Spectral Analysis of Manhattan Street Networks

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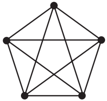
Plan

1. Digraphs and Spectra

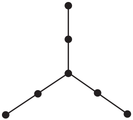
1.1. Graphs and Adjacency Matrices

Definition (graph)

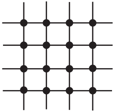
A *graph* is a pair $G = (V, E)$, where V is the set of *vertices* and E the set of *edges*. We write $x \sim y$ (adjacent) if they are connected by an edge.



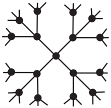
complete graph K_5



star graph



2-dim lattice



homogeneous tree T_4

Definition (adjacency matrix)

The *adjacency matrix* of a graph $G = (V, E)$ is defined by

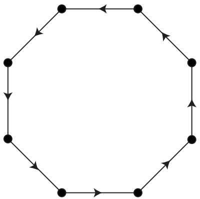
$$A = [A_{xy}]_{x,y \in V} \qquad A_{xy} = \begin{cases} 1, & x \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

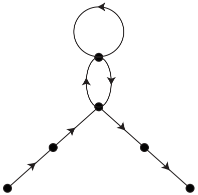
▶ The adjacency matrix possesses all the information of a graph.

1.2. Digraphs and Adjacency Matrices

Definition (digraph)

A *digraph (directed graph)* is a pair $G = (V, E)$, where V is a non-empty set and E is a subset of $V \times V$. An element $x \in V$ is called a *vertex* and $e = (x, y) \in E$ an *arc (arrow)* from x to y. In that case we also write $x \to y$.





- A digraph may have a *loop*, i.e., an arc from a vertex to itself.
- We study *finite* digraphs, i.e., with finite number of vertices.

Definition (adjacency matrix)

The *adjacency matrix* $A = [A_{xy}]_{x,y \in V}$ is defined by $A_{xy} = \begin{cases} 1, & x \to y, \\ 0, & \text{otherwise.} \end{cases}$

1.3. Spectra of Graphs and of Digraphs

A: adjacency matrix of a graph or of a digraph $G = (V, E)$, $|V| = n$.

Characteristic equation

$$\varphi_G(x) = \det(x - A) = \prod_i (x - \lambda_i)^{m_i}$$

$\Longrightarrow$ eigenvalue λ_i with *algebraic multiplicity* m_i

Eigenvalue problem

$$Ax = \lambda x, \qquad W(\lambda_i) = \{x \in \mathbb{C}^n \,;\, Ax = \lambda_i x\}$$

$\Longrightarrow$ eigenvalue λ_i with *geometric multiplicity* $l_i = \dim W(\lambda_i)$

Definition (Spectrum)

$$\mathrm{ASpec}\,(G) = \begin{pmatrix} \cdots & \lambda_i & \cdots \\ \cdots & m_i & \cdots \end{pmatrix}, \qquad \mathrm{GSpec}\,(G) = \begin{pmatrix} \cdots & \lambda_i & \cdots \\ \cdots & l_i & \cdots \end{pmatrix}$$

- $1 \le l_i \le m_i$ and $l_i < m_i$ may happen for a general digraph.
- For a graph (= a symmetric digraph), we have $\mathrm{ASpec}\,(G) = \mathrm{GSpec}\,(G)$.
- The *spectral (eigenvalue) distribution* of G is defined by

$$\mu_G = \frac{1}{|V|} \sum_i m_i \delta_{\lambda_i}$$

2. Product Structure

2.1. Product structures

Spectral graph theory

1. A large graph may consist of smaller components with "product" operation:

$$G = G_1 \text{``} * \text{''} G_2$$

2. This implies a notion of "product" of adjacency matrices:

$$A = A_1 \text{``} * \text{''} A_2$$

3. and a "convolution product" of spectral distributions:

$$\mu_G = \mu_{G_1} \text{``} * \text{''} \mu_{G_2}$$

Quantum probability

1. We have several different concepts of independence.
2. The distribution of the sum of "independent" random variables gives rise to a new notion of "convolution product" of probability distribution.

