{D}^*,\Gamma(E_p))}\, dt$$

Theorem 4.4 (HS criterion) *For any* $\Xi \in L^2(\mathbf{R}, \mathcal{L}_2(\mathbf{D}^*, \Gamma(H)))$ *the annihilation integral* $\delta^-(\Xi)$ *is a Hilbert–Schmidt operator on* $\Gamma(H)$.

Theorem 4.5 *Let* Ξ *be a member of one of the domains of the annihilation integral in the above diagrams. Then it holds that*

$$\langle\!\langle \delta^-(\Xi)\phi, \psi \rangle\!\rangle = \int_{\mathbf{R}} \langle\!\langle \Xi(t)(\nabla\phi(t)), \psi \rangle\!\rangle \, dt$$

for a suitable pair ϕ, ψ. *Therefore,*

$$\delta^-(\Xi)\phi = \int_{\mathbf{R}} \Xi(t)(\nabla\phi(t)) \, dt. \tag{BL2}$$

Remark (BL2) means that our $\delta^-(\Xi)$ coincides with the non-adapted quantum stochastic integrals defined by Belavkin (1991) and Lindsay (1993) when Ξ is in the common domain.

Theorem 4.6 (Relation between creation and annihilation integrals)

$$(\delta^-(\Xi))^* = \delta^+(\Xi^*).$$

- Boundedness criteria follow from those for creation integrals.

4.3 Conservation Integral

The *conservation integral* δ^0 is defined by the adjoint actions of the conservation gradient ∇^0:

$$\begin{array}{ccc} L^2(\mathbf{R}, \mathcal{L}((E), \mathbf{D})) & \longrightarrow & L^2(\mathbf{R}, \mathcal{L}((E), \Gamma(H))) \\ {\scriptstyle \delta^0}\big\downarrow & & {\scriptstyle \delta^0}\big\downarrow \\ \mathcal{L}((E), \Gamma(H)) & \longrightarrow & \mathcal{L}((E), \mathbf{D}^*). \end{array}$$

- We have similar "boundedness criterion" as in the case of creation and annihilation integrals.

Theorem 4.7 *Let Ξ be a member of one of the domains of the conservation integral in the above diagram. Then it holds that*

$$\langle\langle \delta^0(\Xi)\phi, \psi \rangle\rangle = \int_{\mathbf{R}} \langle\langle \Xi(t)\nabla\phi(t), \nabla\psi(t) \rangle\rangle \, dt$$

for a suitable pair ϕ, ψ. Therefore,

$$\delta^0(\Xi)\phi = \delta(\Xi\nabla\phi), \tag{BL3}$$

where $\Xi\nabla\phi$ is defined by $\Xi\nabla\phi(t) = \Xi(t)(\nabla\phi(t))$.

Remark (BL3) means that our $\delta^0(\Xi)$ coincides with the non-adapted quantum stochastic integrals defined by Belavkin (1991) and Lindsay (1993) when Ξ is in the common domain.

4.4 Further extensions in terms of $\mathcal{G}$

The creation integral:

$$\delta^+ : \begin{array}{l} L^2(\mathbf{R}, \mathcal{L}((E)^*, \mathcal{G})) \to \mathcal{L}((E)^*, \mathcal{G}), \\ L^2(\mathbf{R}, \mathcal{L}((E), \mathcal{G}^*)) \to \mathcal{L}((E), \mathcal{G}^*). \end{array}$$

The annihilation integral:

$$\delta^- : \begin{array}{l} L^2(\mathbf{R}, \mathcal{L}(\mathcal{G}^*, (E))) \to \mathcal{L}(\mathcal{G}, (E)^*), \\ L^2(\mathbf{R}, \mathcal{L}(\mathcal{G}, (E)^*)) \to \mathcal{L}(\mathcal{G}, (E)^*), \end{array}$$

The conservation integral:

$$\delta^0 : L^2(\mathbf{R}, \mathcal{L}((E), \mathcal{G}^*)) \to \mathcal{L}((E), \mathcal{G}^*), \qquad \text{etc.}$$

