Contents

8 Homogeneous Trees

8.1 Main Problems

Definition 8.1.1 A connected graph is called a *tree* is it has no cycles. A tree is called *homogeneous* if it is regular.

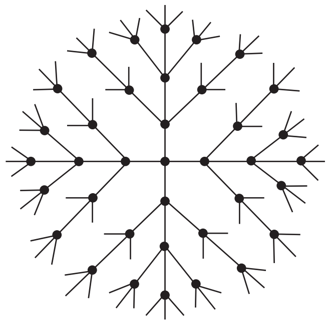


Figure 8.1: Homogeneous tree of degree 4

Let T_κ be the homoeeous tree of deree $\kappa \geq 2$ and $A = A_\kappa$ be the adjacency matrix. We choose and fix a vertex $o \in T_\kappa$ as an origin (root). Our first interest lies in

$$(A^m)_{oo} = \langle \delta_o, A^m \delta_o \rangle = |\{m\text{-step walks from } o \text{ to itself}\}|.$$

In particular, we will study

(1) Integral representation of $(A^m)_{oo}$:

$$(A^m)_{oo} = \int_{-\infty}^{+\infty} x^m \mu_\kappa(dx), \qquad m = 1, 2, \dots,$$

where $\mu_\kappa(dx)$ is a probability distribution on $\mathbf{R}$.

(2) Asymptotic behavior of $(A^m)_{oo}$ for a large κ.

We call μ_κ the *vacuum spectral distribution* of the homogeneous tree T_κ.

8.2 Vacuum Spectral Distribution

As usual, we start with the stratification of the vertex set V of T_κ:

$$V = \bigcup_{n=0}^{\infty} V_n\,, \qquad V_n = \{y \in V\,;\, \partial(o, x) = n\}.$$

It is easy to see that

$$|V_0| = 1, \quad |V_1| = \kappa, \quad |V_2| = \kappa(\kappa - 1), \quad \dots, \quad |V_n| = \kappa(\kappa - 1)^{n-1}. \tag{8.1}$$

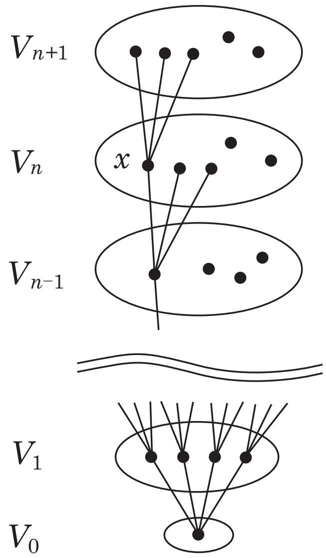


Figure 8.2: Stratification of T_4

Define a unit vector Φ_n in $\ell^2(T_\kappa)$ by

$$\Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} \delta_x$$

and let $\Gamma(T_\kappa)$ be the linear span of $\{\Phi_0, \Phi_1, \dots\}$.

Now let $A = A^+ + A^-$ be the quantum decomposition of the adjacency matrix $A = A_\kappa$. Let us observe the actions of $A^\pm$ on $\Gamma(T_\kappa)$. First, since

$$|V_n|^{1/2} A^+ \Phi_n = \sum_{x \in V_n} A^+ \delta_x = \sum_{y \in V_{n+1}} \delta_y = |V_{n+1}|^{1/2} \Phi_{n+1}\,,$$

we obtain

$$A^+ \Phi_n = \left(\frac{|V_{n+1}|}{|V_n|} \right)^{1/2} \Phi_{n+1}\,.$$

In view of (8.1) we obtain

$$A^+ \Phi_0 = \sqrt{\kappa}\, \Phi_1\,, \quad A^+ \Phi_n = \sqrt{\kappa - 1}\, \Phi_{n+1} \quad (n \geq 1). \tag{8.2}$$

In a similar manner, we consider

$$|V_n|^{1/2} A^- \Phi_n = \sum_{x \in V_n} A^- \delta_x\,.$$

For $n \geq 2$ there are $\kappa - 1$ vertices in V_n which have a common end-vertex in V_{n-1}. Hence, for $n \geq 2$

$$|V_n|^{1/2} A^- \Phi_n = \sum_{x \in V_n} A^- \delta_x = (\kappa - 1) \sum_{y \in V_{n-1}} \delta_y = (\kappa - 1)|V_{n-1}|^{1/2} \Phi_{n-1}\,.$$

While, for $n = 1$ we have

$$|V_1|^{1/2} A^- \Phi_1 = \sum_{x \in V_1} A^- \delta_x = \kappa \delta_o = \kappa \Phi_0 \,.$$

Then, in view of (8.1) we obtain

$$A^- \Phi_0 = 0, \quad A^- \Phi_1 = \sqrt{\kappa}\, \Phi_0 \,, \quad A^- \Phi_n = \sqrt{\kappa - 1}\, \Phi_{n-1} \quad (n \geq 2). \tag{8.3}$$

We see from (8.2) and (8.3) also that $\Gamma(T_\kappa)$ is invariant under the actions of $A^\pm$.

Summing up,

Proposition 8.2.1 *Notations being as above,* $(\Gamma(T_\kappa), \{\Phi_n\}, A^+, A^-)$ *is an interacting Fock space associated with the Jacobi sequence*

$$\omega_1 = \kappa, \quad \omega_2 = \omega_3 = \cdots = \kappa - 1.$$

The vacuum spectral distribution μ_κ is a probability measure whose Jacobi coefficients are

$$\omega_1 = \kappa, \quad \omega_2 = \omega_3 = \cdots = \kappa - 1, \qquad \alpha_1 = \alpha_2 = \cdots = 0.$$

In fact, we have

$$\langle \Phi_0, A^m \Phi_0 \rangle = \langle \Phi_0, (A^+ + A^-)^m \Phi_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu_\kappa(dx), \qquad m = 1, 2, \ldots. \tag{8.4}$$

We also note that μ_κ in (8.4) is uniquely determined since Carleman's condition

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{\omega_n}} = \infty$$

is satisfied.

Summing up,

Proposition 8.2.2 *The vacuum spectral distribution of* T_κ *is a probability distribution* μ_κ *characterized uniquely by its Jacobi coefficients*

$$\omega_1 = \kappa, \quad \omega_2 = \omega_3 = \cdots = \kappa - 1, \qquad \alpha_1 = \alpha_2 = \cdots = 0.$$

Remark 8.2.3 An explicit form of μ_κ is known:

$$\mu_k(dx) = \frac{\kappa}{2\pi} \frac{\sqrt{4(\kappa - 1) - x^2}}{\kappa^2 - x^2} 1_{[-2\sqrt{\kappa-1}, 2\sqrt{\kappa-1}]}(x) dx.$$

The detail will be discussed later.

8.3 Asymptotic Spectral Distribution

We are interested in the asymptotic behavior of μ_κ as $\kappa \to \infty$. Note first that

$$\mathrm{mean}(\mu_\kappa) = \int_{-\infty}^{+\infty} x\mu_\kappa(dx) = (A)_{oo} = 0,$$
$$\mathrm{var}(\mu_\kappa) = \int_{-\infty}^{+\infty} (x - \mathrm{mean}(\mu_\kappa))^2 \mu_\kappa(dx) = (A^2)_{oo} = \deg(o) = \kappa.$$

Therefore,

$$\frac{A}{\sqrt{\kappa}} = \frac{A^+}{\sqrt{\kappa}} + \frac{A^-}{\sqrt{\kappa}}$$

is a reasonable scaling for $\kappa \to \infty$.

It follows from (8.2) and (8.3) that

$$\frac{A^+}{\sqrt{\kappa}}\,\Phi_0 = \Phi_1, \quad \frac{A^+}{\sqrt{\kappa}}\,\Phi_n = \sqrt{\frac{\kappa-1}{\kappa}}\,\Phi_{n+1} \quad (n \geq 1) \tag{8.5}$$

$$\frac{A^-}{\sqrt{\kappa}}\,\Phi_0 = 0, \quad \frac{A^-}{\sqrt{\kappa}}\,\Phi_1 = \Phi_0, \quad \frac{A^-}{\sqrt{\kappa}}\,\Phi_n = \sqrt{\frac{\kappa-1}{\kappa}}\,\Phi_{n-1} \quad (n \geq 2) \tag{8.6}$$

The actions of $\dfrac{A_\kappa^\pm}{\sqrt{\kappa}}$ in the limit as $\kappa \to \infty$ are now easily expected.

Definition 8.3.1 An interacting Fock space associated with the Jacobi sequence $\omega_n \equiv 1$ is called the *free Fock space*. Let $(\Gamma_{\mathrm{free}}, \{\Psi_n\}, B^+, B^-)$ be a free Fock space. Then,

$$B^+\Phi_n = \Phi_{n+1} \quad (n \geq 0), \qquad B^-\Phi_0 = 0, \quad B^-\Phi_n = \Phi_{n-1} \quad (n \geq 1). \tag{8.7}$$

Theorem 8.3.2 (Quantum Central Limit Theorem) *For any $\epsilon_1, \dots, \epsilon_m \in \{\pm\}$ and $m = 1, 2, \dots$ we have*

$$\lim_{\kappa\to\infty} \left\langle \Phi_0, \frac{A_\kappa^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A_\kappa^{\epsilon_1}}{\sqrt{\kappa}}\,\Phi_0 \right\rangle = \langle \Psi_0, B^{\epsilon_m} \cdots B^{\epsilon_1}\Psi_0 \rangle$$

For short, we say that

$$\lim_{\kappa\to\infty} \frac{A_\kappa^\pm}{\sqrt{\kappa}} = B^\pm$$

in the sense of stochastic convergence.

Proof. More generally, we may prove that

$$\lim_{\kappa\to\infty} \left\langle \Phi_i, \frac{A_\kappa^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A_\kappa^{\epsilon_1}}{\sqrt{\kappa}}\,\Phi_j \right\rangle = \langle \Psi_i, B^{\epsilon_m} \cdots B^{\epsilon_1}\Psi_j \rangle \tag{8.8}$$

for any $i, j \geq 0$. The proof is by induction on m. For $m = 1$ we need to prove that

$$\lim_{\kappa\to\infty} \left\langle \Phi_i, \frac{A_\kappa^{\epsilon_1}}{\sqrt{\kappa}}\,\Phi_j \right\rangle = \langle \Psi_i, B^{\epsilon_1}\Psi_j \rangle \tag{8.9}$$

for any $i, j \ge 1$ and $\epsilon_1 = \pm$. Suppose that $\epsilon_1 = +$. By (8.5),

$$\lim_{\kappa\to\infty} \left\langle \Phi_i, \frac{A_\kappa^+}{\sqrt{\kappa}}\,\Phi_0 \right\rangle = \lim_{\kappa\to\infty} \langle \Phi_i, \Phi_1\rangle = \langle \Psi_i, \Psi_1\rangle = \langle \Psi_i, B^+\Psi_0\rangle,$$

$$\lim_{\kappa\to\infty} \left\langle \Phi_i, \frac{A_\kappa^+}{\sqrt{\kappa}}\,\Phi_j \right\rangle = \lim_{\kappa\to\infty} \sqrt{\frac{\kappa-1}{\kappa}}\,\langle \Phi_i, \Phi_{j+1}\rangle = \langle \Psi_i, \Psi_{j+1}\rangle = \langle \Psi_i, B^+\Psi_j\rangle,$$

where $j \ge 1$. Thus, (8.9) is shown for $\epsilon_1 = +$. The case of $\epsilon_1 = -$ is similar.

We now come toe the induction step, but the idea is similar. The detailed proof is left to the reader. ∎

8.4 Vacuum Distribution of Free Fock Space

Let $(\Gamma_{\text{free}}, \{\Psi_n\}, B^+, B^-)$ be a free Fock space.

Lemma 8.4.1 *For $m = 1, 2, \dots,$*

$$\langle \Psi_0, (B^+ + B^-)^{2m-1}\Psi_0\rangle = 0, \tag{8.10}$$

$$\langle \Psi_0, (B^+ + B^-)^{2m}\Psi_0\rangle = \frac{(2m)!}{m!(m+1)!}\,. \tag{8.11}$$

PROOF. We start with

$$\langle \Psi_0, (B^+ + B^-)^k\Psi_0\rangle = \sum \epsilon_1, \dots, \epsilon_k \in \{\pm\}\langle \Psi_0, B^{\epsilon_k}\cdots B^{\epsilon_1}\Psi_0\rangle,$$

where

$$\langle \Psi_0, B^{\epsilon_k}\cdots B^{\epsilon_1}\Psi_0\rangle = \begin{cases} 1, & \text{if } B^{\epsilon_k}\cdots B^{\epsilon_1}\Psi_0 = \Psi_0, \\ 0, & \text{otherwise.} \end{cases}$$

Then (8.10) follows immediately from the actions of $B^\pm$ in (8.7). For $k = 2m$, $B^{\epsilon_{2m}}\cdots B^{\epsilon_1}\Psi_0 = \Psi_0$ occurs if and only if

$$\begin{gathered} \epsilon_1 \ge 0, \\ \epsilon_1 + \epsilon_2 \ge 0, \\ \cdots \\ \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{2m-1} \ge 0, \\ \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{2m-1} + \epsilon_{2m} = 0. \end{gathered}$$

Such a sequence $(\epsilon_1, \epsilon_2, \dots, \epsilon_{2m}) \in \{+,-\}^{2m}$ is called a *Catalan path* of length $2m$ and denote by $\mathcal{C}_m$ the set of such Catalan paths. With this notation we come to

$$\langle \Psi_0, (B^+ + B^-)^{2m}\Psi_0\rangle = |\mathcal{C}_m|.$$

It is then sufficient to show that

$$|\mathcal{C}_m| = \frac{(2m)!}{m!(m+1)!}\,.$$

The proof is given separately. ∎

Lemma 8.4.2 *For $m = 1, 2, \dots$ we have*

$$|\mathcal{C}_m| = \frac{(2m)!}{m!(m+1)!}\,.$$

PROOF. We set

$$\tilde{\mathcal{C}}_m = \left\{\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{2m}) \in \{+,-\}^{2m}\,;\, \epsilon_1 + \cdots + \epsilon_{2m} = 0\right\}.$$

Obviously, $\mathcal{C}_m \subset \tilde{\mathcal{C}}_m$. Each $\epsilon \in \tilde{\mathcal{C}}_m$ corresponds to a path connecting the vertices

$$(0,0),\ (1,\epsilon_1),\ (2,\epsilon_1+\epsilon_2),\ \dots,\ (2m, \epsilon_1+\epsilon_2+\cdots+\epsilon_{2m}) = (2m,0)$$

in order. Since we have

$$|\tilde{\mathcal{C}}_m| = \binom{2m}{m} = \frac{(2m)!}{m!m!}\,,$$

for $|\mathcal{C}_m|$ it is sufficient to count the number of paths in $\tilde{\mathcal{C}}_m \setminus \mathcal{C}_m$. By definition a path $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{2m})$ in $\tilde{\mathcal{C}}_m \setminus \mathcal{C}_m$ has one or more vertices with negative ordinates. Let k be the abscissa of the first such vertex. Then $1 \le k \le 2m-1$. If $k = 1$ we have $\epsilon_1 = -1$. Otherwise,

$$\begin{aligned}&\epsilon_1 \ge 0, \quad \epsilon_1+\epsilon_2 \ge 0, \quad \dots, \quad \epsilon_1+\cdots+\epsilon_{k-1} = 0,\\ &\epsilon_1+\cdots+\epsilon_{k-1}+\epsilon_k = -1.\end{aligned}$$

Let L be the horizontal line passing through $(0,-1)$. Then ϵ has one or more vertices which lie on L and $(k,-1)$ is the first one. Define $\bar{\epsilon}$ to be the path obtained from ϵ by reflecting the first part of ϵ up to $(k,-1)$ with respect to L (see Fig. 8.3). Then $\bar{\epsilon}$ becomes a path

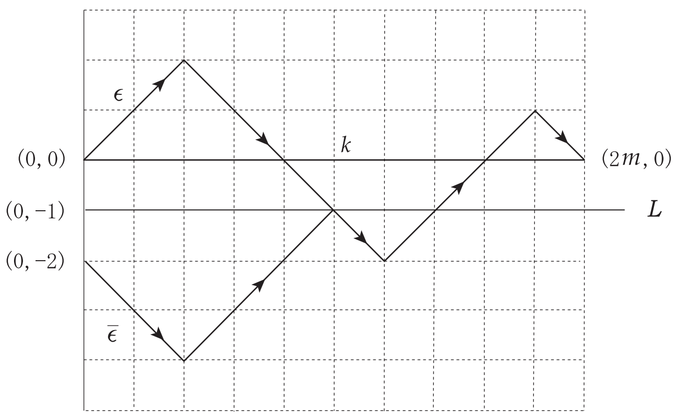


Figure 8.3: Counting the Catalan number

from $(0,-2)$ to $(2m,0)$ passing through $(k,-1)$ as the first meeting point with L. It is easily verified that $\epsilon \leftrightarrow \bar{\epsilon}$ is a one-to-one correspondence between $\tilde{\mathcal{C}}_m \setminus \mathcal{C}_m$ and the set of paths connecting $(0,-2)$ and $(2m,0)$. Obviously, the number of such paths is

$$\binom{2m}{m+1} = \frac{(2m)!}{(m+1)!(m-1)!} = |\tilde{\mathcal{C}}_m \setminus \mathcal{C}_m|.$$

Hence

$$|\mathcal{C}_m| = \frac{(2m)!}{m!m!} - \frac{(2m)!}{(m+1)!(m-1)!} = \frac{(2m)!}{m!(m+1)!},$$

which completes the proof. ∎

Definition 8.4.3 We call

$$C_m = |\mathcal{C}_m| = \frac{(2m)!}{m!(m+1)!}$$

the mth *Catalan number*.

Theorem 8.4.4 *We have*

$$\langle \Psi_0, (B^+ + B^-)^m \Psi_0 \rangle = \frac{1}{2\pi} \int_{-2}^{+2} x^m \sqrt{4-x^2}\, dx, \qquad m = 1, 2, \dots. \tag{8.12}$$

Proof. By direct computation we may obtain

$$\frac{1}{2\pi} \int_{-2}^{+2} x^{2m-1} \sqrt{4-x^2}\, dx = 0,$$
$$\frac{1}{2\pi} \int_{-2}^{+2} x^{2m} \sqrt{4-x^2}\, dx = \frac{(2m)!}{m!(m+1)!}.$$

Then the assertion follows by combining Lemma 8.4.1. ∎

Definition 8.4.5 The probability distribution

$$\frac{1}{2\pi} \sqrt{4-x^2}\, 1_{[-2,2]}(x) dx$$

is called the *Wigner semicircle law*. This is normalized to have mean 0 and variance 1.