$$X = X_1 + X_2 \qquad \Longrightarrow \qquad \mu_X = \mu_{X_1} \text{``} * \text{''} \mu_{X_2}$$

3. Associated quantum central limit theorems

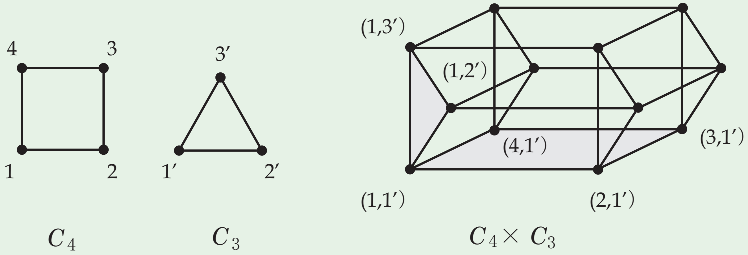
Definition (Direct (Cartesian) product)

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We say that $(x, y), (x', y') \in V = V_1 \times V_2$ are adjacent, $(x, y) \sim (x', y')$, if

$$\text{(i) } x = x' \text{ and } y \sim y'; \quad \text{or} \quad \text{(ii) } x \sim x' \text{ and } y = y'.$$

Then V becomes a graph which is called the *direct product* of G_1 and G_2, and is denoted by $G_1 \times G_2$.

Example ($C_4 \times C_3$)



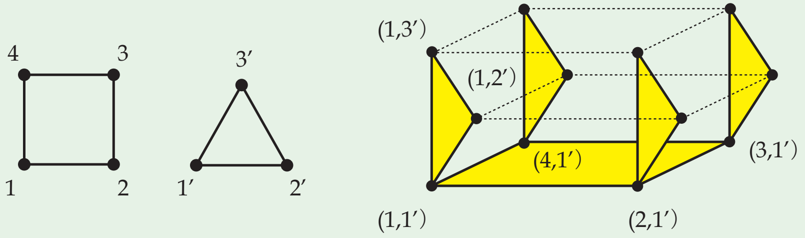
Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We fix a vertix $o_2 \in V_2$. For $(x, y), (x', y') \in V_1 \times V_2$ we write $(x, y) \sim (x', y')$ if

$$\text{(i) } x = x' \text{ and } y \sim y'; \quad \text{or} \quad \text{(ii) } x \sim x' \text{ and } y = y' = o_2.$$

Then $V_1 \times V_2$ becomes a graph, denoted by $G_1 \rhd_{o_2} G_2$, and is called the *comb product*. This is a subgraph of $G_1 \times G_2$.

Example ($C_4 \rhd C_3$)



C_4 C_3 $C_4 \rhd C_3$

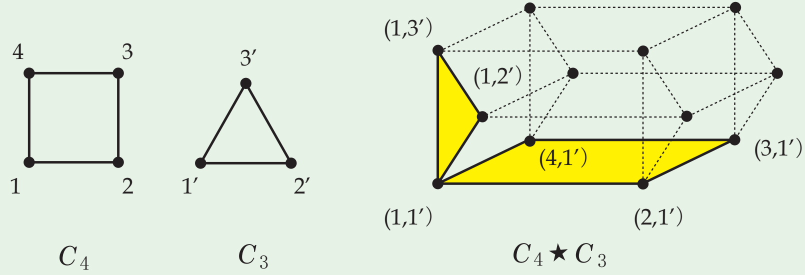
Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with distinguished vertices $o_1 \in V_1$ and $o_2 \in V_2$. Define a subset of $V_1 \times V_2$ by

$$V_1 \star V_2 = \{(x, o_2)\,;\, x \in V_1\} \cup \{(o_1, y)\,;\, y \in V_2\}$$

The induced subgraph of $G_1 \times G_2$ spanned by $V_1 \star V_2$ is called the *star product* of G_1 and G_2 (with contact vertices o_1 and o_2), and is denoted by $G_1 \star G_2 = G_1 \,{}_{o_1}\!\star_{o_2}\, G_2$.