Remark For the conservation integral, we used the estimate of the diagonalized tensor product $\nabla \oslash \nabla$ as follows: For any $p \geq 0$, $q \in \mathbf{R}$, $r > 0$ and $s > 0$ with $p + s > 5/12$,

$$\int_{\mathbf{R}} \|[(\nabla \oslash \nabla)\phi \otimes \psi](t)\|^2_{\mathcal{G}_q \otimes \Gamma(E_p)} dt \leq K(q, r) C(p, s) \|\phi\|^2_{q+r} \|\psi\|^2_{p+s}, \qquad \phi \in \mathcal{G}, \quad \psi \in (E).$$

Hence $\nabla \oslash \nabla$ is a continuous linear map from $\mathcal{G}_{q+r} \otimes_\pi \Gamma(E_{p+s})$ into $L^2(\mathbf{R}, \mathcal{G}_q \otimes \Gamma(E_p))$. Consequently, the conservation gradient

$$\nabla^0 : \mathcal{L}((E)^*, \mathcal{G}) \to L^2(\mathbf{R}, \mathcal{L}((E)^*, \mathcal{G}))$$

becomes a continuous linear map, and so is $\delta^0 = (\nabla^0)^*$.

4.5 The classical-quantum correspondence

(classical random variables) $\subset$ (quantum random variables)

Fundamental Lemma Each $\Phi \in (E)^*$ gives rise to a white noise operator $M_\Phi \in \mathcal{L}((E),(E)^*)$ by multiplication, i.e.,

$$\langle\langle M_\Phi \phi, \psi \rangle\rangle = \langle\langle \Phi, \phi\psi \rangle\rangle, \qquad \phi, \psi \in (E),$$

where $\phi\psi$ is the pointwise product defined through the Wiener–Itô–Segal isomorphism. Moreover, $\Phi \mapsto M_\Phi$ is a continuous linear injection.

Note: $M_\Phi \phi_0 = \Phi$, where $\phi_0 = (1, 0, 0, \dots) \in (E)$ is the vacuum vector.

Theorem 4.8 *Let* $\Phi \in L^2(\mathbf{R}, \mathcal{G}^*)$ *and* $M_\Phi(t) = M_{\Phi(t)}$. *Then* $M_\Phi \in L^2(\mathbf{R}, \mathcal{L}((E), \mathcal{G}^*))$ *and it holds that*

$$\delta^+(M_\Phi)\phi_0 = \delta(\Phi), \qquad \delta^-(M_\Phi)\phi_0 = 0, \qquad \delta^0(M_\Phi)\phi_0 = 0.$$

Proof (1) We can show that $M_\phi \in \mathcal{L}((E), \mathcal{G}^*)$ for any $\phi \in \mathcal{G}^*$.

(2) Hence $M_\Phi \in L^2(\mathbf{R}, \mathcal{L}((E), \mathcal{G}^*))$ and their quantum stochastic integrals are defined.

(3) The three relations follow by direct verification, e.g., for any $\psi \in (E)$ we have

$$
\begin{aligned}
\langle\!\langle \delta^+(M_\Phi)\phi_0,\, \psi \rangle\!\rangle &= \langle\!\langle \mathcal{K}\delta^+(M_\Phi),\, \psi \otimes \phi_0 \rangle\!\rangle = \langle\!\langle \mathcal{K}M_\Phi, (\nabla \otimes I)(\psi \otimes \phi_0) \rangle\!\rangle \\
&= \int_{\mathbf{R}} \langle\!\langle \mathcal{K}M_{\Phi(t)}, (\nabla\psi(t)) \otimes \phi_0 \rangle\!\rangle dt = \int_{\mathbf{R}} \langle\!\langle M_{\Phi(t)}\phi_0,\, \nabla\psi(t) \rangle\!\rangle dt \\
&= \int_{\mathbf{R}} \langle\!\langle \Phi(t), \nabla\psi(t) \rangle\!\rangle dt = \langle\!\langle \delta(\Phi),\, \psi \rangle\!\rangle,
\end{aligned}
$$

which proves the first identity.