Theorem 8.4.6 (Central Limit Theorem) *Let A_κ be the adjacency matrix of a homogeneous tree T_κ. We have*

$$\lim_{\kappa\to\infty} \left(\left(\frac{A_\kappa}{\sqrt{\kappa}} \right)^m \right)_{oo} = \frac{1}{2\pi} \int_{-2}^{+2} x^m \sqrt{4-x^2}\, dx, \qquad m = 1, 2, \dots.$$

PROOF. We start with

$$\left(\left(\frac{A_\kappa}{\sqrt{\kappa}}\right)^m\right)_{oo} = \left\langle \Phi_0, \left(\frac{A^+}{\sqrt{\kappa}} + \frac{A^-}{\sqrt{\kappa}}\right)^m \Phi_0 \right\rangle.$$

Expanding $\left(\frac{A^+}{\sqrt{\kappa}} + \frac{A^-}{\sqrt{\kappa}}\right)^m$ and applying Theorem 8.3.2 (QCT), we obtain

$$\lim_{\kappa\to\infty} \left(\left(\frac{A_\kappa}{\sqrt{\kappa}}\right)^m\right)_{oo} = \langle \Psi_0, (B^+ + B^-)^m \Psi_0 \rangle.$$

Then with the help of Theorem 8.4.4, we get the assertion. ∎

The Wigner semicircle law is a unique probability distribution satisfying (8.12). In other words, the Wigner semicircle law is the solution of a determinate moment problem. In fact, its moments of even orders are given by

$$M_{2m} = C_m = \frac{(2m)!}{m!(m+1)!}$$

and satisfies Carleman's condition:

$$\sum_{m=1}^{\infty} M_{2m}^{-\frac{1}{2m}} = +\infty.$$

Then by a general result on the weak convergence of probability measures, we conclude the following

Corollary 8.4.7 *Let μ_κ be the vacuum spectral distribution of T_κ. Then*

$$\lim_{\kappa\to\infty} \int_{-\infty}^{+\infty} f\left(\frac{x}{\sqrt{\kappa}}\right) \mu_\kappa(dx) = \frac{1}{2\pi} \int_{-2}^{+2} f(x)\sqrt{4-x^2}\, dx$$

for all $f \in C_{bdd}(\mathbf{R})$. In other words, the scaled $\tilde{\mu}_\kappa(dx) = \mu_\kappa(\sqrt{\kappa}\, dx)$ converges weakly to the Wigner semicircle law as $\kappa \to \infty$.

8.5 Exercises

1. Complete the proof of Theorem 8.3.2.
2. Prove that the Wigner semicircle law is the solution of a determinate moment problem.

9 Deformed Vacuum States and Free Poisson Distributions

9.1 Q-Matrix

Let $G = (V, E)$ be a graph and $\partial(x, y)$ denotes the graph distance between two verices $x, y \in V$. For $-1 \le q \le 1$ we define a matrix $Q = Q_q$ with elements

$$(Q)_{xy} = q^{\partial(x,y)}.$$

By definition $Q_0 = I$. We call Q the *Q-matrix* of G.

Let $o \in V$ be a fixed origin of the graph G. A linear functional on the adjacency algebra $\mathcal{A}$ defined by

$$\mathcal{A} \ni a \mapsto \langle a \rangle_q = \langle Q\delta_o, a\delta_o \rangle$$

is called a *deformed vacuum functional.*

Proposition 9.1.1 *The deformed vacuum functional is a state if*

(i) *$QA = AQ$;*

(ii) *Q is a positive definite kernel on V, i.e.,*

$$\sum_{x,y \in V} f(x) q^{\partial(x,y)} f(y) \ge 0$$

for all $f : V \to \mathbf{R}$ with finite supports.

Theorem 9.1.2 *The deformed vacuum functional $\langle \cdot \rangle_q$ on the homogeneous tree T_κ is a state for all $-1 \le q \le 1$.*

Proof. We check the conditions (i) and (ii) in Proposition 9.1.1. First, (i) is clear because T_κ is distance-regular. For (ii) it is sufficient to show that the Q-matrix of a finite tree is positive definite for all $-1 \le q \le 1$. We employ Bożejko's theorem. ∎

Theorem 9.1.3 (Bożejko) *Let V be a set which is a union of two subsets V_1, V_2 whose intersection consists of a single point, say $o \in V$. Namely,*

$$V = V_1 \cup V_2\,, \qquad V_1 \cap V_2 = \{o\}.$$

For $i = 1, 2$ let K_i be a positive definite kernel on V_i and assume that $K_1(o, o) = K_2(o, o) = 1$. Define a $\mathbf{C}$-valued function K on $V \times V$ by

$$K(x,y) = \begin{cases} K_1(x,y), & \text{if } x, y \in V_1\,, \\ K_2(x,y), & \text{if } x, y \in V_2\,, \\ K_1(x,o)K_2(o,y), & \text{if } x \in V_1,\ y \in V_2\,, \\ K_2(x,o)K_1(o,y), & \text{if } x \in V_2,\ y \in V_1\,. \end{cases}$$

Then K is a positive definite kernel on V.

PROOF. Let $f \in C_0(V)$. We may write $f = f_1 + f_2$ with $f_i \in C_0(V_i)$ though uniqueness does not hold for $V_1 \cap V_2 = \{o\}$. Then,

$$\begin{aligned}\sum_{x,y\in V} \overline{f(x)}\, K(x,y) f(y) = &\sum_{x,y\in V_1} \overline{f_1(x)}\, K_1(x,y) f_1(y) \\ &+ \sum_{x,y\in V_2} \overline{f_2(x)}\, K_2(x,y) f_2(y) \\ &+ \sum_{x\in V_1, y\in V_2} \overline{f_1(x)}\, K(x,y) f_2(y) \\ &+ \sum_{x\in V_2, y\in V_1} \overline{f_2(x)}\, K(x,y) f_1(y). \end{aligned} \tag{9.1}$$

We now observe that

$$\left| \sum_{x\in V_1} \overline{f_1(x)}\, K_1(x,o) \right|^2 = \left| \sum_{x,y\in V_1} \overline{f_1(x)}\, K_1(x,y) \delta_o(y) \right|^2 . \tag{9.2}$$

Since K_1 is a positive definite kernel on V_1, the right-hand side in (9.2) is the square of the inner product

$$\langle f_1, \delta_o \rangle_{K_1} = \sum_{x,y\in V_1} \overline{f_1(x)}\, K_1(x,y) \delta_o(y).$$

Then by the Schwarz inequality we have

$$|\langle f_1, \delta_o \rangle_{K_1}|^2 \le \langle f_1, f_1 \rangle_{K_1} \langle \delta_o, \delta_o \rangle_{K_1} .$$

Namely,

$$\begin{aligned}\left| \sum_{x\in V_1} \overline{f_1(x)}\, K_1(x,o) \right|^2 &\le K_1(o,o) \sum_{x,y\in V_1} \overline{f_1(x)}\, K_1(x,y) f_1(y) \\ &= \sum_{x,y\in V_1} \overline{f_1(x)}\, K_1(x,y) f_1(y). \end{aligned} \tag{9.3}$$

Similarly,

$$\left| \sum_{y\in V_2} \overline{f_2(y)}\, K_2(y,o) \right|^2 \le \sum_{x,y\in V_2} \overline{f_2(x)}\, K_2(x,y) f_2(y). \tag{9.4}$$

On the other hand, by definition we have

$$\begin{aligned} &\sum_{x\in V_1, y\in V_2} \overline{f_1(x)}\, K(x,y) f_2(y) \\ &\quad = \sum_{x\in V_1, y\in V_2} \overline{f_1(x)}\, K_1(x,o) K_2(o,y) f_2(y) \\ &\quad = \sum_{x\in V_1} \overline{f_1(x)}\, K_1(x,o) \sum_{y\in V_2} K_2(o,y) f_2(y) \\ &\quad = \sum_{x\in V_1} \overline{f_1(x)}\, K_1(x,o) \overline{\sum_{y\in V_2} \overline{f_2(y)}\, K_2(y,o)} . \end{aligned} \tag{9.5}$$

Similarly,

$$\sum_{x\in V_2, y\in V_1} \overline{f_2(x)}\, K(x,y) f_1(y) = \overline{\sum_{x\in V_1} \overline{f_1(x)}\, K_1(x,o)} \sum_{y\in V_2} \overline{f_2(y)}\, K_2(y,o). \tag{9.6}$$

Combining (9.3)–(9.6), we see that (9.1) becomes

$$\begin{aligned}\sum_{x,y\in V} \overline{f(x)}\, K(x,y) f(y) &\ge \left|\sum_{x\in V_1} \overline{f_1(x)}\, K_1(x,o)\right|^2 + \left|\sum_{y\in V_2} \overline{f_2(y)}\, K_2(y,o)\right|^2 \\ &\quad + \sum_{x\in V_1} \overline{f_1(x)}\, K_1(x,o) \overline{\sum_{y\in V_2} \overline{f_2(y)}\, K_2(y,o)} \\ &\quad + \overline{\sum_{x\in V_1} \overline{f_1(x)}\, K_1(x,o)} \sum_{y\in V_2} \overline{f_2(y)}\, K_2(y,o) \\ &= \left|\sum_{x\in V_1} \overline{f_1(x)}\, K_1(x,o) + \sum_{y\in V_2} \overline{f_2(y)}\, K_2(y,o)\right|^2 \ge 0.\end{aligned}$$

This completes the proof. ∎

Definition 9.1.4 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with fixed origins $o_1 \in V_1$ and $o_2 \in V_2$. The *star product* of G_1 and G_2 is obtained by glueing o_1 and o_2, and is denoted by $G_1 \star G_2$.

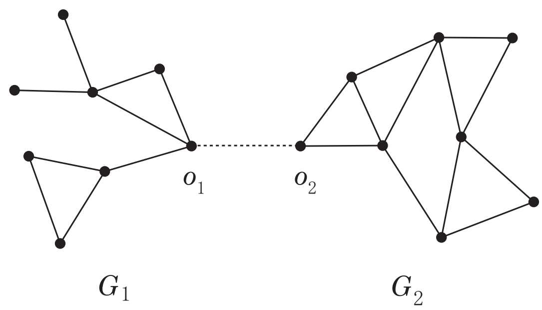


Figure 9.1: Star product $G_1 \star G_2$

Theorem 9.1.5 *Let $G_1 = (V_1, E_2)$ and $G_2 = (V_1, E_2)$ be two graphs with with distance functions ∂_{G_1} and ∂_{G_2}, respectively. Taking origins $o_1 \in V_1$ and $o_2 \in V_2$, we form the star product $G = G_1 \star G_2 = (V, E)$, the distance function of which is denoted by ∂_G. The Q-matrices of G_1, G_2 and G are defined by*

$$Q_1 = (q^{\partial_{G_1}(x,y)})_{x,y\in V_1}, \qquad Q_2 = (q^{\partial_{G_2}(x,y)})_{x,y\in V_2}, \qquad Q = (q^{\partial_G(x,y)})_{x,y\in V}.$$

If both Q_1 and Q_2 are positive definite kernels (for some q) on V_1 and V_2, respectively, then so is Q.

PROOF. It is easy to see by definition of a star product that

$$\partial_G(x,y) = \begin{cases} \partial_{G_1}(x,y), & \text{if } x,y \in V_1\,, \\ \partial_{G_2}(x,y), & \text{if } x,y \in V_2\,, \\ \partial_{G_1}(x,o_1) + \partial_{G_2}(o_2,y), & \text{if } x \in V_1,\ y \in V_2\,, \\ \partial_{G_2}(x,o_2) + \partial_{G_1}(o_1,y), & \text{if } x \in V_2,\ y \in V_1\,. \end{cases}$$

Then Q, Q_1, Q_2 satisfy the conditions of Theorem 9.1.3. ∎

PROOF OF THEOREM 9.1.2 (CONT.) For the graph T_1 (•——•) the Q-matrix takes the form:

$$Q = \begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix}.$$

It is straightforward to see that Q is positive definite for all $q \in [-1,1]$. Since any finite tree is obtained from T_1 by repeated application of star products, the Q-matrix of a finite tree is positive definite for all $q \in [-1,1]$ by Theorem 9.1.5.

Now consider T_κ and $f \in C_0(T_\kappa)$. We may choose a finite sub tree of T_κ, say G, such that f vanishes outside G. Note that the distance function ∂_G is just the restriction of that of T_κ. Hence the Q-matrix of G, say Q_G, is the restriction of that of T_κ. Therefore,

$$\sum_{x,y\in V} \overline{f(x)}\, q^{\partial(x,y)} f(y) = \sum_{x,y\in G} \overline{f(x)}\, q^{\partial_G(x,y)} f(y),$$

which is ≥ 0 since Q_G is positive definite by the above argument. ∎

Definition 9.1.6 The deformed vacuum state $\langle\cdot\rangle_q$ on the adjacency algebra of T_κ is called the *Haagerup state*. In fact, Theorem 9.1.2 is originally due to Haagerup, while our proof is based on Bożejko's argument.

9.2 Spectral Distributions in Haagerup States

Let T_κ be the homogeneous tree of degree κ and consider the Haagerup states $\langle\cdot\rangle_q$ with $-1 \leq q \leq 1$. We are interested in the asymptotics of the spectral distribution $\mu_{\kappa,q}$ determined by

$$\langle A^m \rangle_q = \int_{-\infty}^{+\infty} x^m \mu_{\kappa,q}(dx), \qquad m = 1, 2, \dots.$$

It is reasonable to call $\mu_{\kappa,q}$ a *deformed Kesten distribution*.

We first note the following

Lemma 9.2.1 (1) mean $(\mu_{\kappa,q}) = \langle A \rangle_q = \kappa q$.

(2) var $(\mu_{\kappa,q}) = \Sigma_q^2(A) = \kappa(1-q^2)$.

PROOF. (1) By definition

$$\begin{aligned}\langle A\rangle_q &= \langle Q\delta_o, A\delta_o\rangle = \langle \delta_o, QA\delta_o\rangle = (QA)_{oo} \\ &= \sum_{x\in V}(Q)_{ox}(A)_{xo} = \sum_{x\sim o}(Q)_{ox} = \sum_{x\sim o} q^{\partial(o,x)} \\ &= q|\{x\in V\,;\, x\sim o\}| = q\kappa.\end{aligned}$$

(2) Since

$$\Sigma_q^2(A) = \langle A^2\rangle_q - \langle A\rangle_q^2$$

by definition, we need to compute $\langle A^2\rangle_q$. In a similar manner as in (1) we see that

$$\langle A^2\rangle_q = \kappa(\kappa-1)q^2 + \kappa,$$

from which the assertion follows. ∎

Lemma 9.2.1 suggests that a reasonable object to study is not A itself but the normalized adacency matrix defined by

$$\frac{A-\langle A\rangle_q}{\Sigma_q(A)} = \frac{A-\kappa q}{\sqrt{\kappa(1-q^2)}}\,.$$

We will study

$$\left\langle \left(\frac{A-\kappa q}{\sqrt{\kappa(1-q^2)}}\right)^m \right\rangle_q, \qquad m=1,2,\dots.$$

9.3 Asymptotic Spectral Distribution

Having already chosen an origin o of $T\kappa$, we have the natural stratification and the quantum decomposition of $A = A^+ + A^-$ ($A^\circ = 0$ for a tree). Accordingly, the normalized adjacency matrix is decomposed into three parts:

$$\frac{A-\kappa q}{\sqrt{\kappa(1-q^2)}} = \frac{A^+}{\sqrt{\kappa(1-q^2)}} + \frac{A^-}{\sqrt{\kappa(1-q^2)}} + \frac{-\kappa q}{\sqrt{\kappa(1-q^2)}}\,.$$

For simplicity we introduce $C^\epsilon = C^\epsilon(\kappa, q)$ by

$$C^+ = \frac{A^+}{\sqrt{\kappa(1-q^2)}}\,, \qquad C^- = \frac{A^-}{\sqrt{\kappa(1-q^2)}} \qquad C^\circ = \frac{-\kappa q}{\sqrt{\kappa(1-q^2)}}\,. \tag{9.7}$$

In view of the actios of $A^\pm$ on $\Gamma(T_\kappa)$ given in (8.2) and (8.3), we have

$$C^+\Phi_0 = \frac{1}{\sqrt{1-q^2}}\,\Phi_1, \qquad C^+\Phi_n = \sqrt{\frac{\kappa-1}{\kappa(1-q^2)}}\,\Phi_{n+1} \quad (n\ge 1)$$

$$C^-\Phi_0 = 0, \qquad C^-\Phi_1 = \frac{1}{\sqrt{1-q^2}}\,\Phi_1, \qquad C^-\Phi_n = \sqrt{\frac{\kappa-1}{\kappa(1-q^2)}}\,\Phi_{n-1} \quad (n\ge 2)$$

$$C^\circ\Phi_n = -\sqrt{\frac{q^2\kappa}{1-q^2}}\,\Phi_n \quad (n\ge 0)$$

We are interested in the asymptotics as $\kappa \to \infty$ (the growing trees) so we need to take a suitable balance with q. The reasonable scaling is as follows:

$$\kappa \to \infty, \qquad q\sqrt{\kappa} \to \gamma, \qquad q \to 0, \tag{9.8}$$

where $\gamma \in \mathbf{R}$ is a constant. Under this scaling limit the limit actions of C^ϵ are rather apparent. In particular, in view of the actions of $C\pm$, we expect that the limit is described in terms of the free Fock space.

We need to discuss the mixed moments:

$$\langle C^{\epsilon_m} \cdots C^{\epsilon_1} \rangle_q = \langle Q\Phi_0, C^{\epsilon_m} \cdots C^{\epsilon_1} \Phi_0 \rangle,$$

where the limit actions of $C^{\epsilon_m}, \dots, C^{\epsilon_1}$ are redy observed. Consider the vector $Q\Phi_0$. By definition

$$\begin{aligned} Q\Phi_0 &= \sum_{x \in V} \langle \delta_x, Q\Phi_0 \rangle \delta_x = \sum_{x \in V} (Q)_{xo} \delta_x \\ &= \sum_{x \in V} q^{\partial(x,o)} \delta_x = \sum_{n=0}^{\infty} \sum_{x \in V_n} q^n \delta_x \\ &= \sum_{n=0}^{\infty} q^n |V_n|^{1/2} \Phi_n \end{aligned}$$

Since $|V_n| \sim \kappa^n$ (see (8.1)), under the scaling limit as in (9.8) the coefficient converges:

$$q^n |V_n|^{1/2} \to \gamma^n.$$

Let $(\Gamma_{\text{free}}, \{\Psi_n\}, B^+, B^-)$ be the free Fock space. For $z \in \mathbf{C}$ define

$$\Omega_z = \sum_{n=0}^{\infty} z^n \Psi_n \,. \tag{9.9}$$

This is a formal sum but makes a sense as a linear functional on the $*$-algebra $\mathcal{A}_{\text{free}}$ generated by B^+, B^- and diagonal operators. Namely, for $a \in \mathcal{A}_{\text{free}}$,

$$\langle \Omega_z, a\Psi_0 \rangle = \sum_{n=0}^{\infty} \bar{z}^n \langle \Psi_n, a\Phi_0 \rangle$$

is a finite sum and

$$a \mapsto \langle \Omega_z, a\Psi_0 \rangle$$

is a linear functional on $\mathcal{A}_{\text{free}}$. We call Ω_z is a *coherent vector*.