Example ($C_4 \star C_3$)



2.5. Graph Products and Concepts of Independence

(I) Direct (Cartesian) product $G = G_1 \times G_2$

1. $A = A_1 \otimes I_2 + I_1 \otimes A_2$: sum of commutative (tensor) independent rv's in a suitable state
2. $\mathbf{Spec}(G) = \mathbf{Spec}(G_1) * \mathbf{Spec}(G_2)$ — usual convolution
3. Examples include: integer lattice, Hamming graph, ...
4. Gaussian (normal) distribution in the limit (classical CLT)

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(II) Comb product $G = G_1 \triangleright_{o_2} G_2$

1. $A = A_1 \otimes P_2 + I_1 \otimes A_2$: sum of monotone independent rv's in a sutable state
2. $\mathbf{Spec}(G) = \mathbf{Spec}(G_1) \triangleright \mathbf{Spec}(G_2)$ — monotone convolution
3. Examples include: comb graphs
4. Arcsine law in the limit (monotone CLT)

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4. Arcsine law in the limit (monotone CLT)

(III) Star product $G = G_1 \star G_2 = G_1 \,{}_{o_1}\!\star_{o_2}\, G_2$

1. $A = A_1 \otimes P_2 + P_1 \otimes A_2$: sum of Boolean independent rv's in a suitable state
2. $\mathbf{Spec}(G) = \mathbf{Spec}(G_1) \star \mathbf{Spec}(G_2)$ — Boolean convolution
3. Examples include: star graphs
4. Bernoulli distribution in the limit (Boolean CLT)

cf. (IV) Free product — Free independence — Homogeneous tree

2.6. Summary: Graph Products and Independence in Quantum Probability

graph product	comb	star	direct	free
independence	monotone	Boolean	commutative	free
CLM	arcsine	Bernoulli	Gaussian	Wigner
examples	comb graph	star graph	integer lattice	homogeneous tree

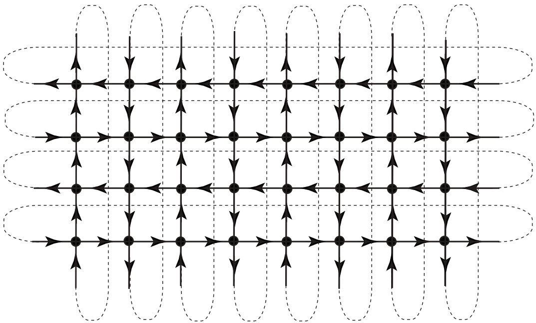
A. Hora and N. Obata:
Quantum Probability and Spectral Analysis of Graphs, Springer, 2007.

More varieties of graph products studied by Wysoczański, Lenczewski, ...

$\Rightarrow$ What about digraphs?

3. Manhattan Products of Digraphs

$$M(2m, 2n) = C_{2m} \# C_{2n}$$

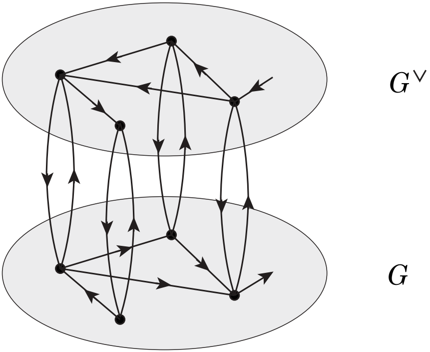


1. F. Comellas, C. Dalfó, M. A. Fiol and M. Mitjana: *A spectral study of the Manhattan networks*, Electronic Notes in Discrete Mathematics **29** (2007) 267–271.
2. F. Comellas, C. Dalfó, M. A. Fiol and M. Mitjana: *The spectra of Manhattan street networks*, Linear Algebra Appl. **429** (2008), 1823–1839.

3.2. $G\#C_2$

G: a digraph

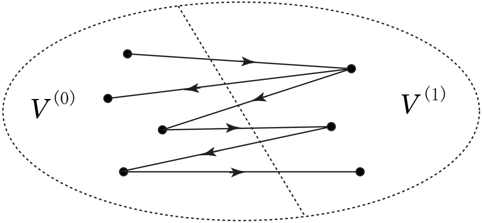
C_2: a symmetric digraph on two vertices



3.3. Bipartite Digraphs

Definition

A digraph $G = (V, E)$ is called *bipartite* if the vertex set admits a partition $V = V^{(0)} \cup V^{(1)}$ such that every arc bridges $V^{(0)}$ and $V^{(1)}$.