5 Quantum White Noise Derivatives

5.1 Motivation

Every white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ is a "function" of quantum white noise:

$$\Xi = \Xi(a_s, a_t^* \,;\, s \in \mathbf{R},\, t \in \mathbf{R})$$

Fock expansion of a white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$:

$$\Xi = \sum_{l,m=0}^{\infty} \int_{\mathbf{R}^{l+m}} \kappa_{l,m}(s_1,\dots,s_l,t_1,\dots,t_m) a_{s_1}^* \dots a_{s_l}^* a_{t_1} \dots a_{t_m} \, ds_1 \dots ds_l dt_1 \dots dt_m$$

where $\kappa \in \mathcal{S}'_{\mathrm{sym}}(\mathbf{R}^l) \otimes \mathcal{S}'_{\mathrm{sym}}(\mathbf{R}^m)$.

- What are the derivatives with respect to a_t and a_t^*? say, $D_t^- = \dfrac{\partial}{\partial a_t}$, $D_t^+ = \dfrac{\partial}{\partial a_t^*}$
- We expect:

$$D_t^-[a(f)] = D_t^-\left[\int_{\mathbf{R}} f(s) a_s \, ds\right] = f(t)I, \qquad D_t^+[a(f)] = D_t^+\left[\int_{\mathbf{R}} f(s) a_s \, ds\right] = 0, \qquad \text{etc.}$$

- We wish to apply these derivetives to the quantum stochastic integrals:

$$\delta^-(\Xi) = \int_{\mathbf{R}} \Xi(t) a_t \, dt, \qquad \delta^+(\Xi) = \int_{\mathbf{R}} a_t^* \Xi(t) \, dt, \qquad \delta^0(\Xi) = \int_{\mathbf{R}} a_t^* \Xi(t) a_t \, dt$$

5.2 Creation- and Annihilation-Derivatives

Definition The *creation derivative* and *annihilation derivative* of a white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ with respect to $\zeta \in E$ are defined respectively by

$$D_\zeta^+\Xi = [a(\zeta),\Xi] = a(\zeta)\Xi - \Xi a(\zeta), \qquad D_\zeta^-\Xi = -[a^*(\zeta),\Xi] = \Xi a^*(\zeta) - a^*(\zeta)\Xi.$$

Note: The commutators are well-defined white noise operators, i.e., belongs to $\mathcal{L}((E),(E)^*)$.

- $D^\pm : E \times \mathcal{L}((E),(E)^*) \ni (\zeta,\Xi) \mapsto D_\zeta^\pm\Xi \in \mathcal{L}((E),(E)^*)$ is a continuous bilinear map.

Theorem 5.1 *Let* $\zeta \in E$ *and* $\Xi \in L^2(\mathbf{R},\mathcal{L}((E),(E)^*))$. *Then we have*

$$D_\zeta^+(\delta^+(\Xi)) = \delta^+(D_\zeta^+\Xi) + \int_{\mathbf{R}} \zeta(t)\Xi(t)dt, \qquad D_\zeta^-(\delta^+(\Xi)) = \delta^+(D_\zeta^-\Xi).$$

$$D_\zeta^+(\delta^-(\Xi)) = \delta^-(D_\zeta^+\Xi), \qquad D_\zeta^-(\delta^-(\Xi)) = \delta^-(D_\zeta^-\Xi) + \int_{\mathbf{R}} \zeta(t)\Xi(t)dt,$$

$$D_\zeta^+(\delta^0(\Xi)) = \delta^0(D_\zeta^+\Xi) + \delta^-(\zeta\Xi), \qquad D_\zeta^-(\delta^0(\Xi)) = \delta^0(D_\zeta^-\Xi) + \delta^+(\zeta\Xi).$$

Proof is more or less by direct computation.

⋆ Taking $\zeta = \delta_t$, we could get $\Xi(t)$ from $\delta^\epsilon(\Xi)$.