Remark 9.3.1 (1) The infinite series (9.9) converges in norm for $|z| < 1$.

(2) Ω_z is an eigenvector of B^-, i.e., $B^-\Omega_z = z\Omega_z$. More precisely, $\langle \Omega_z, B^+\Psi_n \rangle = \langle z\Omega_z, \Psi_n \rangle$ for n.

Theorem 9.3.2 (Quantum Central Limit Theorem) *Let $A = A_\kappa$ be the adjacency matrix of T_κ and define $C^\epsilon = C^\epsilon(\kappa, q)$ as in (9.7). Let $(\Gamma_{\text{free}}, \{\Psi_n\}, B^+, B^-)$ be the free Fock space and set $B^\circ = -\gamma I$ (scalar operator). Then*

$$\lim \langle C^{\epsilon_m} \cdots C^{\epsilon_1} \rangle_q = \langle \Omega_\gamma, B^{\epsilon_m} \cdots B^{\epsilon_1} \Psi_0 \rangle_{\text{free}},$$

where the limit is taken as $\kappa \to \infty$, $q \to 0$ with $q\sqrt{\kappa} \to \gamma \in \mathbf{R}$ (constant).

Proof. The proof is already cer by the bove argument. ∎

Corollary 9.3.3 *For the normalized adjacency matrix of T_κ we have*

$$\lim \left\langle \left(\frac{A - \kappa q}{\sqrt{\kappa(1-q^2)}} \right)^m \right\rangle_q = \langle \Omega_\gamma, (B^+ + B^- - \gamma I)^m \Psi_0 \rangle_{\text{free}}, \qquad m = 1, 2, \dots .$$

9.4 Free Poisson Distributions

In this section we meet one of the most basic result on the free Fock space. Let P be the vacuum projection, i.e.,

$$P\Psi_0 = \Psi_0, \qquad P\Psi_n = 0 \quad (n \ge 1).$$

Note that $B^+B^- = I - P$.

Lemma 9.4.1 *For $z \in \mathbf{C}$ and $m = 1, 2, \dots$ we have:*

$$\langle \Omega_{\bar z}, (B^+ + B^-)^m \Psi_0 \rangle = \langle \Psi_0, (B^+ + B^- + zP)^m \Psi_0 \rangle, \tag{9.10}$$

$$\langle \Omega_{\bar z}, (B^+ + B^- - z)^m \Psi_0 \rangle = \langle \Phi_0, (B^+ + B^- - zB^+B^-)^m \Psi_0 \rangle, \tag{9.11}$$

where $\Omega_{\bar z}$ is the coherent vector with parameter $\bar z$.

Proof. (9.11) follows from (9.10). In fact,

$$\begin{aligned}
&\langle \Omega_{\bar z}, (B^+ + B^- - z)^m \Psi_0 \rangle \\
&\quad = \sum_{n=0}^{m} \binom{m}{n} (-z)^{m-n} \langle \Omega_{\bar z}, (B^+ + B^-)^n \Psi_0 \rangle \\
&\quad = \sum_{n=0}^{m} \binom{m}{n} (-z)^{m-n} \langle \Psi_0, (B^+ + B^- + zP)^n \Psi_0 \rangle \\
&\quad = \langle \Psi_0, (B^+ + B^- + zP - z)^m \Psi_0 \rangle.
\end{aligned}$$

Since $B^+B^- = 1 - P$, the last expression becomes

$$= \langle \Psi_0, (B^+ + B^- - zB^+B^-)^m \Psi_0 \rangle,$$

which proves (9.11). The proof of (9.10) is left to the reader. ∎

In particular, for any $\gamma \in \mathbf{R}$ there exists a probability measure μ_γ such that

$$\langle \Omega_\gamma, (B^+ + B^- - \gamma)^m \Psi_0 \rangle = \langle \Psi_0, (B^+ + B^- - \gamma B^+ B^-)^m \Psi_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu_\gamma(dx)$$

for $m = 1, 2, \dots$. In fact, the Jacobi coefficients of μ_γ is given by

$$\omega_1 = \omega_2 = \cdots = 1, \qquad \alpha_1 = 0, \quad \alpha_2 = \alpha_3 = \cdots = -\gamma. \tag{9.12}$$

Then, Corollary 9.3.3 yields the following

Theorem 9.4.2 (CLT) *For the normalized adjacency matrix of T_κ we have*

$$\lim \left\langle \left(\frac{A - \kappa q}{\sqrt{\kappa(1-q^2)}} \right)^m \right\rangle_q = \int_{-\infty}^{+\infty} x^m \mu_\gamma(dx), \qquad m = 1, 2, \dots,$$

where μ_γ is uniquely determined by the Jacobi coefficients given by (9.12).

We are now in a good position to give the following

Definition 9.4.3 Let $(\Gamma_{\text{free}}, \{\Psi_n\}, B^+, B^-)$ be the free Fock space and $\lambda > 0$ a constant. The vacuum spectral distribution of $(B^+ + \sqrt{\lambda})(B^- + \sqrt{\lambda})$ is called the *free Poisson distribution* or *Marchenko–Pastur distribution* with parameter λ. In other words, the free Poisson distribution with parameter λ is a probability measure ν_λ uniquely specified by

$$\langle \Psi_0, ((B^+ + \sqrt{\lambda})(B^- + \sqrt{\lambda}))^m \Psi_0 \rangle = \int_{-\infty}^{+\infty} x^m \nu_\lambda(dx), \quad m = 1, 2, \dots. \tag{9.13}$$

Lemma 9.4.4 (1) mean $(\nu_\lambda) =$ var $(\nu_\lambda) = \lambda$.

(2) *The Jacobi coefficients of ν_λ are given by*

$$\omega_1 = \omega_2 = \cdots = \lambda, \qquad \alpha_1 = \lambda, \quad \alpha_2 = \alpha_3 = \cdots = \lambda + 1. \tag{9.14}$$

PROOF. (1) follows from (2) since mean $(\nu_\lambda) = \alpha 1$ and var $(\nu_\lambda) = \omega 1$.
(2) Note that

$$(B^+ + \sqrt{\lambda})(B^- + \sqrt{\lambda}) = \sqrt{\lambda}\, B^+ + \sqrt{\lambda}\, B^- + (\lambda + B^+ B^-).$$

Since

$$\sqrt{\lambda}\, B^+ \Phi_n = \sqrt{\lambda}\, \Phi_{n+1}, \qquad n \geq 0,$$

we obtain $\omega_1 = \omega_2 = \cdots = \lambda$. Similarly, from

$$(\lambda + B^+ B^-)\Phi_0 = \lambda \Phi_0, \qquad (\lambda + B^+ B^-)\Phi_n = (\lambda + 1)\Phi_n \quad (n \geq 1)$$

we see that $\alpha_1 = \lambda$ and $\alpha_2 = \alpha_3 = \cdots = \lambda + 1$. ∎

Comparing (9.12) and (9.14), we claim the following

Theorem 9.4.5 *For $\gamma \neq 0$, μ_γ is obtained from the free Poisson distribution ν_{1/γ^2} with parameter $1/\gamma^2$ by reflection and normalization. For $\gamma = 0$, μ_γ is the Wigner semicircle law.*

PROOF. Use Exercise 3. ∎

Remark 9.4.6 The density function of the free Poisson distribution is given explicitly. For $\lambda > 0$ we set

$$\rho_\lambda(x) = \begin{cases} \dfrac{\sqrt{4\lambda - (x-1-\lambda)^2}}{2\pi x}, & (1-\sqrt{\lambda})^2 \le x \le (1+\sqrt{\lambda})^2, \\ 0, & \text{otherwise.} \end{cases}$$

The free Poisson distribution with parameter λ is given by

$$\begin{cases} (1-\lambda)\delta_0 + \rho_\lambda(x)dx, & 0 < \lambda < 1, \\ \rho_\lambda(x)dx, & \lambda \ge 1. \end{cases}$$

9.5 Exercises

1. Show that $\mathrm{var}\,(\mu_{\kappa,q}) = \kappa(1-q^2)$. [Lemma 9.2.1]
2. Show that

$$\langle \Omega_{\bar{z}}, (B^+ + B^-)^m \Psi_0 \rangle = \langle \Psi_0, (B^+ + B^- + zP)^m \Psi_0 \rangle, \quad z \in \mathbf{C}, \quad m = 1, 2, \dots .$$

[Lemma 9.4.1 (1)]

3. Let μ be a probability distribution and $(\{\omega_n\}, \{\alpha_n\})$ the Jacobi coefficients. Show the following:

(1) The Jacobi coefficients of the translated $\mu(dx - s)$ are given by $(\{\omega_n\}, \{\alpha_n + s\})$, $s \in \mathbf{R}$.

(2) The Jacobi coefficients of the scaled $\mu(\lambda^{-1}dx)$ are given by $(\{\lambda^2\omega_n\}, \{\lambda\alpha_n\})$, $\lambda \in \mathbf{R}$, $\lambda \neq 0$.

(3) In particular, the Jacobi coefficients of the reflected $\mu(-dx)$ are given by $(\{\omega_n\}, \{-\alpha_n\})$.

10 Stieltjes Transform and Continued Fraction

10.1 Overview

Let $\mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ be the set of probability measures on $\mathbf{R}$ having finite moments of all orders. With each $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ we associate the *moment sequence* $\{M_0(\mu) = 1, M_1(\mu), M_2(\mu), \dots\}$ defined by

$$M_m(\mu) = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots. \tag{10.1}$$

By the Gram-Schmidt orthogonalization we consruct the orthogonal polynomials $\{P_0(x) = 1, P_1(x), \dots, P_n(x) = x^n + \cdots\}$ from which we obtain the Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$.

For an infinite sequence of real numbers $\{M_0 = 1, M_1, M_2, \dots\}$ we define the *Hankel determinants* by

$$\Delta_m = \det \begin{bmatrix} M_0 & M_1 & \dots & M_m \\ M_1 & M_2 & \dots & M_{m+1} \\ \vdots & \vdots & & \vdots \\ M_m & M_{m+1} & \dots & M_{2m} \end{bmatrix}, \qquad m = 0, 1, 2, \dots. \tag{10.2}$$

Let $\mathfrak{M}$ be the set of infinite sequences of real numbers $\{M_0 = 1, M_1, M_2, \dots\}$ satisfying one of the following two conditions:

(i) [infinite type] $\Delta_m > 0$ for all $m = 0, 1, 2, \dots$;

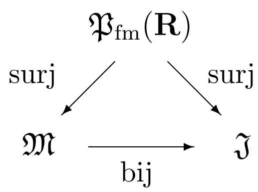
(ii) [finite type] there exists $m_0 \geq 1$ such that $\Delta_0 > 0, \Delta_1 > 0, \dots, \Delta_{m_0 - 1} > 0$ and $\Delta_{m_0} = \Delta_{m_0+1} = \cdots = 0$.

Let $\mathfrak{J}$ be the set of pairs of sequences $(\{\omega_n\}, \{\alpha_n\})$ satisfying one of the following conditions:

(i) [infinite type] $\{\omega_n\}$ is a Jacobi sequence of infinite type and $\{\alpha_n\}$ is an infinite sequence of real numbers;

(ii) [finite type] $\{\omega_n\}$ is a Jacobi sequence of finite type and $\{\alpha_n\}$ is a finite real sequence $\{\alpha_1, \dots, \alpha_{m_0}\}$, where $m_0 \geq 1$ is the smallest number such that $\omega_{m_0} = 0$.

We have the following diagram:



We shall discuss how to recover $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ from $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$ when the uniqueness holds.

10.2 Finite Jacobi Matrices

Let $(\{\omega_n\},\{\alpha_n\}) \in \mathfrak{J}$ and set

$$T = T_n = \begin{bmatrix} \alpha_1 & \sqrt{\omega_1} & & & & & \\ \sqrt{\omega_1} & \alpha_2 & \sqrt{\omega_2} & & & & \\ & \sqrt{\omega_2} & \alpha_3 & \sqrt{\omega_3} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \sqrt{\omega_{n-2}} & \alpha_{n-1} & \sqrt{\omega_{n-1}} \\ & & & & & \sqrt{\omega_{n-1}} & \alpha_n \end{bmatrix}, \tag{10.3}$$

whenever $\omega_{n-1} > 0$. A matrix of the form (10.3) is called a *Jacobi matrix (of finite type)*.

We set

$$e_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Proposition 10.2.1

$$\langle e_0, (z-T)^{-1}e_0\rangle = \frac{1}{z-\alpha_1} \, \frac{\omega_1}{-\, z-\alpha_2} \, \frac{\omega_2}{-\, z-\alpha_3} \, - \cdots - \, \frac{\omega_{n-1}}{z-\alpha_n}. \tag{10.4}$$

Proof. We set

$$(z-T)^{-1}e_0 = f = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix}.$$

First note that

$$\langle e_0, (z-T)^{-1}e_0\rangle = \langle e_0, f\rangle = f_0\,.$$

On the other hand, we see from $(z-T)f = e_0$ that

$$\begin{cases} (z-\alpha_1)f_0 - \sqrt{\omega_1}\, f_1 = 1, \\ -\sqrt{\omega_i}\, f_{i-1} + (z-\alpha_{i+1})f_i - \sqrt{\omega_{i+1}}\, f_{i+1} = 0, \qquad i = 1,2,\ldots,n-2, \\ -\sqrt{\omega_{n-1}}\, f_{n-2} + (z-\alpha_n)f_{n-1} = 0. \end{cases} \tag{10.5}$$

From the first relation in (10.5) we obtain

$$f_0 \left\{ (z-\alpha_1) - \sqrt{\omega_1}\, \frac{f_1}{f_0} \right\} = 1,$$

and hence

$$f_0 = \cfrac{1}{z - \alpha_1 - \sqrt{\omega_1}\,\dfrac{f_1}{f_0}}\,. \tag{10.6}$$

Similarly, from (10.5) we obtain

$$-\sqrt{\omega_i}\, f_{i-1} + f_i \left\{ (z - \alpha_{i+1}) - \sqrt{\omega_{i+1}}\, \frac{f_{i+1}}{f_i} \right\} = 0\,,$$

and therefore

$$\sqrt{\omega_i}\, \frac{f_i}{f_{i-1}} = \cfrac{\omega_i}{z - \alpha_{i+1} - \sqrt{\omega_{i+1}}\, \dfrac{f_{i+1}}{f_i}}\,. \tag{10.7}$$

Finally, from (10.5) we have

$$\sqrt{\omega_{n-1}}\, \frac{f_{n-1}}{f_{n-2}} = \frac{\omega_{n-1}}{z - \alpha_n}\,. \tag{10.8}$$

Combining (10.6)–(10.8), we come to

$$f_0 = \frac{1}{z-\alpha_1} \mathbin{-} \frac{\omega_1}{z-\alpha_2} \mathbin{-} \frac{\omega_2}{z-\alpha_3} \mathbin{-} \cdots \mathbin{-} \frac{\omega_{n-1}}{z-\alpha_n}\,,$$

from which (10.4) follows. ∎

Proposition 10.2.2 *For $k = 1, 2, \dots, n$ we define monic polynomials $P_k(z) = z^k + \cdots$ and $Q_{k-1}(z) = z^{k-1} + \cdots$ by*

$$\frac{1}{z-\alpha_1} \mathbin{-} \frac{\omega_1}{z-\alpha_2} \mathbin{-} \frac{\omega_2}{z-\alpha_3} \mathbin{-} \cdots \mathbin{-} \frac{\omega_{k-1}}{z-\alpha_k} = \frac{Q_{k-1}(z)}{P_k(z)}\,. \tag{10.9}$$

Then, the following recurrence relations are satisfied:

$$\begin{cases} P_0(z) = 1, \quad P_1(z) = z - \alpha_1\,, \\ P_k(z) = (z - \alpha_k) P_{k-1}(z) - \omega_{k-1} P_{k-2}(z), \quad k = 2, 3, \dots, n, \end{cases} \tag{10.10}$$

$$\begin{cases} Q_0(z) = 1, \quad Q_1(z) = z - \alpha_2\,, \\ Q_k(z) = (z - \alpha_{k+1}) Q_{k-1}(z) - \omega_k Q_{k-2}(z), \quad k = 2, 3, \dots, n-1. \end{cases} \tag{10.11}$$

PROOF. By induction, see also Exercise 1. ∎

Proposition 10.2.3 (Determinantal formula) *For $k = 1, 2, \dots, n$ it holds that*

$$P_k(z) = \det \begin{bmatrix} z-\alpha_1 & -\sqrt{\omega_1} & & & & \\ -\sqrt{\omega_1} & z-\alpha_2 & -\sqrt{\omega_2} & & & \\ & -\sqrt{\omega_2} & z-\alpha_3 & -\sqrt{\omega_3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\sqrt{\omega_{k-2}} & z-\alpha_{k-1} & -\sqrt{\omega_{k-1}} \\ & & & & -\sqrt{\omega_{k-1}} & z-\alpha_k \end{bmatrix} = \det(z - T_k).$$

For $k = 2, 3, \dots, n$ it holds that

$$Q_{k-1}(z) = \det \begin{bmatrix} z-\alpha_2 & -\sqrt{\omega_2} & & & \\ -\sqrt{\omega_2} & z-\alpha_3 & -\sqrt{\omega_3} & & \\ & \ddots & \ddots & \ddots & \\ & & -\sqrt{\omega_{k-2}} & z-\alpha_{k-1} & -\sqrt{\omega_{k-1}} \\ & & & -\sqrt{\omega_{k-1}} & z-\alpha_k \end{bmatrix}.$$

Proof. By expanding the determinants in the last column one can check easily that these determinants satisfy the recurrence relations in (10.10) and (10.11). ∎

We now need spectral properties of the Jacobi matrix T.

Proposition 10.2.4 *Every eigenvalue of $T = T_n$ is real and simple. Moreover,*

$$\operatorname{Spec} T_n = \{\lambda \in \mathbf{C}\,;\, P_n(\lambda) = 0\}. \tag{10.12}$$

Proof. Since T is an $n \times n$ real symmetric matrix, it has n real eigenvalues. (10.12) is obvious from $\det(z - T_n) = P_n(z)$, see Proposition 10.2.3.