- A bipartite digraph does not contain a cycle of odd degree. More generally, a bipartite digraph does not contain a colliding cycle of odd degree.
- Examples: paths P_m, cycles of even degree C_{2m}, etc.

3.4. Manhattan Product of Bipartite Digraphs

$G_i = (V_i, E_i)$: bipartite digraphs
$G_i^\vee = (V_i, E_i^\vee)$: opposite digraphs
$V = V_1 \times V_2$

Define arcs in V as follows:

(i) If $y = y' \in V_2^{(0)}$ and $x \to x'$ then $(x, y) \to (x', y')$.

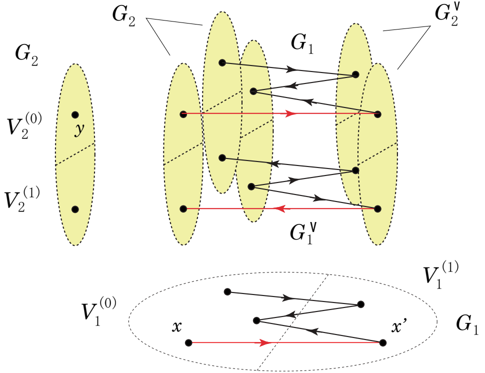
(i′) If $y = y' \in V_2^{(1)}$ and $x \to x'$ then $(x', y') \to (x, y)$.

(ii) If $x = x' \in V_1^{(0)}$ and ....

(ii′) If $x = x' \in V_1^{(1)}$ and ....

The digraph obtained in this manner is called the *Manhattan product* and is denoted by

$$G = G_1 \# G_2\,.$$



Theorem

The adjacency matrix A of $G = G_1 \# G_2$ satisfies

$$(A)_{(x,y)(x',y')} = \delta_{xx'}(t^{\pi_1(x)}(A_2))_{yy'} + (t^{\pi_2(y)}(A_1))_{xx'}\delta_{yy'},$$

for $x, x' \in V_1$ and $y, y' \in V_2$, where $t(A) = A^T$ and π_i is the "parity function".

cf) For a direct product $G_1 \times G_2$ we have

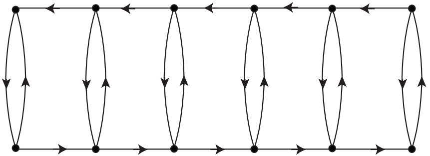
$$(A)_{(x,y)(x',y')} = \delta_{xx'}(A_2)_{yy'} + (A_1)_{xx'}\delta_{yy'}\,,$$

which means that

$$A = I \otimes A_2 + A_1 \otimes I$$

$\Longrightarrow$ commutative independence in quantum probability.

4. Spectral Analysis of Manhattan Products of Digraphs



The characteristic polynomial of $P_n \# C_2$ is given by (B: adjacency matrix of P_n)

$$\begin{aligned}\varphi_n(x) &= \det(x - A) = \det \begin{bmatrix} x - B & -I \\ -I & x - B^T \end{bmatrix} \\ &= \det((x - B)(x - B^T) - I) = x^2 \varphi_{n-1}(x) - x^2 \varphi_{n-2}(x), \end{aligned}$$

$$\varphi_1(x) = x^2 - 1, \qquad \varphi_2(x) = x^4 - 2x^2.$$

Using the recurrence relation of the Chebyshev polynomial of the second kind, we obtain

$$\varphi_n(x) = x^{n-1} \tilde{U}_{n+1}(x), \quad \text{where} \quad \tilde{U}_n(2\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}\,.$$

4.1. $P_n \# C_2$ (cont)