Definition A white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ is called *pointwisely differentiable* if there exists a measurable map $t \mapsto D_t^{\pm}\Xi \in \mathcal{L}((E),(E)^*)$ such that

$$\langle\!\langle (D_\zeta^{\pm}\Xi)\phi_\xi,\, \phi_\eta \rangle\!\rangle = \int_{\mathbf{R}_+} \langle\!\langle (D_t^{\pm}\Xi)\phi_\xi,\, \phi_\eta \rangle\!\rangle \zeta(t)dt, \qquad \zeta \in H,\ \xi,\eta \in E.$$

Then $D_t^+\Xi$ and $D_t^-\Xi$ are called the *pointwise creation-derivative* and *pointwise annihilation-derivative* of Ξ, respectively.

Theorem 5.2 *For $\Xi \in L^2(\mathbf{R}, \mathcal{L}(\mathcal{G},\mathcal{G}^*))$, the quantum stochastic integrals $\delta^\epsilon(\Xi)$ are pointwisely differentiable. Moreover, for a.e. $t \in \mathbf{R}_+$ we have*

$$\begin{aligned} D_t^+(\delta^+(\Xi)) &= \delta^+(D_t^+\Xi) + \Xi(t)\,, & D_t^-(\delta^+(\Xi)) &= \delta^+(D_t^-\Xi), \\ D_t^+(\delta^-(\Xi)) &= \delta^-(D_t^+\Xi), & D_t^-(\delta^-(\Xi)) &= \delta^-(D_t^-\Xi) + \Xi(t)\,, \\ D_t^+(\delta^0(\Xi)) &= \delta^0(D_t^+\Xi) + \Xi(t)a_t\,, & D_t^-(\delta^0(\Xi)) &= \delta^0(D_t^-\Xi) + a_t^*\Xi(t)\,. \end{aligned}$$

5.3 Application to Quantum Martingales

Representation Theorem [Parthasarathy–Sinha (1986), Ji (2003)] A regular quantum martingale $\{M_t\}_{t\in\mathbf{R}_+} \subset \mathcal{L}(\mathcal{G}_p(\mathbf{R}_+), \mathcal{G}_q(\mathbf{R}_+))$ admits an integral representation:

$$M_t = \lambda I + \int_0^t (E dA + F dA^* + G d\Lambda),$$

where $\{E_t\}, \{F_t\}, \{G_t\}$ in $\mathcal{L}(\mathcal{G}_p(\mathbf{R}_+), \mathcal{G}_q(\mathbf{R}_+))$ are adapted processes and $\lambda \in \mathbf{C}$.

Theorem 5.3 *The integrands of M_t is obtained by*

$$E_s = D_s^- \left[M_s - \int_0^s a_u^* (D_u^+ M_u) du \right],$$

$$F_s = D_s^+ \left[M_s - \int_0^s (D_u^- M_u) a_u du \right],$$

$$G_s = D_s^+ \left[\int_0^s \left\{ D_u^- \left(M_u - \int_0^u E_v a_v dv - \int_0^u a_v^* F_v dv \right) \right\} du \right].$$

Summary

1. We introduced quantum stochastic gradients ∇^+, ∇^-, ∇^0

2. We introduced quantum stochastic integrals $\delta^+(\Xi)$, $\delta^-(\Xi)$, $\delta^0(\Xi)$ by the adjoint actions of quantum stochastic gradients.

3. We discussed regularity properties (HS criteria, boundedness criteria, norm estimates) of $\delta^\epsilon(\Xi)$.

4. We introduced quantum white noise derivatives $D_t^+(\Xi)$ and $D_t^-(\Xi)$.

5. We discussed differentiability of quantum stochastic integrals and application to quantum martingales.

References

[1] U. C. Ji and N. Obata: *Quantum stochastic gradients*, preprint, 2007.

[2] U. C. Ji and N. Obata: *Annihilation-derivative, creation-derivetive and representation of quantum martingales*, preprint, 2007.

[3] Further works in progress.