We prove that every eigenspace of T is of one dimension. Let λ be an eigenvalue of T and f a correswponding eigenvector. We write

$$f = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix}.$$

Then $(\lambda - T)f = 0$ is equivalent to the following

$$\begin{cases} (\lambda-\alpha_1)f_0 - \sqrt{\omega_1}\, f_1 = 0, \\ -\sqrt{\omega_i}\, f_{i-1} + (\lambda-\alpha_{i+1})f_i - \sqrt{\omega_{i+1}}\, f_{i+1} = 0, \quad i = 1, 2, \dots, n-2, \\ -\sqrt{\omega_{n-1}}\, f_{n-2} + (\lambda - \alpha_n) f_{n-1} = 0. \end{cases} \tag{10.13}$$

Now let h, g be two eigenvectors corresponding to λ. Choose $(\alpha, \beta) \in \mathbf{R}^2$, $(\alpha, \beta) \neq (0, 0)$, such that $\alpha g_0 + \beta h_0 = 0$. Since $f = \alpha g + \beta h$ satisfies $(\lambda - T)f = 0$, we have (10.13). Note that $f_0 = 0$. Then, succesive application of (10.13) implies $f_1 = \cdots = f_{n-1} = 0$. Thus we have $f = 0$, which means that g and h are linearly dependent. Consequently, the eigenspace corresponding to λ is of one dimension. ∎

Proposition 10.2.5 *For $\lambda \in \operatorname{Spec} T$ we put*

$$f(\lambda) = \begin{bmatrix} P_0(\lambda) \\ P_1(\lambda)/\sqrt{\omega_1} \\ \vdots \\ P_{n-1}(\lambda)/\sqrt{\omega_{n-1}\cdots\omega_1} \end{bmatrix}. \tag{10.14}$$

Then $f(\lambda) \neq 0$ and $Tf(\lambda) = \lambda f(\lambda)$. Namely, $f(\lambda)$ is an eigenvector associated with λ.

PROOF. $f(\lambda) \neq 0$ is obvious since $P_0(\lambda) = 1$. In view of (10.10) we obtain

$$
\begin{aligned}
&P_0(\lambda) = 1, \\
&P_1(\lambda) = \lambda - \alpha_1\,, \\
&P_k(\lambda) = (\lambda - \alpha_k)P_{k-1}(\lambda) - \omega_{k-1}P_{k-2}(\lambda), \quad k = 2, 3, \dots, n-1, \\
&0 = (\lambda - \alpha_n)P_{n-1}(\lambda) - \omega_{n-1}P_{n-2}(\lambda).
\end{aligned}
$$

The last identity comes from $P_n(\lambda) = \det(\lambda - T) = 0$. Then a simple computation yields

$$
\begin{aligned}
&\sqrt{\omega_1}\,\frac{P_1(\lambda)}{\sqrt{\omega_1}} = \lambda - \alpha_1 = (\lambda - \alpha_1)P_0(\lambda), \\
&\sqrt{\omega_k}\,\frac{P_k(\lambda)}{\sqrt{\omega_k \cdots \omega_1}} = (\lambda - \alpha_k)\frac{P_{k-1}(\lambda)}{\sqrt{\omega_{k-1} \cdots \omega_1}} - \sqrt{\omega_{k-1}}\,\frac{P_{k-2}(\lambda)}{\sqrt{\omega_{k-2} \cdots \omega_1}}\,,
\end{aligned}
$$

for $k = 2, 3, \dots, n-1$, and

$$
0 = (\lambda - \alpha_n)\frac{P_{n-1}(\lambda)}{\sqrt{\omega_{n-1} \cdots \omega_1}} - \sqrt{\omega_{n-1}}\,\frac{P_{n-2}(\lambda)}{\sqrt{\omega_{n-2} \cdots \omega_1}}\,.
$$

The above relations are combined into a single identity: $(\lambda - T)f(\lambda) = 0$. ∎

Proposition 10.2.6 *Define a measure μ on $\mathbf{R}$ by*

$$
\mu = \sum_{\lambda \in \operatorname{Spec} T} \|f(\lambda)\|^{-2}\delta_\lambda\,, \tag{10.15}
$$

where $f(\lambda) \in \mathbf{R}^n$ is given by (10.14). *Then, $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ and*

$$
\langle e_0, (z - T)^{-1}e_0 \rangle = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z - x}\,. \tag{10.16}
$$

PROOF. Since every eigenvalue of T is simple (Proposition 10.2.4), we see from Proposition 10.2.5 that $\{\|f(\lambda)\|^{-1}f(\lambda)\,;\, \lambda \in \operatorname{Spec} T\}$ becomes a complete orthonormal basis of $\mathbf{C}^n$. Hence

$$
\begin{aligned}
\langle e_0, (z - T)^{-1}e_0 \rangle &= \sum_{\lambda \in \operatorname{Spec} T} \langle e_0, \|f(\lambda)\|^{-1}f(\lambda)\rangle \langle \|f(\lambda)\|^{-1}f(\lambda), (z - T)^{-1}e_0 \rangle \\
&= \sum_{\lambda \in \operatorname{Spec} T} \|f(\lambda)\|^{-2}\langle e_0, f(\lambda)\rangle \langle (\bar{z} - T)^{-1}f(\lambda), e_0 \rangle \\
&= \sum_{\lambda \in \operatorname{Spec} T} \|f(\lambda)\|^{-2}(z - \lambda)^{-1}.
\end{aligned}
$$

where we used $\langle e_0, f(\lambda)\rangle = P_0(\lambda) = 1$ and $(\bar{z} - T)^{-1}f(\lambda) = (\bar{z} - \lambda)^{-1}f(\lambda)$. Then, in view of (10.15) we obtain

$$
\langle e_0, (z - T)^{-1}e_0 \rangle = \sum_{\lambda \in \operatorname{Spec} T} \frac{\|f(\lambda)\|^{-2}}{z - \lambda} = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z - x}\,,
$$

which proves (10.16).

We need to show that $\mu(\mathbf{R}) = 1$. This may be proved by observing asymptotics of both sides of (10.16). In fact, with the help of Propositions 10.2.1 and 10.2.2 we see that

$$\lim_{\substack{z\to\infty \\ \mathrm{Re}\, z=0}} z\langle e_0, (z-T)^{-1}e_0\rangle = \lim_{\substack{z\to\infty \\ \mathrm{Re}\, z=0}} \frac{zQ_{n-1}(z)}{P_n(z)} = 1, \tag{10.17}$$

where we applied the fact that both $zQ_{n-1}(z)$ and $P_n(z)$ are monic polynomials of degree n. On the other hand,

$$\lim_{\substack{z\to\infty \\ \mathrm{Re}\, z=0}} z\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \int_{-\infty}^{+\infty} \mu(dx) = \mu(\mathbf{R}) \tag{10.18}$$

by the dominated convergence theorem. We see from (10.17) and (10.18) that $\mu(\mathbf{R}) = 1$.

∎

Definition 10.2.7 For any probability measure μ (not necessarily having moments) the integral

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x}, \qquad \mathrm{Im}\, z \neq 0$$

converges and $G_\mu(z)$ becomes a holomorphic function in $\{\mathrm{Im}\, z \neq 0\} = \mathbf{C}\backslash\mathbf{R}$. We call $G_\mu(z)$ the *(Cauchy-) Stieltjes transform* of μ.

Theorem 10.2.8 *Let* $\alpha_1, \dots, \alpha_n \in \mathbf{R}$ *and* $\omega_1 > 0, \dots, \omega_{n-1} > 0$. *Then the polynomials* $P_0(z), P_1(z), \dots, P_{n-1}(z)$ *defined by the recurrence relation* (10.10) *are the orthogonal polynomials associated with* μ *defined in* (10.15). *Therefore, the Jacobi coefficients of* μ *is* $(\{\alpha_1, \dots, \alpha_n\}, \{\omega_1, \dots, \omega_{n-1}\})$. *Moreover, the Stieltjies transform* $G_\mu(z)$ *admits a continued fraction expansion:*

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \cdots - \frac{\omega_{n-1}}{z-\alpha_n}.$$

Proof. By using the recurrence formula (10.10) we may see easily that

$$P_0(T)e_0 = e_0\,, \quad P_k(T)e_0 = \sqrt{\omega_k \cdots \omega_1}\, e_k\,, \qquad k = 1, 2, \dots, n-1. \tag{10.19}$$

On the other hand, for any polynomials p, q with real coefficients we have

$$\begin{aligned}
\langle p(T)e_0, q(T)e_0\rangle &= \sum_{\lambda\in\mathrm{Spec}\, T} \langle p(T)e_0, \|f(\lambda)\|^{-1}f(\lambda)\rangle\langle \|f(\lambda)\|^{-1}f(\lambda), q(T)e_0\rangle \\
&= \sum_{\lambda\in\mathrm{Spec}\, T} \|f(\lambda)\|^{-2}\langle e_0, p(T)f(\lambda)\rangle\langle q(T)f(\lambda), e_0\rangle \\
&= \sum_{\lambda\in\mathrm{Spec}\, T} \|f(\lambda)\|^{-2}p(\lambda)q(\lambda)\langle e_0, f(\lambda)\rangle\langle f(\lambda), e_0\rangle \\
&= \sum_{\lambda\in\mathrm{Spec}\, T} \|f(\lambda)\|^{-2}p(\lambda)q(\lambda) \\
&= \int_{-\infty}^{+\infty} p(x)q(x)\mu(dx).
\end{aligned}$$

Hence, in particular,

$$\int_{-\infty}^{+\infty} P_j(x)P_k(x)\mu(dx) = \langle P_j(T)e_0, P_k(T)e_0\rangle = \omega_j \cdots \omega_1 \langle e_j, e_k\rangle$$

so that $P_0(z), P_1(z), \dots, P_{n-1}(z)$ are the orthogonal polynomials associated with μ. ∎

10.3 General Case

Let $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$ be of infinite type. Then for any n, defining a Jacobi matrix T_n as in (10.3), we obtain a probability measure μ_n and the polynomials $\{P_0(x), P_1(x), \dots, P_n(x)\}$ as in the previous section. Since these polynomials are defined by the recurrence relation with $(\{\omega_n\}, \{\alpha_n\})$, $\{P_0(x), P_1(x), \dots, P_n(x)\}$ are common for all μ_m for $m \ge n$. Consequently, given $(\{\omega_n\}, \{\alpha_n\})$, we have an infinite sequence of probability measures mu_n, and an infinite sequence polynomials

$$P_0(x) = 1, \quad P_1(x), \dots, P_n(x) = x^n + \cdots, \quad , \dots .$$

Lemma 10.3.1 *Let $\mu \in \mathfrak{P}_{\rm fm}(\mathbf{R})$ be a probability measure whose Jacobi coefficients are $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$. Then, for any $m = 1, 2, \dots$ we have*

$$\lim_{n\to\infty} M_m(\mu_n) = M_m(\mu).$$

Proof. In general, $M_m(\nu)$ is described by the first m terms of the Jacobi coefficients of ν. Suppose that $n \ge m$. Then we see that

$$M_m(\mu_n) = M_m(\mu_{n+1}) = \cdots = M_m(\mu),$$

from which the assertion is clear. ∎

Theorem 10.3.2 *Let $\mu \in \mathfrak{P}_{\rm fm}(\mathbf{R})$ be the solution of a determinate moment problem and $(\{\omega_n\}, \{\alpha_n\})$ be the Jacobi coefficients. Then the Stieltjies transform $G_\mu(z)$ admits a continued fraction expansion:*

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z-\alpha_1 -} \, \frac{\omega_1}{z-\alpha_2 -} \, \frac{\omega_2}{z-\alpha_3 - \cdots -} \, \frac{\omega_{n-1}}{z-\alpha_n} \,,$$

where the right-hand side converges in $\{\operatorname{Im} z \neq 0\}$.

Proof. By Theorem 10.2.8 we have

$$\int_{-\infty}^{+\infty} \frac{\mu_n(dx)}{z-x} = \frac{1}{z-\alpha_1 -} \, \frac{\omega_1}{z-\alpha_2 -} \, \frac{\omega_2}{z-\alpha_3 - \cdots -} \, \frac{\omega_{n-1}}{z-\alpha_n} \,.$$

On the other hand, it follows from Lemma 10.3.1 and the assumption that μ_n converges to μ weakly. Since $x \mapsto 1/(z-x)$ is a bounded continuous function on $\mathbf{R}$, we have

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \frac{\mu_n(dx)}{z-x} = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} \,.$$

This completes the proof. ∎

10.4 Exercises

1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers. Define a linear fractional transformations by

$$\tau_k(w) = \frac{a_k}{b_k + w}, \qquad w \in \mathbf{C} \cup \{\infty\}.$$

Define $\{A_n\}$ and $\{B_n\}$ respectively by the following recurrence relations:

$$\begin{cases} A_{-1} = 1, \quad A_0 = 0, \\ A_n = b_n A_{n-1} + a_n A_{n-2}\,, \qquad n = 1, 2, \dots, \end{cases}$$

$$\begin{cases} B_{-1} = 0, \quad B_0 = 1, \\ B_n = b_n B_{n-1} + a_n B_{n-2}\,, \qquad n = 1, 2, \dots \end{cases}$$

Prove that

(1) $\tau_1 \cdots \tau_n(w) = \dfrac{A_n + A_{n-1}w}{B_n + B_{n-1}w}$

(2) $\dfrac{a_1}{b_1} + \dfrac{a_2}{b_2} + \dfrac{a_3}{b_3} + \dots + \dfrac{a_n}{b_n} = \dfrac{A_n}{B_n}$

2. The *Kesten distribution* with parameters (p, q), $p > 0$, $q \ge 0$, is a probability distribution whose Jacobi coefficients are given by

$$\omega_1 = p, \quad \omega_2 = \omega_3 = \dots = q, \qquad \alpha_n \equiv 0.$$

Prove that the Stieltjes transform is given by

$$G(z) = \frac{1}{z} - \frac{p}{z} - \frac{q}{z} - \frac{q}{z} - \dots = -\frac{1}{2}\,\frac{(p-2q)z + p\sqrt{z^2 - 4q}}{p^2 - (p-q)z^2}\,.$$

11 Growing Regular Graphs

11.1 Actions of A^ϵ Revisited

Let $G = (V, E)$ be a graph of degree κ and $o \in V$ a fixed origin. Let

$$V = \bigcup_{n=0}^{\infty} V_n \,, \qquad A = A^+ + A^- + A^\circ$$

be the stratification and the quantum decomposition of the adjacency matrix A. For $n = 0, 1, 2, \dots$ we define a unit vector by

$$\Phi_n = \frac{1}{\sqrt{|V_n|}} \sum_{x \in V_n} \delta_x$$

and denote $\Gamma = \Gamma(G)$ by the linear space spanned by $\{\Phi_0, \Phi_1, \dots\}$.

It is important to study the actions of the quantum components A^ϵ on the vectors Φ_n. We need notation. For $x \in V$ and $\epsilon \in \{+, -, \circ\}$ we set

$$\omega_\epsilon(x) = |\{y \in V \,;\, y \sim x,\ \partial(o, y) = \partial(o, x) + \epsilon\}|.$$

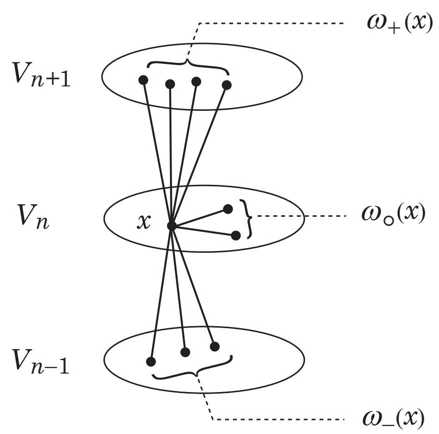


Figure 11.1: $\omega_\epsilon(x)$

By definition we have

$$\sqrt{|V_n|}\, A^+ \Phi_n = \sum_{x \in V_n} A^+ \delta_x = \sum_{y \in V_{n+1}} \omega_-(y) \delta_y \tag{11.1}$$

The mean value of $\omega_-(y)$, where y runs over V_{n+1}, is defined by

$$M(\omega_- | V_{n+1}) = \frac{1}{|V_{n+1}|} \sum_{y \in V_{n+1}} \omega_-(y).$$

With this (11.1) becomes

$$\begin{aligned}\sqrt{|V_n|}\, A^+\Phi_n &= \sum_{y\in V_{n+1}} M(\omega_-|V_{n+1})\delta_y + \sum_{y\in V_{n+1}} (\omega_-(y) - M(\omega_-|V_{n+1}))\delta_y \\ &= M(\omega_-|V_{n+1})\sqrt{|V_{n+1}|}\,\Phi_{n+1} + \sum_{y\in V_{n+1}} (\omega_-(y) - M(\omega_-|V_{n+1}))\delta_y\,.\end{aligned}$$

Hence,

$$A^+\Phi_n = M(\omega_-|V_{n+1})\sqrt{\frac{|V_{n+1}|}{|V_n|}}\,\Phi_{n+1} + E^+\Phi_n \tag{11.2}$$

where

$$E^+\Phi_n = \frac{1}{\sqrt{|V_n|}} \sum_{y\in V_{n+1}} (\omega_-(y) - M(\omega_-|V_{n+1}))\delta_y\,. \tag{11.3}$$

We regard the first term in the right-hand side of (11.2) as the principal part of the action of A^+ and the second the error part. The norm of the error part is given by

$$\begin{aligned}\|E^+\Phi_n\|^2 &= \left\| \frac{1}{\sqrt{|V_n|}} \sum_{y\in V_{n+1}} (\omega_-(y) - M(\omega_-|V_{n+1}))\delta_y \right\|^2 \\ &= \frac{1}{|V_n|} \sum_{y\in V_{n+1}} \{\omega_-(y) - M(\omega_-|V_{n+1})\}^2 \\ &= \frac{|V_{n+1}|}{|V_n|}\,\Sigma^2(\omega_-|V_{n+1}),\end{aligned} \tag{11.4}$$

where

$$\Sigma^2(\omega_-|V_{n+1}) = \frac{1}{|V_{n+1}|} \sum_{y\in V_{n+1}} \big\{\omega_-(y) - M(\omega_-|V_{n+1})\big\}^2$$

is the variance of $\omega_-(y)$, where $y \in V_{n+1}$.

In a similar manner, we have

$$A^-\Phi_n = M(\omega_+|V_{n-1})\sqrt{\frac{|V_{n-1}|}{|V_n|}}\,\Phi_{n-1} + E^-\Phi_n\,, \tag{11.5}$$

$$A^\circ\Phi_n = M(\omega_\circ|V_n)\Phi_n + E^\circ\Phi_n\,, \tag{11.6}$$

where the error terms are given by

$$E^-\Phi_n = \frac{1}{\sqrt{|V_n|}} \sum_{y\in V_{n-1}} (\omega_+(y) - M(\omega_+|V_{n-1}))\delta_y\,.$$

$$E^\circ\Phi_n = \frac{1}{\sqrt{|V_n|}} \sum_{y\in V_n} (\omega_\circ(y) - M(\omega_\circ|V_n))\delta_y\,.$$

The norms of these error terms are

$$\|E^{-}\Phi_n\|^2 = \frac{|V_{n-1}|}{|V_n|}\,\Sigma^2(\omega_+|V_{n-1}), \tag{11.7}$$

$$\|E^{\circ}\Phi_n\|^2 = \Sigma^2(\omega_\circ|V_n). \tag{11.8}$$

(11.2), (11.5) and (11.6) are unified into the following form:

$$A^{\epsilon}\Phi_n = M(\omega_{-\epsilon}|V_{n+\epsilon})\sqrt{\frac{|V_{n+\epsilon}|}{|V_n|}}\,\Phi_{n+\epsilon} + E^{\epsilon}\Phi_n\,, \qquad \epsilon\in\{+,-,\circ\}. \tag{11.9}$$

If all the error terms vanish, the actions of the quantum components A^ϵ gives rise to an interacting Fock space structure and the spectral distributions of A is obtained by means of quantum probabilistic techniques. Such a case happens typically for homogeneous trees and distance-regular graphs.