Theorem (H.-O. Lee: Master Thesis (2011))

$$\mathrm{ASpec}(P_n \# C_2) = \left\{ 2\cos\frac{k\pi}{n+2}\,;\ k=1,2,\dots,n+1 \right\} \cup \begin{pmatrix} 0 \\ n-1 \end{pmatrix},$$

where every non-zero eigenvalue has algebraic multiplicity one. The geometric multiplicity of zero eigenvalue is $n-1$.

cf) $\mathrm{ASpec}(P_n \times C_2) = \begin{pmatrix} -1 & 1 \\ n & n \end{pmatrix}$

Theorem (H.-O. Lee: Master Thesis (2011))

The asymptotic (algebraic) spectral distribution of $P_n \# C_2$ is given by

$$\frac{1}{2}\,\delta_0 + \frac{1}{2}\,\rho(x)dx, \qquad \rho(x) = \frac{1}{\pi\sqrt{4-x^2}}\,\chi_{(-2,2)}(x)$$

is the arcsine law with mean 0 and variance 2.

4.2. $M(2m, 2n) = C_{2m} \# C_{2n}$

Theorem (Comellas–Dalfó–Fiol–Mitjana (2007))

The $\mathbf{GSpec}$ *of the two-dimensional Manhattan network* $M(2m, 2n) = C_{2m} \# C_{2n}$ *is given by*

$$0 \quad \textit{and} \quad \pm\sqrt{2\cos\frac{2\pi k}{m} + 2\cos\frac{2\pi l}{m}}\,, \quad \begin{matrix} 0 \le k \le m-1, \\ 0 \le l \le n-1, \end{matrix}$$

with the geometric multiplicity of every non-zero eigenvalue coincides with the times it appears in the above expression. In particular,

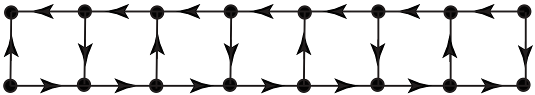
$$l(0) \ge 2mn \left(= \frac{2m \times 2n}{2} \right),$$

and equality happens when both m, n *are odd numbers.*

- Proof by constructing eigenvecotors from those of C_{2m} and C_{2n} .

A natural question: what about $P_m \# P_n$?
(removing the periodic boundary condition)

4.3. $P_n \# P_2$



The adjacency matrix is given by

$$A = \begin{bmatrix} B & P \\ Q & B^T \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & & & 1 \\ & & & & & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \end{bmatrix}, \quad Q = I - P.$$

We have immediately

$$\varphi_n(x) = \det(x - A) = x^n \det(x - B^T - Q(x - B)^{-1} P)$$

Theorem (O. IIS **18** (2012))

For $n = 1, 2, \dots$, *the characteristic polynomial* $\varphi_n(x)$ *of* $P_n \# P_2$ *is given by*

$$\varphi_{2n-1}(x) = x^{2n-2}\tilde{U}_n(x^2),$$
$$\varphi_{2n}(x) = x^{2n}\tilde{U}_n(x^2),$$

where $\tilde{U}_n$ *is defined by* $\tilde{U}_n(2\cos\theta) = \dfrac{\sin(n+1)\theta}{\sin\theta}$ $(= U(\cos\theta)$ *is the Chebyshev polynomial of the second kind).*

Proof. By standard cofactor expansion we obtain

$$\begin{aligned}
\varphi_1(x) &= x^2, \\
\varphi_2(x) &= x^4, \\
\varphi_3(x) &= x^6 - x^2, \\
\varphi_4(x) &= x^8 - x^4, \\
\varphi_n(x) &= x^4\varphi_{n-2} - x^4\varphi_{n-4}(x), \quad n \geq 5.
\end{aligned}$$

Then the result is easily verified.