When we consider a growing graphs, we may expect the error terms vanish asymptotically. Slightly more precise, when we consider a growing graph $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$, for capturing the asymptotic properties of $A^{(\nu)}$ it is necessarily to take the normalization. Let us consider the mean and variance of $A = A^{(\nu)}$ in a state φ_ν:

$$\langle A\rangle_\nu = \varphi_\nu(A), \qquad \Sigma_\nu^2(A) = \varphi_\nu((A-\langle A\rangle_\nu)^2).$$

The normalized adjacency matrix is given by

$$\frac{A-\langle A\rangle_\nu}{\Sigma_\nu(A)} = \frac{A^+}{\Sigma_\nu(A)} + \frac{A^-}{\Sigma_\nu(A)} + \frac{A^\circ-\langle A\rangle_\nu}{\Sigma_\nu(A)}\,. \tag{11.10}$$

It follows from (11.9) that

$$\frac{A^\epsilon}{\Sigma_\nu(A)}\,\Phi_n = M(\omega_{-\epsilon}|V_{n+\epsilon})\frac{1}{\Sigma_\nu(A)}\sqrt{\frac{|V_{n+\epsilon}|}{|V_n|}}\,\Phi_{n+\epsilon} + \frac{1}{\Sigma_\nu(A)}\,E^\epsilon\Phi_n \tag{11.11}$$

and

$$\left\|\frac{1}{\Sigma_\nu(A)}\,E^\epsilon\Phi_n\right\|^2 = \frac{1}{\Sigma_\nu^2(A)}\,\frac{|V_{n+\epsilon}|}{|V_n|}\,\Sigma^2(\omega_{-\epsilon}|V_{n+\epsilon}). \tag{11.12}$$

If theses errors in (11.12) vanish in the limit, the asymptotiv actions of the quantum components in the right-hand side in (11.10) are described in terms of an interacting Fock space.

11.2 An Exampe: $\mathbb{Z}^N$ as $N\to\infty$

Let us consider the integer lattice $\mathbb{Z}^N$. Each $x\in\mathbb{Z}^N$ is expressible in the form:

$$x = (\xi_1,\xi_2,\dots,\xi_N) = \sum_{i=1}^N \xi_i e_i\,, \qquad \xi_i\in\mathbb{Z},$$

where $e_1, \dots, e_N$ are the standard basis. Taking $o = (0, 0, \dots, 0)$ to be the origin, we introduce the stratification and the quantum decomposition of the adjacency matrix:

$$\mathbb{Z}^N = \bigcup_{n=0}^{\infty} V_n \,, \qquad A = A^+ + A^-.$$

Here we note that $A^\circ = 0$ since there is no edge lying in a stratum V_n.

We consider the vacuum state at $o \in V$. Then,

$$\langle A \rangle = 0, \qquad \Sigma^2(A) = \langle A^2 \rangle = \deg(o) = 2N.$$

Let us check (11.11) and (11.12). It is clear that $\omega_\pm(y)$ is not necessarily constant on each V_n, but for a large N it is "almost" constant. In fact, for a large N, most generic vertices in V_n have the form

$$y = \pm e_{i_1} \pm e_{i_2} \pm \cdots \pm e_{i_n} \,, \quad 1 \le i_1 < i_2 < \cdots < i_n \le N, \tag{11.13}$$

namely, in the right hand side each e_i appears with multiplicity at most one. Note that

$$|V_n| = \binom{N}{n} 2^n + O(N^{n-1}) = \frac{(2N)^n}{n!} + O(N^{n-1}), \tag{11.14}$$

where the principal term corresponds to the number of generic vertices. For a generic $y \in V_n$ we have $\omega_-(y) = n$, since a vertex $x \in V_{n-1}$ which is adjacent to y is obtained by subtracting one e_{i_k} from the right hand side of (11.13). Hence,

$$M(\omega_-|V_n) = n, \qquad M(\omega_+|V_n) = 2N - n$$

and

$$\Sigma^2(\omega_-|V_n) = \frac{1}{|V_n|}(n-1)^2 O(N^{n-1}) = O(N^{-1}).$$

Moreover,

$$\sqrt{\frac{|V_{n+1}|}{|V_n|}} = \frac{\sqrt{2N}}{\sqrt{n+1}} + O(N^{-1/2})$$

Then, (11.11) and (11.12) become

$$\frac{A^+}{\sqrt{2N}} \Phi_n = \frac{n+1}{\sqrt{2N}} \left(\frac{\sqrt{2N}}{\sqrt{n+1}} + O(N^{-1/2}) \right) \Phi_{n+1} + \frac{1}{\sqrt{2N}} E^\epsilon \Phi_n \,,$$

$$\left\| \frac{1}{\sqrt{2N}} E^\epsilon \Phi_n \right\|^2 = \frac{1}{2N} \left(\frac{2N}{n+1} + O(1) \right) O(N^{-1}) = O(N^{-1}).$$

Consequently,

$$\frac{A^+}{\sqrt{2N}} \Phi_n = \sqrt{n+1}\, \Phi_{n+1} + O(N^{-1/2}). \tag{11.15}$$

In a similar manner, we obtain

$$\frac{A^-}{\sqrt{2N}}\,\Phi_n = \sqrt{n}\;\Phi_{n-1} + O(N^{-1}). \tag{11.16}$$

This means that $\Gamma(\mathbb{Z}^N)$ is asymptotically invariant under the (normalized) quantum components. It is immediately seen from (11.15) and (11.16), at a formal level at least, that the actions of normalized quantum components in the limit coincides with those of the annihilation and creation operators of the Boson Fock space $\Gamma_{\mathrm{Boson}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$, i.e.,

$$\lim_{N\to\infty} \frac{A_N^\pm}{\sqrt{2N}} = B^\pm.$$

Hence, still at a formal level, we obtain

$$\lim_{N\to\infty} \left\langle \left(\frac{A_N}{\sqrt{2N}}\right)^m \right\rangle = \langle \Psi_0, (B^+ + B^-)^m \Psi_0 \rangle, \quad m = 1, 2, \dots.$$

As is well known, the right hand side is the mth moment of the standard Gaussian distribution. Consequently,

$$\lim_{N\to\infty} \left\langle \left(\frac{A_N}{\sqrt{2N}}\right)^m \right\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^m e^{-x^2/2} dx, \quad m = 1, 2, \dots.$$

In other words, the asymptotic spectral distribution of the adjacency matrix of the integer lattice in the vacuum state is the standard Gaussian distribution.

11.3 QCLT for a Growing Regular Graph

We now consider a growing regular graph $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$, where each $G^{(\nu)}$ is given an origin o_ν. The degree is denoted by $\kappa = \kappa_\nu$. We are interested in the case where

(A1) $\lim_\nu \kappa(\nu) = \infty$.

We will concentrate on the vacuum states at $o \in V$. Then the normalized adjacency matrix is given by

$$\frac{A}{\sqrt{\kappa}} = \frac{A^+}{\sqrt{\kappa}} + \frac{A^-}{\sqrt{\kappa}} + \frac{A^\circ}{\sqrt{\kappa}}\,.$$

For notational simplicity we sometimes omit the suffix ν.

Statistics of $\omega_\epsilon(x)$ will play a crucial role. Whenever $V_n \neq \varnothing$, we define

$$M(\omega_\epsilon|V_n) = \frac{1}{|V_n|} \sum_{x\in V_n} \omega_\epsilon(x),$$

$$\Sigma^2(\omega_\epsilon|V_n) = \frac{1}{|V_n|} \sum_{x\in V_n} \big\{\omega_\epsilon(x) - M(\omega_\epsilon|V_n)\big\}^2,$$

$$L(\omega_\epsilon|V_n) = \max\{\omega_\epsilon(x)\,;\, x \in V_n\}.$$

Namely, $M(\omega_\epsilon|V_n)$ is the mean value of $\omega_\epsilon(x)$ when x runs over V_n, and $\Sigma^2(\omega_\epsilon|V_n)$ its variance. Both $\Sigma^2(\omega_\epsilon|V_n)$ and $L(\omega_\epsilon|V_n)$ indicate fluctuation of $\omega_\epsilon(x)$.

We now claim the essential conditions for our QCT. These conditions are found through detailed observation of the argument in the previous section.

(A2) For each $n = 1, 2, \dots$ there exists a limit

$$\omega_n = \lim_\nu M(\omega_-|V_n^{(\nu)}) < \infty. \tag{11.17}$$

Moreover,

$$\lim_\nu \Sigma^2(\omega_-|V_n^{(\nu)}) = 0, \tag{11.18}$$

$$W_n \equiv \sup_\nu L(\omega_-|V_n^{(\nu)}) < \infty. \tag{11.19}$$

(A3) For each $n = 0, 1, 2, \dots$ there exists a limit

$$\alpha_{n+1} = \lim_\nu M\left(\frac{\omega_\circ}{\sqrt{\kappa(\nu)}} \,\middle|\, V_n^{(\nu)}\right) = \lim_\nu \frac{M(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty. \tag{11.20}$$

Moreover,

$$\lim_\nu \Sigma^2\left(\frac{\omega_\circ}{\sqrt{\kappa(\nu)}} \,\middle|\, V_n^{(\nu)}\right) = \lim_\nu \frac{\Sigma^2(\omega_\circ|V_n^{(\nu)})}{\kappa(\nu)} = 0, \tag{11.21}$$

$$\sup_\nu \frac{L(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty. \tag{11.22}$$

Remark 11.3.1 Condition (A2) for $n = 1$ and (A3) for $n = 0$ are automatically satisfied. Note also that $\omega_1 = 1$ and $\alpha_1 = 0$.

Remark 11.3.2 If $G^{(\nu)}$ happens to be a finite graph, $V_n^{(\nu)} = \varnothing$ occurs for some n. Then $M(\omega_\epsilon|V_n^{(\nu)})$ is defined only up to a certain n. This causes, however, no difficulty for defining ω_n and α_n for all n (Exercise 2)

The meaning of (A1) is clear. Condition (A2) means that, in each stratum most of the vertices have the same number of downward edges, and as the graph grows the fluctuation of that number tends to zero. Condition (A3) is for edges lying in each stratum. The number of such edges may increase as the graph grows, but the growth rate is bounded by $\kappa(\nu)^{1/2}$. We roughly see from conditions (A1), (11.19) and (11.22) that, for a "generic" vertex $x \in V_n^{(\nu)}$,

$$\omega_+(x) = O(\kappa(\nu)), \qquad \omega_\circ(x) = O(\kappa(\nu)^{1/2}), \qquad \omega_-(x) = O(1),$$

as the graph grows.

Theorem 11.3.3 (QCLT) *Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph satisfying conditions (A1)–(A3) and A_ν its adjacency matrix. Let $(\Gamma, \{\Psi_n\}, B^+, B^-)$ be the interacting Fock space associated with $\{\omega_n\}$ and B° the diagonal operator defined by $\{\alpha_n\}$, where $\{\omega_n\}$ and $\{\alpha_n\}$ are given in conditions (A1)–(A3). Then we have*

$$\lim_\nu \left\langle \Phi_j^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle = \langle \Psi_j, B^{\epsilon_m} \cdots B^{\epsilon_1} \Psi_n \rangle, \tag{11.23}$$

for any $\epsilon_1, \dots, \epsilon_m \in \{+, -, \circ\}$, $m = 1, 2, \dots,$ and $j, n = 0, 1, 2, \dots$.

The proof owes to precise estimates the error terms in terms of the variances $\Sigma_\nu^2(\omega_\epsilon | V_n)$. The proof is omitted, instead we only record a noticeable result which is used in the proof.

Lemma 11.3.4 *If a growing regular graph $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ satisfies conditions (A1)–(A3), we have*

$$\lim_\nu \frac{|V_n^{(\nu)}|}{\kappa(\nu)^n} = \frac{1}{\omega_n \cdots \omega_1}, \qquad n = 1, 2, \dots. \tag{11.24}$$

11.4 Coxeter Graphs

The Coxeter groups provide interesting examples of growing Cayley graphs. Let Σ be a countable infinite set. A function $m : \Sigma \times \Sigma \to \{1, 2, \dots\} \cup \{\infty\}$ is called a *Coxeter matrix* if (i) $m(s, s) = 1$ for all $s \in \Sigma$, and (ii) $m(s, t) = m(t, s) \geq 2$ for $s \neq t$. Let $\Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma$ be an increasing sequence of subsets of Σ such that $|\Sigma_N| = N$ and $\Sigma = \bigcup \Sigma_N$. For each $N \geq 1$ let G_N be the group generated by Σ_N subject only to the relations:

$$(st)^{m(s,t)} = e, \qquad s, t \in \Sigma_N, \tag{11.25}$$

where e stands for the unit. In case of $m(s, t) = \infty$ we understand that st is of infinite order. The pair (G_N, Σ_N) is called a *Coxeter system* of rank N (i.e., $|\Sigma_N| = N$), and G_N is called a *Coxeter group*. It is known that each $s \in \Sigma_N$ has order two, namely, is not reduced to the unit (this is not very trivial). The corresponding Cayley graph is denoted by the same symbol (G_N, Σ_N). We consider the family of Cayley graphs (G_N, Σ_N), $N = 1, 2, \dots,$ as a growing regular graph.

In fact, it is shown that the inclusion $\Sigma_N \to \Sigma_{N+1}$ extends uniquely an injective homomorphism $G_N \to G_{N+1}$. The inductive limit group, denoted simply by G, is called the *infinite Coxeter group* associated with a Coxeter matrix $\{m(s, t)\}$.

By definition each $g \in G_N$, $g \neq e$, admits an expression of the form

$$x = s_1 s_2 \cdots s_r, \qquad s_i \in \Sigma_N.$$

If r is as small as possible, the expression is called a *reduced expression* and the number $r = |x|$ is called the *length* of x. The length function is well defined on G.

Lemma 11.4.1 *For any $s \in \Sigma$ and $x \in G$ we have*

$$|sx| = |x| \pm 1, \qquad |xs| = |x| \pm 1.$$

Proof. There exists a unique homomorphism (character) $\chi : G \to \{\pm 1\}$ such that

$$\chi(s) = -1, \qquad s \in \Sigma.$$

For any $x \in G$, taking a reduced expression $x = s_1 s_2 \cdots s_r$, $r = |x|$, we have

$$\chi(x) = \chi(s_1)\chi(s_2)\cdots\chi(s_r) = (-1)^{|x|}, \qquad x \in G.$$

Using this formula, we obtain

$$\chi(sx) = (-1)^{|sx|},$$

and

$$\chi(sx) = \chi(s)\chi(x) = (-1)(-1)^{|x|} = (-1)^{|x|+1}.$$

Therefore, $|sx| \equiv |x| + 1 \pmod 2$. This must be compatible with the triangle inequality (Exercise 3)

$$|x| - 1 \le |sx| \le |x| + 1, \qquad x \in G.$$

Thus $|sx| = |x| \pm 1$ follows. ∎ ∎

Lemma 11.4.2 (Deletion condition) *Let $g \in G$ be expressed in the form*

$$g = s_1 s_2 \cdots s_m, \qquad s_i \in \Sigma. \tag{11.26}$$

If $|g| < m$, then there exist a pair of indices $1 \le i < j \le m$ such that

$$g = s_1 \cdots \check{s}_i \cdots \check{s}_j \cdots s_m,$$

where $\check{s}$ stands for deletion. Therefore, given $g \in G$ of the form (11.26), *its reduced expression is obtained by deleting even number of s_i appearing therein.*

The proof is omitted. The deletion condition is quite useful in the study of the Coxeter groups. Lemma 11.4.1 is also an immediate consequence.

Lemma 11.4.3 *If $s_1, s_2, \dots, s_n \in \Sigma$ are mutually distinct, then $g = s_1 s_2 \cdots s_n$ is a reduced expression.*

Proof. For $n = 1$ the assertion is obvious since Σ is injectively contained in G. Let $n \ge 2$. Suppose that $g = s_1 s_2 \cdots s_n$ is not a reduced expression though $s_1, s_2, \dots, s_n \in \Sigma$ are mutually distinct. Then by Lemma 11.4.2,

$$s_1 \cdots s_n = s_1 \cdots \check{s}_i \cdots \check{s}_j \cdots s_n$$

and hence

$$s_i = s_{i+1} \cdots s_{j-1} s_j s_{j-1} \cdots s_{i+1}.$$

Since the right hand side is of length 1, deleting an even number of elements from the right hand side leads to a reduced expression. The obtained reduced expression should be one of $\{s_{i+1}, \dots, s_j\}$. This contradicts the assumption that $s_1, \dots, s_n$ are mutually distinct. ∎

We next study the Coxeter group associated with a Coxeter matrix satisfying $m(s,t) \ge 3$ for any pair $s \ne t$. In that case, the Cayley graph has no cycle with length less than six, i.e., contains neither triangle, square, nor pentagon.

Lemma 11.4.4 *Assume that $m(s,t) \ge 3$ for any pair $s \ne t$. If $s_1, \dots, s_n \in \Sigma$ are mutually distinct and the relation*

$$s_1 \cdots s_n = sx$$

holds for some $s \in \Sigma$ and $x \in G$ of length $n-1$, then $s = s_1$.

PROOF. We prove the assertion by induction on n. For $n = 1$ the assertion is obvious. Assume that $s_1 s_2 = sx$ holds where $s_1, s_2 \in \Sigma$ are mutually distinct, $s \in \Sigma$ and $x \in G$ of length 1. From $s = s_1 s_2 x$ we see easily that $s = s_1$ or $s = s_2$ or $s = x$. If $s = x$ happens, we have $s_1 = s_2$ which yields contradiction. If $s = s_2$ happens, $x = s_1$ and $(s_1 s_2)^2 = e$ which is again contradiction. Consequently, $s = s_1$.

Assume that the assertion is valid up to $n-1$, $n \ge 2$. Since

$$ss_1 \cdots s_n = x \tag{11.27}$$

is of length $n-1$, deleting two elements from the left hand side we obtain a reduced expression of x. If these two elements are chosen from $\{s_1, \dots, s_n\}$, say, s_i, s_j $(i < j)$, we come back to

$$s_1 \cdots \check{s}_i \cdots \check{s}_j \cdots s_n = sx = s_1 \cdots s_n,$$

which is a reduced expression by Lemma 11.4.3. This is contradiction. Hence, to get a reduced expression of x in (11.27), we need to delete s and s_i for some $i = 1, \dots, n$. In that case we come to

$$s_1 \cdots \check{s}_i \cdots s_n = x,$$

and hence

$$ss_1 \cdots s_{i-1} = s_1 \cdots s_i. \tag{11.28}$$

If $1 \le i \le n-1$, by the assumption of induction we have $s = s_1$. Suppose $i = n$, i.e.,

$$ss_1 \cdots s_{n-1} = s_1 \cdots s_n.$$

By a simple argument with the deletion condition we see that $s \in \{s_1, \dots, s_n\}$. If $s = s_j$, $1 \le j \le n-1$, then

$$s_n = s_{n-1} \cdots s_1 s_j s_1 \cdots s_{n-1},$$

which implies that s_n coincides with some of $\{s_1, \dots s_{n-1}\}$. But this contradicts the assumption. Hence $s = s_n$, i.e.,

$$s_n s_1 \cdots s_{n-1} = s_1 \cdots s_n. \tag{11.29}$$

We shall prove that this does not occur. Note first that (11.29) is equivalent to the following

$$(s_{n-2} \cdots s_1) s_n (s_1 \cdots s_{n-2}) s_{n-1} = s_{n-1} s_n.$$

Since this is of length 2, deleting an even number of elements from the left hand side, we obtain a reduced expression of length 2, say, tt'. This is the case of $n = 2$ so we know that $t = s_{n-1}$. But this is impossible. ∎

Consider the Cayley graph (G_N, Σ_N) with $e \in G_N$ being an origin. We consider as usual the stratification

$$G_N = \bigcup_{n=0}^{\infty} V_n^{(N)}.$$

Statistics of $\omega_-(x)$ is of importance. We see from Lemma 11.4.1 that $\omega_\circ(x) = 0$ for all $x \in G_N$.

Lemma 11.4.5 *Assume that $m(s,t) \ge 3$ for any pair $s,t \in \Sigma$, $s \ne t$. Then, for any $n = 1, 2, \dots$ we have*

$$\lim_{N\to\infty} \frac{|\{x \in V_n^{(N)}\,;\, \omega_-(x) = 1\}|}{|V_n^{(N)}|} = 1. \tag{11.30}$$

Lemma 11.4.6 *Assume that $m(s,t) \ge 3$ for any pair $s,t \in \Sigma$, $s \ne t$. Then $\omega_-(x) \le 2$ for all $x \in G$.*

The proofs are not difficult and omitted.

Theorem 11.4.7 (QCLT for Coxeter groups) *Let (G, Σ) be an infinite Coxeter group with a Coxeter matrix $\{m(s,t)\}$ such that $m(s,t) \ge 3$ for any pair $s,t \in \Sigma$, $s \ne t$. Let $\Sigma_1 \subset \Sigma_2 \subset \cdots$ be an increasing sequence of subsets of Σ such that $\bigcup_{N=1}^\infty \Sigma_N = \Sigma$, and consider the Cayley graph of the Coxeter group (G_N, Σ_N) and its adjacency matrix A_N. Then $A_N^\circ = 0$ and*

$$\lim_{N\to\infty} \frac{A_N^\pm}{\sqrt{|\Sigma_N|}} = B^\pm,$$

in the sense of stochastic convergence with respect to the vacuum state, where $B^\pm$ are the annihilation and creation operators in the free Fock space.

Proof. It is sufficient to show conditions (A1)–(A3). First (A1) is obvious, since the degree of G_N is $|\Sigma_N|$ which tends to the infinity by assumption. Conditions (11.17) and (11.18) in (A2) follow from Lemma 11.4.5. Moreover, the sequence therein is $\omega_n \equiv 1$ so that the limit is described by the free Fock space. Condition (11.19) in (A2) follows from Lemma 11.4.6 with $W_n = 2$. Finally, (A3) is obvious since $\omega_\circ(x) = 0$ for all $x \in G_N$. Consequently, our assertion is an immediate consequence of Theorem 11.3.3. ∎

It is known that the symmetric group $S(N)$ is generated by the successive transpositions

$$\sigma_1 = (12), \quad \sigma_2 = (23), \quad \dots, \quad \sigma_{N-1} = (N-1\,N).$$

We set $\Sigma_N = \{\sigma_1, \sigma_2, \dots, \sigma_{N-1}\}$. Then $(S(N), \Sigma_N)$ becomes a Coxeter group. Note that the Coxeter matrix is given by

$$m(i,j) = \begin{cases} 3, & |i-j| = 1, \\ 2, & |i-j| \ge 2. \end{cases} \tag{11.31}$$

Therefore, Theorem 11.4.7 is not applicable. Instead, we have the following

Theorem 11.4.8 (QCLT for symmetric groups) *Let A_N be the adjacency matrix of the Cayley graph $(S(N), \Sigma_N)$. Then $A_N^\circ = 0$ and*

$$\lim_{N\to\infty} \frac{A_N^\pm}{\sqrt{N-1}} = B^\pm,$$

in the sense of stochastic convergence with respect to the vacuum state, where $B^\pm$ are the annihilation and creation operators in the Boson Fock space.

11.5 Exercises

1. Let $\mathcal{G} = (V, E)$ be a regular graph with degree κ. Prove that

(1) $M(\omega_+|V_n) + M(\omega_-|V_n) + M(\omega_\circ|V_n) = \kappa$.

(2) $\Sigma(\omega_+|V_n) \le \Sigma(\omega_-|V_n) + \Sigma(\omega_\circ|V_n)$.

2. Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph satisfying conditions (A1)–(A3). Prove that $(\{\omega_n\}, \{\alpha_n\})$ defined therein is a Jacobi coefficient of infinite type.

3. Let (G, Σ) be a Coxeter group. Prove the following properties for the length function.

(1) $|x| = |x^{-1}|$ for $x \in G$.

(2) For $x \in G$, $|x| = 1$ if and only if $x \in \Sigma$.

(3) $|x| - |y| \le |xy| \le |x| + |y|$ for $x, y \in G$.

(4) $|x| - 1 \le |sx| \le |x| + 1$ for $x \in G$ and $s \in \Sigma$.

4. Prove (11.31).

12 Graph Products

12.1 Motivation

A growing graph models a revolution of networks in the real world.

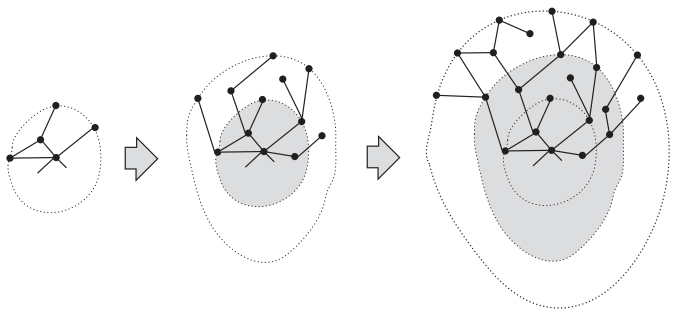


Figure 12.1: Growing graph

It would be interesting if the growing graph $G^{(\nu)}$ is considered as an analogue of an independent increment process in classical probability theory. It is our hope that the evolution is formulated as

$$G^{(\nu)} = G^{(\nu-1)} \sharp H^{(\nu)}, \tag{12.1}$$

where $\sharp H^{(\nu)}$ is an operation to form a new graph $G^{(\nu)}$ and $H^{(\nu)}$ is given at each evolution step. We hope that $H^{(\nu)}$ shares a common sprit with independent random variables.

In this chapter we discuss graph products. Given two graphs G_1 and G_2, we form a new graph $G_1 \sharp G_2$ as a "product." This graph product gives rise to a product of the adjacency matrices

$$A = A_1 \sharp A_2 \,. \tag{12.2}$$

When the evoluton of graphs is formulated in terms of a graph product, (12.1) yields

$$A^{(\nu)} = A^{(\nu-1)} \sharp B^{(\nu)} = \cdots = (\cdots((A^{(0)} \sharp B^{(0)}) \sharp B^{(1)}) \cdots) \sharp B^{(\nu)}.$$

We may expect that the spectral properties of $A^{(\nu)}$ follow from the study of some interrelation among $B^{(\nu)}$ with respect to the operation $\sharp$. From this aspect various types of independence in quantum probability would be useful.

12.2 Direct (Cartesian) Products

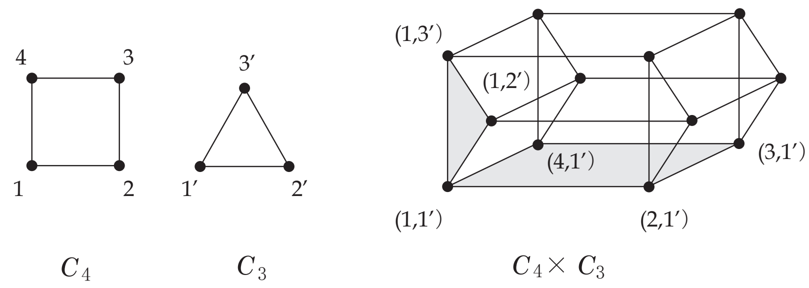
Definition 12.2.1 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. For $(x, y), (x', y') \in V_1 \times V_2$ we write $(x, y) \sim (x', y')$ if one of the following conditions is satisfied:

(i) $x = x'$ and $y \sim y'$;

(ii) $x \sim x'$ and $y = y'$.

Then $V_1 \times V_2$ becomes a graph in such a way that $(x, y), (x', y') \in V_1 \times V_2$ are adjacent if $(x, y) \sim (x', y')$. This graph is called the *direct product* of G_1 and G_2, and is denoted by $G_1 \times G_2$.

Example 12.2.2 $C_4 \times C_3$



Lemma 12.2.3 (1) $G_1 \times G_2 \cong G_2 \times G_1$.

(2) $(G_1 \times G_2) \times G_3 \cong G_1 \times (G_2 \times G_3)$.

PROOF. Straightforward. ∎

Example 12.2.4 $\mathbb{Z}^N \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$ (N times)

The adjacency matrix A_i acts on $C(V_i)$ by usual matrix multiplication, hence the adjacency matrix A of the direct product $G_1 \times G_2$ acts on $C(V_1 \times V_2) \cong C(V_1) \otimes C(V_2)$, where the canonical isomorphism is defined by the correspondence of basis $\delta_{(x,y)} \mapsto \delta_x \otimes \delta_y$.

Theorem 12.2.5 *As an operator acting on $C(V_1) \otimes C(V_2)$, the adjacency matrix A of the direct product $G_1 \times G_2$ is of the form:*

$$A = A_1 \otimes E_2 + E_1 \otimes A_2 \,, \tag{12.3}$$

where E_i is the identity matrix on $C(V_i)$.

PROOF. We see that

$$(A_1 \otimes E_2)_{(x,y),(x',y')} = (A_1)_{xx'}(E_2)_{y,y'} = \begin{cases} 1, & \text{if } x \sim x' \text{ and } y = y', \\ 0, & \text{otherwise.} \end{cases}$$

Similarly,

$$(E_1 \otimes A_2)_{(x,y),(x',y')} = (E_1)_{xx'}(A_2)_{y,y'} = \begin{cases} 1, & \text{if } x = x' \text{ and } y \sim y', \\ 0, & \text{otherwise.} \end{cases}$$

Since the two conditions (i) $x \sim x'$ and $y = y'$; (ii) $x = x'$ and $y \sim y'$ do not occur simultaneously, we have

$$(A_1 \otimes E_2 + E_1 \otimes A_2)_{(x,y),(x',y')} = \begin{cases} 1, & \text{if } (x,y) \sim (x',y'), \\ 0, & \text{otherwise.} \end{cases}$$

This means that $A_1 \otimes E_2 + E_1 \otimes A_2$ coincides with the adjacency matrix of $G_1 \times G_2$. ∎

Theorem 12.2.6 *Let $G = G_1 \times G_2$. Then,*

$$\partial_G((x,y),(x',y')) = \partial_{G_1}(x,x') + \partial_{G_2}(y,y'). \tag{12.4}$$

Proof. Set $s = \partial_G((x,y),(x',y'))$. Then we may find a sequence of vertices of $G_1 \times G_2$ such that

$$(x,y) = (x_0,y_0) \sim (x_1,y_1) \sim (x_2,y_2) \sim \cdots \sim (x_{s-1},y_{s-1}) \sim (x_s,y_s) = (x',y').$$

Then, every pair of consecutive vertices in the sequence

$$x = x_0, \quad x_1, \quad x_2, \quad \dots, \quad x_{s-1}\,, \quad x_s = x'$$

are identical or adjacent. Hence, reducing consecutively identical vertices into one vertex, we obtain a walk connecting x and x', of which the length is, say, α. Similarly, from

$$y = y_0, \quad y_1, \quad y_2, \quad \dots, \quad y_{s-1}\,, \quad y_s = y'$$

we obtain a walk connecting y and y', of which the length is, say, β. By the definition of a direct product graph, $x_i = x_{i+1}$ happens if and only if $y_i \sim y_{i+1}$. Hence

$$\alpha + \beta = s.$$

Since $\partial_{G_1}(x,x') \le \alpha$ and $\partial_{G_2}(y,y') \le \beta$, we have

$$\partial_{G_1}(x,x') + \partial_{G_2}(y,y') \le \alpha + \beta = s.$$

That $\partial_{G_1}(x,x') + \partial_{G_2}(y,y') \ge s$ is shown by constructing a walk. ∎

An interesting consequence is the following

Theorem 12.2.7 *Let Q_1, Q_2 and Q be the Q-matrices of graphs G_1, G_2 and $G = G_1 \times G_2$, with a common parameter q. Then*

$$Q = Q_1 \otimes Q_2\,.$$

Proof. First by definition

$$(Q)_{(x,y),(x',y')} = q^{\partial_G((x,y),(x',y'))}.$$

Applying Theorem 12.2.6, we obtain

$$= q^{\partial_{G_1}(x,x')} q^{\partial_{G_2}(y,y')} = (Q_1)_{xx'}(Q_2)_{yy'} = (Q_1 \otimes Q_2)_{(x,y),(x',y')}.$$

Therefore, $Q = Q_1 \otimes Q_2$. ∎

Positivity of the Q-matrices are of importance. For a graph G we set

$$q[G] = \{-1 \le q \le 1\,;\, Q_q \text{ is strictly positive definite}\},$$
$$\tilde{q}[G] = \{-1 \le q \le 1\,;\, Q_q \text{ is positive definite}\}.$$

Theorem 12.2.8 *Let $G = G_1 \times G_2$.*

(1) $q[G] = q[G_1] \cap q[G_2]$.

(2) $\tilde{q}[G] = \tilde{q}[G_1] \cap \tilde{q}[G_2]$.

PROOF. We see from Theorem 12.2.7 that the eigenvalues of Q are of the form $\alpha\beta$, where α and β are eigenvalues of Q_1 and Q_2, respectively.

(1) Let $q \in q[G_1] \cap q[G_2]$, namely, $Q_i = Q_i(q)$ is a strictly positive definite kernel for G_i. Since the eigenvalues of Q_i are all positive. every eigenvalues of Q are also positive. Therefore, $q[G_1] \cap q[G_2] \subset q[G]$.

We show that Q contains Q_1 as a principal submatrix. Take a vetex $o_2 \in V_2$ and set

$$W = \{(x, o_2)\,;\, x \in V_1\}.$$

Let H_1 be the induced subgraph of $G_1 \times G_2$ spanned by W. Then, H_1 is isomorphic to G_1 and $\partial_H = \partial_{G_1}$ coincides with the restriction of ∂_G to H. Hence Q_1 is regarded as a principal submatrix of Q. The situation is similar for Q_2. Now let $q \in q[G]$. Then Q is strictly positive definite so are all the principal submatrices. In particular, so are Q_1 and Q_2. Consequently, $q[G] \subset q[G_1] \cap q[G_2]$.

(2) The proof is similar. Let $q \in \tilde{q}[G_1] \cap \tilde{q}[G_2]$, namely, $Q_i = Q_i(q)$ is a positive definite kernel for G_i. Since the eigenvalues of Q_i are all non-negative, every eigenvalues of Q are also non-negative. Therefore, $\tilde{q}[G_1] \cap \tilde{q}[G_2] \subset \tilde{q}[G]$.

The second half is also similar to the argument in (1). ∎

Example 12.2.9 The Hamming graph $H(d, N)$ is isomorphic to the direct product of d copies of the perfect graph K_N. In fact, for two vertices $x = (\xi_1, \dots, \xi_d)$ and $y = (\eta_1, \dots, \eta_d)$, the Hamming distance is one if and only if

$$\begin{array}{lllll} \xi_1 \sim \eta_1, & \xi_2 = \eta_2, & \xi_3 = \eta_3, & \dots & \xi_d = \eta_d \quad \text{or} \\ \xi_1 = \eta_1, & \xi_2 \sim \eta_2, & \xi_3 \sim \eta_3, & \dots & \xi_d = \eta_d \quad \text{or} \\ \dots & & & & \\ \xi_1 = \eta_1, & \xi_2 = \eta_2, & \xi_3 = \eta_3, & \dots & \xi_d \sim \eta_d. \end{array}$$

This condition is equivalent to that $x = (\xi_1, \dots, \xi_d)$ and $y = (\eta_1, \dots, \eta_d)$ are adjacent in the direct product $K_N \times \dots \times K_N$ (d times).

Theorem 12.2.10 *Let $G = G_1 \times G_2$ be a direct product of two graphs and $A = A_1 \otimes E + E \otimes A_2$ be the adjacency matrix expressed as an operator on $C(V_1) \otimes C(V_2)$, see Theorem 12.2.5. Then $A = A_1 \otimes E + E \otimes A_2$ is a sum of commutative independent random variables with respect to $\langle\cdot\rangle_q$.*

PROOF. The details are omitted. We only observe that

$$\langle (A_1 \otimes E)^\alpha (E \otimes A_2)^\beta \rangle_q = \langle (A_1 \otimes E)^\alpha \rangle_q \langle (E \otimes A_2)^\beta \rangle_q\,.$$

In fact, by Theorem 12.2.7 we have

$$\begin{aligned}\langle (A_1\otimes E)^\alpha (E\otimes A_2)^\beta\rangle_q &= \langle Q(\delta_o\otimes\delta_o), (A_1\otimes E)^\alpha (E\otimes A_2)^\beta(\delta_o\otimes\delta_o)\rangle\\ &= \langle Q_1\delta_o, A_1^\alpha E^\beta\delta_o\rangle\langle Q_2\delta_o, E^\alpha A_2^\beta\delta_o\rangle\\ &= \langle Q_1\delta_o, A_1^\alpha\delta_o\rangle\langle Q_2\delta_o, A_2^\beta\delta_o\rangle\\ &= \langle Q_1\delta_o, A_1^\alpha\delta_o\rangle\langle Q_2\delta_o, \delta_o\rangle\langle Q_1\delta_o, \delta_o\rangle\langle Q_2\delta_o, A_2^\beta\delta_o\rangle\\ &= \langle (A_1\otimes E)^\alpha\rangle_q\langle (E\otimes A_2)^\beta\rangle_q\,.\end{aligned}$$

∎

12.3 Star Products

Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graphs. Fix vertices $o_1\in V_1$ and $o_2\in V_2$. For $(x,y),(x',y')\in V_1\times V_2$ we write $(x,y)\sim(x',y')$ if one of the following conditions is satisfied:

(i) $x=x'=o_1$ and $y\sim y'$;

(ii) $x\sim x'$ and $y=y'=o_2$.