4.3. $P_n \# P_2$ (cont)

Corollary

For $P_n \# P_2$ *we have:*

(1) Concentration of zero-eigenvalues:

$$m(0) \sim n, \qquad \textit{i.e.,} \quad \lim_{n\to\infty} \frac{m(0)}{2n} = \frac{1}{2}$$

(2) Asymptotic spectral distribution as $n \to \infty$ *is given by*

$$\frac{1}{2}\,\delta_0 + \frac{1}{4}\,\rho(x)\delta(y)dxdy + \frac{1}{4}\,\rho(y)\delta(x)dxdy, \quad z = x + iy,$$

where

$$\rho(x) = \frac{2|x|}{\pi\sqrt{4-x^4}}\,\chi_{[-\sqrt{2},+\sqrt{2}]}(x)$$

Proof. Straightforward from

$$\varphi_{2n-1}(x) = x^{2n-2}\tilde{U}_n(x^2), \quad \varphi_{2n}(x) = x^{2n}\tilde{U}_n(x^2)$$

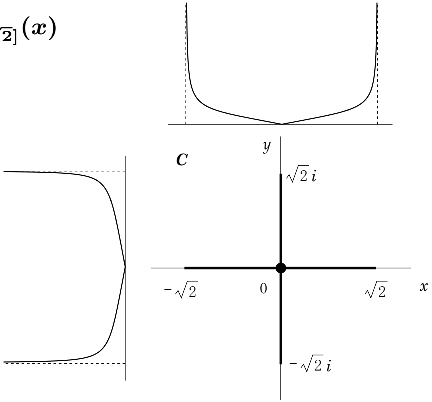
and the spectral distribution associated with $\tilde{U}(x)$ (see §5.1).

Asymptotic spectral distribution of $P_n \# P_2$ as $n \to \infty$

$$\frac{1}{2}\,\delta_0 + \frac{1}{4}\,\rho(x)\delta(y)dxdy + \frac{1}{4}\,\rho(y)\delta(x)dxdy, \quad z = x + iy,$$

where

$$\rho(x) = \frac{2|x|}{\pi\sqrt{4 - x^4}}\chi_{[-\sqrt{2},+\sqrt{2}]}(x)$$

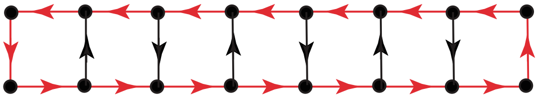


▶ Open question: For general m, n determine $\mathbf{ASpec}\,(P_n \# P_m)$.
Only numerical calculation was done [B. J. Choi (2012)]. Looks very interesting!

4.4. Non-standard Manhattan Product $P_n\#'P_2$

Non-standard Manhattan product $P_n\#'P_m$ only for even m, n

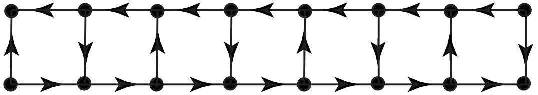
- Non-standard Manhattan product is characterized by the "beltway"



$P_8\#'P_2$

- Results for non-standard case are similar to the standard case [O. IIS **18** (2012)].

cf) Standard Manhattan product $P_8\#P_2$



$P_8\#P_2$

5. Applications to Coupled Oscillators

A basic model of coupled oscillators on a digraph G is described by

$$\begin{cases} \dot{\phi}_k = \omega_k - c \displaystyle\sum_{j \in V} A_{kj} \sin(\phi_k - \phi_j), \\ \dot{\omega}_k = -s \displaystyle\sum_{j \in V} A_{kj} \sin(\phi_k - \phi_j) \end{cases}$$

where $A = [A_{kj}]$ is the adjacency matrix of G, and $c > 0$, $s > 0$.

Motivation: Kuramoto model (1975) + self-adaptive dynamics

- We focus on the *linearized system*:

Coupled oscillators on a digraph

Let $G = (V, E)$ be a digraph and $L = D - A$ its Laplacian. Then

$$\begin{cases} \dot{\phi} = \omega - cL\phi \\ \dot{\omega} = -sL\phi \end{cases} \qquad \text{or} \qquad \begin{bmatrix} \dot{\phi} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -cL & 1 \\ -sL & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \omega \end{bmatrix}$$

For a large class of graphs (symmetric digraphs with no loops) and for a large enough coupling constant $c > 0$ we observe the *synchronization*, namely,

$$\lim_{t\to\infty} \omega_k(t) = \bar{\omega}$$

exists and the limit is independent of $k \in V$. Moreover,

$$\bar{\omega} = \frac{1}{|V|} \sum_{k\in V} \omega_k(0) \quad \text{(independent of the network topology!)}$$

J. Rodriguez and M-O. Hongler:

Networks of self-adaptive dynamical systems, IMA J. Appl. Math. (2012)

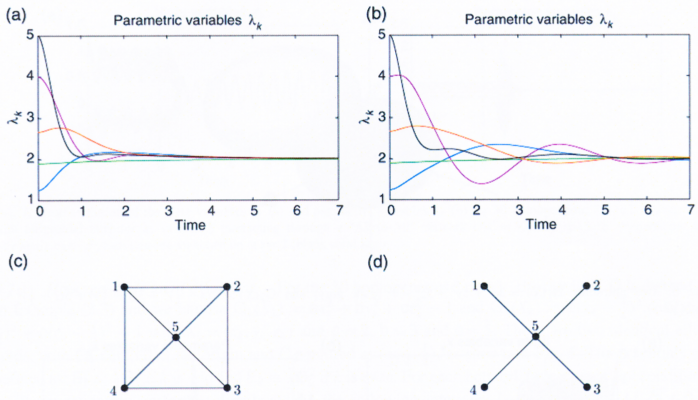


FIG. 4. Time evolution of the parametric variables λ_k (a and b) for five Hopf oscillators interacting through a 'Crystal' network (c) (AC = 3) and an 'All-to-One' network (d) (AC = 1), respectively.

5.3. Coupled Oscillators on Digraphs

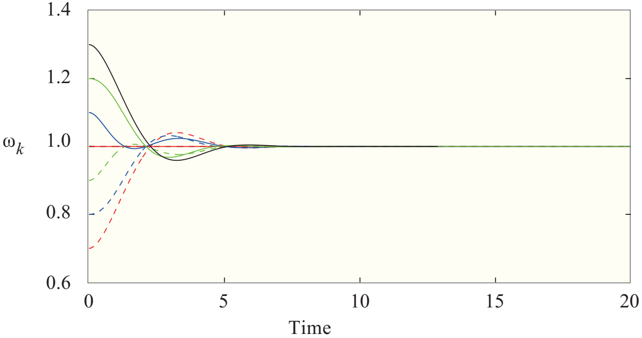
Question

1. Does synchronization occur for coupled oscillators on a digraph?
2. If so, find condition to have synchronization.

- The Laplacian $\boldsymbol{L}$ is no longer symmetric
  $\Longrightarrow$ complex spectrum + non-diagonalizable

Time evolution of ω_k on $P_4 \# P_2$ with initial conditions:

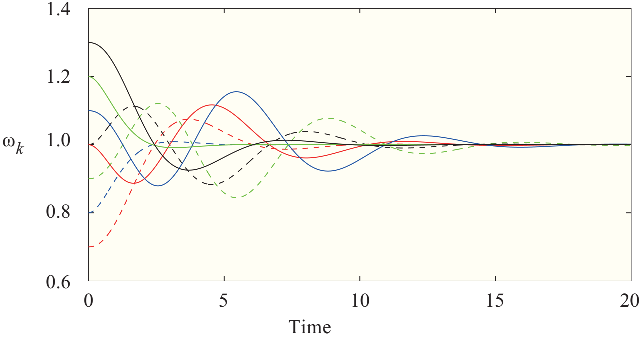
$$\phi_k(0) = 0 \quad \text{for all } k,$$
$$\omega_1(0) = 0.7, \omega_2(0) = 0.8, \omega_3(0) = 0.9, \omega_4(0) = 1$$
$$\omega_5(0) = 1, \omega_6(0) = 1.1, \omega_7(0) = 1.2, \omega_8(0) = 1.3.$$



vertex 1 = dashed red vertex 2 = dashed blue vertex 3 = dashed green vertex 4 = dashed black vertex 5 = red vertex 6 = blue vertex 7 = green vertex 8 = black