Then $V_1\times V_2$ becomes a graph in such a way that $(x,y),(x',y')\in V_1\times V_2$ are adjacent if $(x,y)\sim(x',y')$. This graph is denoted by $\tilde G$ for the moment.

Lemma 12.3.1 *As an operator on $C(V_1)\otimes C(V_2)$ the adjacency matrix of $\tilde G$ is given by*

$$\tilde A = A_1\otimes P_2 + P_1\otimes A_2$$

where $P_i : C(V_i)\to C(V_2)$ is the projection defined by

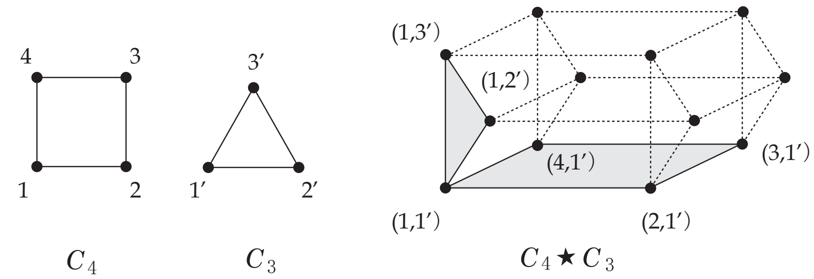
$$P_i\delta_x = \begin{cases}\delta_{o_i}, & \textit{if } x=o_i,\\ 0, & \textit{otherwise.}\end{cases}$$

Definition 12.3.2 Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graphs with fixed origins $o_1\in V_1$ and $o_2\in V_2$. Set

$$V_1\star V_2 = \{(x,o_2)\,;\,x\in V_1\}\cup\{(o_1,y)\,;\,y\in V_2\}$$

The induced subgraph of $\tilde G$ spanned by $V_1\star V_2$ is called the *star product* of G_1 and G_2 (with contact vertices o_1 and o_2), and is denoted by $G_1\star G_2 = G_1\,{}_{o_1}\!\star_{o_2}\,G_2$.

Example 12.3.3 $C_4\star C_3$



Lemma 12.3.4 (1) $G_1 \star G_2 \cong G_1 \star G_2$.

(2) $(G_1 \star G_2) \star G_3 \cong G_1 \star (G_2 \star G_3)$.

PROOF. Exercises. ∎

As usual, we regard the adjacency matrix A_i as an operator acting on $C(V_i)$. Then the adjacency matrix $\tilde{A}$ of $\tilde{G}$ is an operator acting on $C(V_1 \times V_2) = C(V_1) \otimes C(V_2)$. Since $G_1 \star G_2$ is an induced subgraph, its adjacency matrix A is a just a submatrix of the adjacency matrix of the direct product $G_1 \times G_2$.

Theorem 12.3.5 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with fixed origins $o_1 \in V_1$ and $o_2 \in V_2$. Let A be the adjacency matrix of the star product $G_1 \star G_2$. Then, as an operator acting on $C(V_1 \star V_2)$ we have*

$$A = (A_1 \otimes P_2 + P_1 \otimes A_2) \restriction_{C(V_1 \star V_2)}$$

PROOF. It follows from the above argument that $A = A_{G_1 \times G_2} \restriction_{C(V_1 \star V_2)}$. By Theorem 12.2.5 we see that

$$A = A_{G_1 \times G_2} \restriction_{C(V_1 \star V_2)} = (A_1 \otimes E_2 + E_1 \otimes A_2) \restriction_{C(V_1 \star V_2)}$$

It is easily verified by definition that

$$(A_1 \otimes E_2 + E_1 \otimes A_2) \restriction_{C(V_1 \star V_2)} = (A_1 \otimes P_2 + P_1 \otimes A_2) \restriction_{C(V_1 \star V_2)},$$

which completes the proof. ∎

We now consider the graph distance of the star product.

Lemma 12.3.6 *Let $G = G_1 \star G_2$. Then,*

$$\partial_G = \partial_{G_1 \times G_2} \restriction_{V_1 \star V_2} .$$

PROOF. Take a pair of vertices of $G_1 \star G_2$. For $(x, o_2), (x', o_2)$ we have

$$\begin{aligned} \partial_G((x, o_2), (x', o_2)) &= \partial_{G_1}(x, x') \\ &= \partial_{G_1}(x, x') + \partial_{G_2}(o_2, o_2) \\ &= \partial_{G_1 \times G_2}((x, o_2), (x', o_2)). \end{aligned}$$

For $(x, o_2), (o_1, y)$ we have

$$\begin{aligned} \partial_G((x, o_2), (o_1, y)) &= \partial_G((x, o_2), (o_1, o_2)) + \partial_G((o_1, o_2), (o_1, y)) \\ &= \partial_{G_1}(x, o_1) + \partial_{G_2}(o_2, y) \\ &= \partial_{G_1 \times G_2}((x, o_2), (o_1, y)). \end{aligned}$$

∎

Theorem 12.3.7 *The Q-matrix of the star product $G = G_1 \star G_2$ is a principal submatrix of the Q-matrix of $G_1 \times G_2$ as follows:*

$$Q_{G_1 \star G_2} = Q_{G_1 \times G_2} \restriction_{C(V_1 \star V_2)}$$

PROOF. An immediate consequence from Lemma 12.3.6. ∎

Theorem 12.3.8 *Let $G = G_1 \star G_2$.*

(1) $q[G] = q[G_1] \cap q[G_2]$.

(2) $\tilde{q}[G] = \tilde{q}[G_1] \cap \tilde{q}[G_2]$.

PROOF. (1) Supose $q \in q[G_1] \cap q[G_2]$. We see from Theorem 12.2.8 that $Q_{G_1 \times G_2}(q)$ is strictly positive definite. Since $Q_{G_1 \star G_2}$ is a principal submatrix by Theorem 12.3.7, it is also strictly positive definite. Namely, $q[G_1] \cap q[G_2] \subset q[G]$.

Conversely, let $q \in q[G]$. Then $Q_{G_1 \star G_2}(q)$ is strictly positive definite. Since G_i is isometrically imbedded in $G_1 \star G_2$, its Q-matrix is a principal submatrix of $Q_{G_1 \star G_2}(q)$. Therefore, $Q_{G_i}(q)$ is also a strictly positive definite. Thus, $q[G] \subset q[G_1] \cap q[G_2]$.

(2) is proved similarly. ∎

Remark 12.3.9 Theorem 12.3.8 was implicitly mentioned in Secton 9.1. The above argument does not require Theorem 9.1.3.

Theorem 12.3.10 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with fixed origins $o_1 \in V_1$ and $o_2 \in V_2$. Let A be the adjacency matrix of the star product $G_1 \star G_2$. Then, as an operator acting on $C(V_1 \star V_2)$*

$$A = (A_1 \otimes P_2 + P_1 \otimes A_2) \restriction_{C(V_1 \star V_2)}$$

is a sum of Boolean independent random variables with respect to the vacuum state at (o_1, o_2), see also Theorem 12.3.5.

PROOF. Detailed argument is left to the reader. We only show that

$$\langle (A_1 \otimes P_2)^\alpha (P_1 \otimes A_2)^\beta (A_1 \otimes P_2)^\gamma \rangle = \langle (A_1 \otimes P_2)^\alpha \rangle \langle (P_1 \otimes A_2)^\beta \rangle \langle (A_1 \otimes P_2)^\gamma \rangle.$$

In fact, we first observe that

$$\langle (A_1 \otimes P_2)^\alpha (P_1 \otimes A_2)^\beta (A_1 \otimes P_2)^\gamma \rangle = \langle \delta_{o_1}, A_1^\alpha P_1 A_1^\gamma \delta_{o_1} \rangle \langle \delta_{o_2}, P_2 A_2^\beta P_2 \delta_{o_2} \rangle. \tag{12.5}$$

Here $P_1 A_1^\gamma \delta_{o_1} = \langle \delta_{o_1}, A_1^\gamma \delta_{o_1} \rangle \delta_{o_1}$ so that

$$\langle \delta_{o_1}, A_1^\alpha P_1 A_1^\gamma \delta_{o_1} \rangle = \langle \delta_{o_1}, A_1^\alpha \delta_{o_1} \rangle \langle \delta_{o_1}, A_1^\gamma \delta_{o_1} \rangle. \tag{12.6}$$

On the other hand,

$$\langle \delta_{o_2}, P_2 A_2^\beta P_2 \delta_{o_2} \rangle = \langle \delta_{o_2}, A_2^\beta \delta_{o_2} \rangle. \tag{12.7}$$

Incerting (12.6) and (12.7) into (12.5), we obtain the desired relation. ∎

12.4 Comb Products

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We fix a vertix $o_2 \in V_2$. For $(x, y), (x', y') \in V_1 \times V_2$ we write $(x, y) \sim (x', y')$ if one of the following conditions is satisfied:

(i) $x = x'$ and $y \sim y'$;

(ii) $x \sim x'$ and $y = y' = o_2$.

Then $V_1 \times V_2$ becomes a graph in such a way that $(x, y), (x', y') \in V_1 \times V_2$ are adjacent if $(x, y) \sim (x', y')$. This graph is denoted by $G_1 \rhd_{o_2} G_2$ and is called the *comb product*.

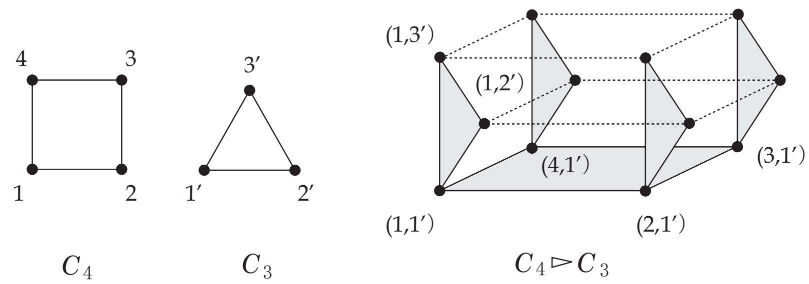
Lemma 12.4.1 *As an operator on $C(V_1) \otimes C(V_2)$ the adjacency matrix of $G_1 \rhd_{o_2} G_2$ is given by*

$$A = A_1 \otimes P_2 + E_1 \otimes A_2$$

where $P_2 : C(V_2) \to C(V_2)$ is the projection onto the space spanned by δ_{o_2} and E_1 is the identity matrix acting on $C(V_1)$.

Proof. Exercise. ∎

Example 12.4.2 $C_4 \rhd C_3$



The comb product is not commutative, but associative.

Lemma 12.4.3 $(G_1 \rhd G_2) \rhd G_3 \cong G_1 \rhd (G_2 \rhd G_3)$.

Theorem 12.4.4 *Let $G = G_1 \rhd G_2$.*

(1) $q[G] = q[G_1] \cap q[G_2]$.

(2) $\tilde{q}[G] = \tilde{q}[G_1] \cap \tilde{q}[G_2]$.

Proof. Since

$$G_1 \rhd G_2 \cong (\cdots((G_1 \star \overbrace{G_2) \star G_2 \star \cdots) \star G_2}^{|V_1| \text{ times}},$$

the assertion follows from Theorem 12.3.8. ∎

Theorem 12.4.5 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with fixed origins $o_2 \in V_2$. Let A be the adjacency matrix of the comb product $G_1 \rhd G_2$. Then, as an operator acting on $C(V_1 \times V_2)$*

$$A = A_1 \otimes P_2 + E_1 \otimes A_2$$

is a sum of monotone independent random variables with respect to the vacuum state at (o_1, o_2), see also Lemma 12.4.1.

13 Random Graphs

13.1 The Erdős–Rényi Random Graph

For an integer $n \geq 1$ we fix a set V of n elements, say,

$$V = \{0, 1, 2, \dots, n-1\}.$$

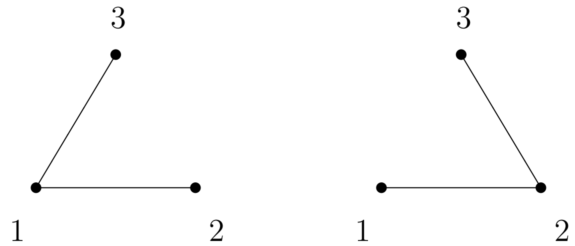
We set

$$\mathcal{G} = \{G = (V, E)\,;\, E \subset V \text{ with } |E| = 2\},$$

which is the set of graphs whose vertex set is V. Note that

$$|\mathcal{G}| = 2^{\binom{n}{2}}.$$

Here we remark that, for example, the following two graphs are distinguished in $\mathcal{G}$ though they are isomorphic.



Given a constant number $0 < p < 1$, we define a probability measure on $\mathcal{G}$ by

$$P(\{G\}) = p^{e(G)}(1-p)^{\binom{n}{2} - e(G)},$$

where $e(G)$ stands for the set of edges of G. It is easily checked that $P(\{G\}) > 0$ for all $G \in \mathcal{G}$ and

$$\sum_{P \in \mathcal{G}} P(\{G\}) = 1.$$

Definition 13.1.1 The probability space $(\mathcal{G}, P)$ is called the *Erdős–Rényi random graph* and is denoted by $\mathcal{G}(n, p)$.

The random graph $\mathcal{G} = \mathcal{G}(n, p)$ is generated in such a way that for a pair of vertices we decide by a coin toss whether to draw an edge or not. This is seen also in terms of the adjacency matrix. The adjacency matrix of $G \in \mathcal{G}$ is denoted by A_G. Since $\mathcal{G}$ is equipped with a probability P, $\{A_G\,;\, G \in \mathcal{G}\}$ becomes a random matrix. This random matrix $A = (A_{ij})$ possesses the following properties:

(i) A_{ij} is a random variables with values in $\{0, 1\}$.

(ii) $A_{ii} = 0$ for all i.

(iii) $A_{ij} = A_{ji}$ for all $i \neq j$.

(iv) $P(A_{ij} = 1) = p$ and $P(A_{ij} = 0) = 1 - p$ for all $i \neq j$.

(v) $\{A_{ij}\,;\,0 \le i < j \le n-1\}$ is independent.

Lemma 13.1.2 *The mean degree of $\mathcal{G}(n,p)$ is given by*

$$\bar{d}(\mathcal{G}(n,p)) = \frac{1}{n}\sum_{i\in V}\mathbf{E}(\deg_G(i)) = (n-1)p.$$

PROOF. For simplicity we write $A = A_G$ and $A = (A_{ij})$. For $i \in V$ we have

$$\deg_G(i) = \sum_{j\in V} A_{ij}\,.$$

Hence

$$\bar{d}(G) = \frac{1}{n}\sum_{i\in V}\deg_G(i) = \frac{1}{n}\sum_{i,j\in V} A_{ij}\,.$$

Taking the mean value over $\mathcal{G}$, we come to

$$\mathbf{E}(\bar{d}(G)) = \frac{1}{n}\sum_{i,j\in V}\mathbf{E}(A_{ij}) = \frac{1}{n}\sum_{i\neq j} p = \frac{1}{n}\,(n^2-n)p,$$

which proves the assertion. ∎

13.2 Mean Eigenvalue Distribution

For $G \in \mathcal{G} = \mathcal{G}(n,p)$ let μ_G denote the eigenvalue distribution. It is known that μ_G is characterized by

$$M_m(\mu_G) = \int_{-\infty}^{+\infty} x^m \mu_G(dx) = \frac{1}{n}\mathrm{Tr}\,(A^m), \qquad m = 1, 2, \dots.$$

Since G is a random graph, we may think of the mean eigenvalue distribution:

$$\mu_{n,p} = \mathbf{E}(\mu_G) = \sum_{G\in\mathcal{G}} P(\{G\})\mu_G\,.$$

Obviously, $\mu_{n,p}$ is a finite sum of δ-measures. We are interested in the following questions:

(1) Find a good expression of $\mu_{n,p}$.

(2) Asymptotics of $\mu_{n,p}$ as $n \to \infty$.

(3) In particular, in the *sparse limit*, i.e., as

$$n \to \infty, \quad p \to 0, \quad np \to \lambda \text{ (constant)}. \tag{13.1}$$

We see from Lemma 13.1.2 that in the sparse limit the number of vertices tends to the infinity while the mean degree remains finite.