Time evolution of ω_k on $P_4 \#' P_2$ with initial conditions:

$$\phi_k(0) = 0 \quad \text{for all } k,$$
$$\omega_1(0) = 0.7, \omega_2(0) = 0.8, \omega_3(0) = 0.9, \omega_4(0) = 1$$
$$\omega_5(0) = 1, \omega_6(0) = 1.1, \omega_7(0) = 1.2, \omega_8(0) = 1.3.$$



vertex 1 = dashed red vertex 2 = dashed blue vertex 3 = dashed green vertex 4 = dashed black vertex 5 = red vertex 6 = blue vertex 7 = green vertex 8 = black

5.6. Asymptotic Stability (Toward Synchronization)

Our system is:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\omega} \end{bmatrix} = M \begin{bmatrix} \phi \\ \omega \end{bmatrix}, \qquad M = \begin{bmatrix} -cL & 1 \\ -sL & 0 \end{bmatrix}$$

Lemma

Let $L \in M(n, \mathbb{C})$, $c > 0$, $s > 0$ and M as above. Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be the eigenvalues of L. Then, the eigenvalues of M are exhausted by $\mu_\pm(\lambda_1), \dots, \mu_\pm(\lambda_n)$, where

$$\mu_\pm(\lambda) = \frac{c}{2}\left(-\lambda \pm \sqrt{\lambda^2 - \frac{4s}{c^2}\lambda}\right).$$

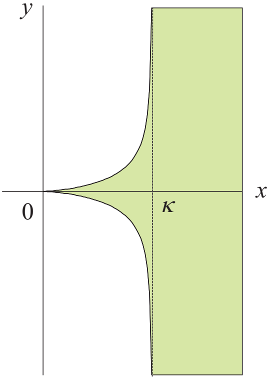
For $\kappa > 0$ let $\Lambda(\kappa)$ be the domain defined by

$$\Lambda(\kappa) = \{\lambda = x + iy \in \mathbb{C}\,;\, x^3 + (x-\kappa)y^2 > 0,\ x > 0\}$$

Theorem

If all non-zero eigenvalues of L are in $\Lambda(s/c^2)$, then $\mathrm{Re}\,\mu < 0$ for every non-zero eigenvalue μ of M.

$\Lambda(\kappa) = \{\lambda = x + iy \in \mathbb{C}\,;\, x^3 + (x-\kappa)y^2 > 0,\ x > 0\}$



$$\begin{bmatrix} \dot{\phi} \\ \dot{\omega} \end{bmatrix} = M \begin{bmatrix} \phi \\ \omega \end{bmatrix}, \qquad M = \begin{bmatrix} -cL & 1 \\ -sL & 0 \end{bmatrix}$$

Theorem (O. and J. Rodriguez (2013))

Let L be the Laplacian of a digraph G. Assume

1. $\mathrm{Re}\,\lambda > 0$ *for all non-zero eigenvalue λ of L;*
2. *the algebraic and geometric multiplicities of zero-eigenvalue coincide.*

Then, choosing $s > 0$ and $c > 0$ with s/c^2 is sufficient small, every eigenvalue μ of M satisfies $\mathrm{Re}\,\mu < 0$.

Hence the solution to our system satisfies

$$\lim_{t\to\infty} \begin{bmatrix} \phi(t) \\ \omega(t) \end{bmatrix} = \lim_{t\to\infty} e^{tM} \begin{bmatrix} \phi(0) \\ \omega(0) \end{bmatrix} = (\text{constant vector}).$$

Open Problem: $\lim_{t\to\infty} \omega_k(t) = \bar{\omega}$ (independent of k).

Digraphs satisfying the condition in the above theorem: $C_{2m}\#C_{2n}$, $P_{2n}\#P_2$ (for small n)

Some Open Questions

1. Determine the spectra of (the adjacency matrix) of the Manhattan street digraphs $P_n \# P_m$ (standard) and $P_n \#' P_m$ (non-standard) for general m, n.
2. Work in progress for $P_n \# P_3$ (standard) and $P_n \#' P_3$ (non-standard).
3. Determine the spectra of (the Laplacian) of the Manhattan street digraphs $P_n \# P_m$ (standard) and $P_n \#' P_m$ (non-standard) for general m, n.
4. Work in progress for $P_{2n} \# P_2$ (standard) .
5. Prove synchronization for a large class of Manhattan products of digraphs.