Lemma 13.2.1 *Let $o \in V$ be fixed. Let ν_G be the spectral distribution of G in the vacuum state at $o \in V$. Namely,*

$$M_m(\nu_G) = \int_{-\infty}^{+\infty} x^m \nu_G(dx) = \langle \delta_o, A_G^m \delta_o \rangle, \qquad m = 1, 2, \dots .$$

Then the mean eigenvalue distribution coincides with $\mathbf{E}(\nu_G)$, i.e.,

$$\mu_{n,p} = \mathbf{E}(\mu_G) = \mathbf{E}(\nu_G).$$

PROOF. Let $i \in V$, $i \neq o$. Let $\sigma : V \to V$ be the transposition of i and o. Then σ induces naturally a transformation on $\mathcal{G}$, which is denoted by $\tilde{\sigma}$. Then,

$$A_{\tilde{\sigma}(G)} = \sigma^{-1} A_G \sigma$$

so that

$$G \cong \tilde{\sigma}(G)$$

and $\tilde{\sigma} : \mathcal{G} \to \mathcal{G}$ is measure-preserving, i.e.,

$$P(\tilde{\sigma}^{-1}(G)) = P(G), \qquad G \in \mathcal{G}.$$

With these observation, we have

$$\begin{aligned} \langle \delta_i, A_G^m \delta_i \rangle &= \langle \delta_{\sigma(o)}, A_G^m \delta_{\sigma(o)} \rangle = \langle \sigma\delta_o, A_G^m \sigma\delta_o \rangle \\ &= \langle \delta_o, \sigma^{-1} A_G^m \sigma \delta_o \rangle = \langle \delta_o, A_{\tilde{\sigma}(G)}^m \delta_o \rangle. \end{aligned}$$

Taking the mean values in both sides and using the invariance of P, we get

$$\mathbf{E}(\langle \delta_i, A_G^m \delta_i \rangle) = \mathbf{E}(\langle \delta_o, A_{\tilde{\sigma}(G)}^m \delta_o \rangle) = \mathbf{E}(\langle \delta_o, A_G^m \delta_o \rangle).$$

Consequently,

$$\begin{aligned} M_m(\mu_{n,p}) &= \frac{1}{n} \mathbf{E}(\mathrm{Tr}\,(A_G^m)) = \frac{1}{n} \sum_{i=0}^{n-1} \langle \delta_i, A_G^m \delta_i \rangle \\ &= \mathbf{E}(\langle \delta_o, A_G^m \delta_o \rangle) = M_m(\mathbf{E}(\nu_G)). \end{aligned}$$

This proves that $\mu_{n,p} = \mathbf{E}(\mu_G) = \mathbf{E}(\nu_G)$. ∎

13.3 Computing the Moments

It follows from Lemma 13.2.1 that

$$M_m(\mu_{n,p}) = \mathbf{E}((A^m)_{00}), \qquad m = 1, 2, \dots . \tag{13.2}$$

The main task is to compute $M_m(\mu_{n,p})$. For $m = 1$ we have obviously,

$$M_1(\mu_{n,p}) = \mathbf{E}((A)_{00}) = 0 \tag{13.3}$$

For $m = 2$ we have

$$M_2(\mu_{n,p}) = \mathbf{E}((A^2)_{00}) = \sum_{i \in V} \mathbf{E}(A_{0i}A_{i0}).$$

Here we note that $A_{00} = 0$ and $A_{0i} = A_{i0}$. Moreover, for $i \neq 0$, we have $A_{0i}^2 = A_{0i}$. Thus,

$$M_2(\mu_{n,p}) = \sum_{i \neq 0} \mathbf{E}(A_{0i}) = (n-1)p. \tag{13.4}$$

For $m = 3$ we have

$$M_3(\mu_{n,p}) = \mathbf{E}((A^3)_{00}) = \sum_{i,j \in V} \mathbf{E}(A_{0i}A_{ij}A_{j0}).$$

Because A is the adjacency matrix we need only to take the case $0 \neq i \neq j \neq 0$ into account. This case is equivalent to say that $0, i, j$ are distinct. Then

$$M_3(\mu_{n,p}) = \sum_{0 \neq i \neq j \neq 0} \mathbf{E}(A_{0i}A_{ij}A_{j0}) = \sum_{0 \neq i \neq j \neq 0} \mathbf{E}(A_{0i})\mathbf{E}(A_{ij})\mathbf{E}(A_{j0}) = (n-1)(n-2)p^3.$$

For a general m we start with

$$M_m(\mu_{n,p}) = \mathbf{E}((A^m)_{00}) = \sum_{0 \neq i_1 \neq \cdots \neq i_{m-1} \neq 0} \mathbf{E}(A_{0i_1}A_{i_1i_2} \cdots A_{i_{m-1}0}). \tag{13.5}$$

Our goal is to obtain a concise expression of (13.5). We need notation. For $m \geq 2$ let $\mathcal{W}(V, m)$ be the set of sequences of elements in V of the form:

$$[i] : (i_0 \equiv)\ 0 \neq i_1 \neq i_2 \neq \cdots \neq i_{m-1} \neq 0\ (\equiv i_m). \tag{13.6}$$

Given $[i] \in \mathcal{W}(V, m)$ as in (13.6), let $G[i]$ denote the underlying graph. Namely, its vertex set $V(G[i])$ is defined to be the set of elements appearing in the sequence $[i]$ (including 0). Two distinct vertices $j, j' \in V(G[i])$ are adjacent by definition if there exists $0 \leq s \leq m-1$ such that $\{i_s, i_{s+1}\} = \{j, j'\}$. Thus the edge set $E(G[i])$ is defined. It is then obvious that $[i]$ becomes an m-step walk in the graph $G[i]$ starting from and terminating at 0 and passing through all the edges. We will assign a label to every edge of $G[i]$. For $e = \{j, j'\} \in E(G[i])$ define

$$\kappa(e) = |\{0 \leq s \leq m-1\,;\ \{i_s, i_{s+1}\} = \{j, j'\}\}|. \tag{13.7}$$

Namely, $\kappa(e)$ is the number of how many times the walk $[i]$ passes through the edge e.

Lemma 13.3.1 *For $m = 1, 2, \ldots$ we have*

$$\mathbf{E}((A^m)_{00}) = \sum_{[i] \in \mathcal{W}(V,m)} p^{e(G[i])}, \tag{13.8}$$

where $e(G[i])$ is the number of edges of $G[i]$.

PROOF. Let $m \ge 2$ and consider a general term in (13.5):

$$\mathbf{E}(A_{0i_1}A_{i_1i_2}\cdots A_{i_{m-1}0}), \qquad [i] \in \mathcal{W}(V,m).$$

On computing the above expectation we need to note that $A_{jj'} = A_{j'j}$ appears with multiplicities inside the bracket. So, writing

$$A_{0i_1}A_{i_1i_2}\cdots A_{i_{m-1}0} = \prod_{0\le j<j'\le n-1} A_{jj'}^{s_{jj'}}, \qquad s_{jj'} = 0,1,2,\dots,$$

we apply independence condition (A3) to obtain the factorization:

$$\mathbf{E}(A_{0i_1}A_{i_1i_2}\cdots A_{i_{m-1}0}) = \prod_{0\le j<j'\le n-1} \mathbf{E}(A_{jj'}^{s_{jj'}}).$$

Obviously, $s_{jj'} \ge 1$ occurs only when $\{j,j'\} \in E(G[i])$ and $s_{jj'} = \kappa(\{j,j'\})$. In this case, since $A_{jj'}$ is $\{0,1\}$-valued,

$$\mathbf{E}(A_{jj'}^{s_{jj'}}) = \mathbf{E}(A_{jj'}) = p.$$

Consequently,

$$\mathbf{E}(A_{0i_1}A_{i_1i_2}\cdots A_{i_{m-1}0}) = p^{e(G[i])},$$

and, taking a sum over $[i] \in \mathcal{W}(V,m)$, we obtain (13.8). ∎

We proceed to compute the right hand side of (13.8). Let $m \ge 2$. A *labeled rooted graph of size* m, denoted by $\mathcal{L} = (\mathcal{V},\mathcal{E},o,\kappa)$, consists of

(L1) a connected graph $(\mathcal{V},\mathcal{E})$ with $2 \le |\mathcal{V}| \le m$;

(L2) a distinguished vertex $o \in \mathcal{V}$ which is called the root;

(L3) a map $\kappa : \mathcal{E} \to \{1,2,\dots,m\}$ such that $\sum_{e\in\mathcal{E}} \kappa(e) = m$.

The map κ is called the *label* of $\mathcal{L}$. For $\mathcal{L} = (\mathcal{V},\mathcal{E},o,\kappa)$ we set

$$v(\mathcal{L}) = |\mathcal{V}|, \qquad e(\mathcal{L}) = |\mathcal{E}|.$$

We note an obvious inequality:

$$v(\mathcal{L}) - 1 \le e(\mathcal{L}) \le m, \tag{13.9}$$

where the first one follows by connectivity of $(\mathcal{V},\mathcal{E})$ and the second from (L3).

Two labeled rooted graphs are called *isomorphic* if there exists a graph-isomorphism preserving the root and label. Let Λ_m denote the complete set of representatives of labeled rooted graphs of size m up to isomorphisms.

For $[i] \in \mathcal{W}(V,m)$, $m \ge 2$, the underlying graph $G[i]$ is naturally equipped with structure of a labeled rooted graph of size m, which is denoted by $\mathcal{L}[i] = (G[i],0,\kappa)$, where the label κ is defined in (13.7). Noting that the product factor in (13.8) is constant on $[i]$'s generating isomorphic $\mathcal{L}[i]$'s, we obtain the following

Lemma 13.3.2 *For $m = 1, 2, \dots$ we have*

$$\mathbf{E}((A^m)_{00}) = \sum_{\mathcal{L} \in \Lambda_m} |\{[i] \in \mathcal{W}(V, m)\,;\, \mathcal{L}[i] \cong \mathcal{L}\}| p^{e(\mathcal{L})} \tag{13.10}$$

Finally, we study the combinatorial number appearing in the above formula. We need further notation. A *unicursal walk* on $\mathcal{L} = (\mathcal{V}, \mathcal{E}, o, \kappa) \in \Lambda_m$ is a walk on the graph $(\mathcal{V}, \mathcal{E})$ from the root o to itself such that every edge $e \in \mathcal{E}$ is passed through as many times as $\kappa(e)$. It follows from (L3) that a unicursal walk is necessarily of m-step. Let $u(\mathcal{L})$ denote the number of unicursal walks in $\mathcal{L}$.

Theorem 13.3.3 *For $m = 1, 2, \dots$ we have*

$$M_m(\mu_{n,p}) = \mathbf{E}((A^m)_{00}) = \sum_{\mathcal{L} \in \Lambda_m} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L}; n) p^{e(\mathcal{L})}, \tag{13.11}$$

where

$$t(\mathcal{L}; n) = (n-1)(n-2)\cdots(n-(v(\mathcal{L})-1)).$$

Proof. By Lemma 13.3.2 we need only to show that

$$|\{[i] \in \mathcal{W}(V, m)\,;\, \mathcal{L}[i] \cong \mathcal{L}\}| = |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L}; n), \qquad \mathcal{L} \in \Lambda_m.$$

Let $\mathcal{L} = (\mathcal{V}, \mathcal{E}, o, \kappa) \in \Lambda_m$ be fixed. First we choose $\varphi : \mathcal{V} \to V$ such that $\varphi(o) = 0$. There are $t(\mathcal{L}; n)$ choices of such maps. Each unicursal walk on $\mathcal{L}$ gives rise to $[i] \in \mathcal{W}(V, m)$. But the same $[i]$ is obtained with multiplicity $|\mathrm{Aut}\,(\mathcal{L})|$. ∎

For example,

$$\begin{aligned} M_4(\mu_{n,p}) &= (n-1)p + 2(n-1)(n-2)p^2 + (n-1)(n-2)(n-3)p^4 \\ M_5(\mu_{n,p}) &= 5(n-1)(n-2)p^3 \\ &\qquad + 5(n-1)(n-2)(n-3)p^4 + (n-1)(n-2)(n-3)(n-4)p^5 \end{aligned}$$

Formulae equivalent to (13.11) have been implicitly or explicitly used in computation of moments of a random matrix, see e.g., Bauer–Golinelli (2001), Hiai–Petz (2000), Wigner (1955, 1957, 1958).

13.4 The Sparse Limit.

Using Theorem 13.3.3 we will calculate the sparse limit:

$$M_m = \lim M_m(\mu_{n,p}),$$

where the limit is taken as (13.1). In view of (13.11) we need only to consider

$$(n-1)(n-2)\cdots(n-(v(\mathcal{L})-1))\, p^{e(\mathcal{L})} \sim n^{v(\mathcal{L})-1-e(\mathcal{L})} (np)^{e(\mathcal{L})}. \tag{13.12}$$

If $\mathcal{L}$ is not a tree, i.e., contains a cycle, then we have $v(\mathcal{L}) \le e(\mathcal{L})$ and (13.12) vanishes in the sparse limit. If $\mathcal{L}$ is a tree, we have $v(\mathcal{L}) = e(\mathcal{L}) + 1$. In this case, (13.12) implies that

$$\lim (n-1)(n-2)\cdots(n-(v(\mathcal{L})-1))\, p^{e(\mathcal{L})} = \lambda^{e(\mathcal{L})}.$$

Thus, we have

$$M_m = \lim M_m(\mu_{n,p}) = \sum_{\mathcal{L}\in\Lambda_m^*} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L})\, \lambda^{e(\mathcal{L})},$$

where

$$\Lambda_m^* = \{\mathcal{L} \in \Lambda_m\,;\, \mathcal{L} \text{ is a tree}\}.$$

Since a tree admits no unicursal walk of odd steps, for an odd m we have $u(\mathcal{L}) = 0$ so the odd moments vanish.

Theorem 13.4.1 *Let M_m be the sparse limit of the m-th moment of mean spectral distribution the Erdős–Rényi random graph $\mathcal{G}(n,p)$. Then for an odd m we have*

$$M_m = 0,$$

and for an even m,

$$M_m = \sum_{\mathcal{L}\in\Lambda_m^*} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L})\, \lambda^{e(\mathcal{L})}. \tag{13.13}$$

We have

$$M_2 = \lambda, \qquad M_4 = \lambda + 2\lambda^2, \qquad M_6 = \lambda + 6\lambda^2 + 5\lambda^3$$

13.5 Partition Statistics and Approximations.

There is another expression for M_m in Theorem 13.4.1. In fact, we are interested only in the moments of even orders. Using Lemma 13.3.1 we start with

$$M_{2m} = \lim \sum_{[i]\in\mathcal{W}(V,2m)} p^{e(G[i])}. \tag{13.14}$$

Taking Theorem 13.4.1 into account, for the limit in the right hand side it is sufficient to take the sum over $[i] \in \mathcal{W}(V, 2m)$ whose underlying graph $G[i]$ is a tree.

Let $[i] \in \mathcal{W}(V, 2m)$ be given as

$$[i] : 0 \equiv i_0 \ne i_1 \ne i_2 \ne \cdots \ne i_{2m-1} \ne i_{2m} \equiv 0,$$

and assume that $G[i]$ is a tree. We associate a partition ϑ of $\{1, 2, \dots, 2m\}$. For $s, t \in \{1, 2, \dots, 2m\}$ we write $s \sim t$ if $\{i_{s-1}, i_s\} = \{i_{t-1}, i_t\}$. Then $s \sim t$ becomes an equivalence relation, which in turn yields a partition of $\{1, 2, \dots, 2m\}$, denoted by $\vartheta = \vartheta[i]$. Let $\mathcal{P}_{\mathrm{T}}(2m)$

denote the set of all partitions of $\{1, 2, \dots, 2m\}$ obtained in this way. Obviously, for $\vartheta = \vartheta[i]$ we have $e(G[i]) = |\vartheta|$. Then (13.14) becomes

$$\begin{aligned} M_{2m} &= \lim \sum_{\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)} \sum_{\substack{[i] \in \mathcal{W}(V,2m) \\ \vartheta[i]=\vartheta}} p^{|\vartheta|} \\ &= \lim \sum_{\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)} (n-1)(n-2)\cdots(n-|\vartheta|)\, p^{|\vartheta|} \\ &= \sum_{\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)} \lambda^{|\vartheta|}. \end{aligned}$$

Summing up,

Theorem 13.5.1 *The sparse limit of the $2m$-th moment of mean spectral distribution of the Erdős–Rényi random graph $\mathcal{G}(n, p)$ is given by*

$$M_{2m} = \sum_{\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)} \lambda^{|\vartheta|}. \tag{13.15}$$

It is obvious by construction that each block of $\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)$ consists of even number of points. Let $\mathcal{P}_{\mathrm{NC}}(2m)$ be the set of non-crossing partitions of $\{1, 2, \dots, 2m\}$ and set

$$\mathcal{P}_{\mathrm{TNC}}(2m) = \{\vartheta \in \mathcal{P}_{\mathrm{NC}}(2m)\,;\ \text{each } v \in \vartheta \text{ consists of even number of points}\}.$$

It is then shown that $\mathcal{P}_{\mathrm{TNC}}(2m) \subset \mathcal{P}_{\mathrm{T}}(2m)$. However, $\mathcal{P}_{\mathrm{T}}(2m)$ contains some crossing partitions too. This would hinder us from getting an explicit expression of the limit distribution. An analytical approach, which yields also an implicit description of the limit distribution, is found in Dorogovtsev–Goltsev–Mendes–Samukhin (2003).

We show two approximations for the limit distribution whose m-th moment is M_m.

Proposition 13.5.2 *Let $\pi_{\lambda/2}$ be the free Poisson distribution with parameter $\lambda/2$ and $\pi^{\vee}_{\lambda/2}$ its reflection, i.e., $\pi^{\vee}_{\lambda/2}(dx) = \pi_{\lambda/2}(-dx)$. Then*

$$M_{2m-1}(\pi_{\lambda/2} \boxplus \pi^{\vee}_{\lambda/2}) = 0, \quad M_{2m}(\pi_{\lambda/2} \boxplus \pi^{\vee}_{\lambda/2}) = \sum_{\vartheta \in \mathcal{P}_{\mathrm{TNC}}(2m)} \lambda^{|\vartheta|}. \tag{13.16}$$

PROOF. The free Poisson distribution $\pi_{\lambda/2}$ is characterized by the constant free cumulants $r_k(\pi_{\lambda/2}) = \lambda/2$. Then, $r_k(\pi^{\vee}_{\lambda/2}) = (-1)^k \lambda/2$ and

$$r_k(\pi_{\lambda/2} \boxplus \pi^{\vee}_{\lambda/2}) = r_k(\pi_{\lambda/2}) + r_k(\pi^{\vee}_{\lambda/2}) = \begin{cases} \lambda, & k \text{ is even,} \\ 0, & k \text{ is odd.} \end{cases}$$

Applying the free moment–cumulant formula:

$$M_k = \sum_{\vartheta \in \mathcal{P}_{\mathrm{NC}}(k)} \prod_{v \in \vartheta} r_{|v|}\,,$$

we have

$$M_{2m-1} = 0, \quad M_{2m} = \sum_{\vartheta \in \mathcal{P}_{\mathrm{TNC}}(2m)} \lambda^{|\vartheta|},$$

which completes the proof. ∎

Comparing (13.15) and (13.16), we can expect that the sparse limit of mean spectral distribution of the Erdős–Rényi random graph is a kind of deformation of the free Poisson distributions.

Next we look for the leading term of M_{2m} for a large λ. In fact,

$$\begin{aligned} M_{2m} &= \sum_{k=1}^{m} |\{\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)\,;\, |\vartheta| = k\}|\, \lambda^k \\ &= |\{\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)\,;\, |\vartheta| = m\}|\, \lambda^m + O(\lambda^{m-1}) \\ &= |\mathcal{P}_{\mathrm{NCP}}(2m)|\, \lambda^m + O(\lambda^{m-1}), \end{aligned}$$

where $\mathcal{P}_{\mathrm{NCP}}(2m)$ stands for the set of non-crossing pair partitions of $\{1, 2, \dots, 2m\}$. The number $|\mathcal{P}_{\mathrm{NCP}}(2m)|$ is well known as the Catalan number and is the $2m$-th moment of the Wigner semicircle law.

Proposition 13.5.3 *For the m-th moment of mean spectral distribution the Erdős–Rényi random graph $\mathcal{G}(n,p)$ we have*

$$\lim_{\lambda\to\infty} \lim \lambda^{-m/2} M_m(\mu_{n,p}) = \frac{1}{2\pi} \int_{-2}^{+2} x^m \sqrt{4 - x^2}\, dx, \qquad m = 1, 2, \dots,$$

where the second limit is the sparse limit as in (13.1)*.*

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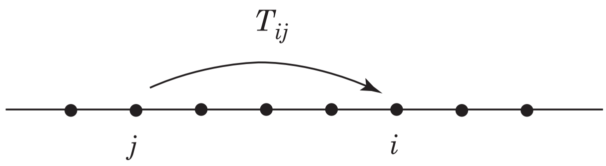
14 Quantum Walks

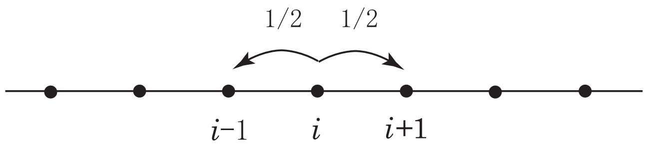
14.1 From Random Walks to Quantum Walks

A random walk on $\mathbb{Z}$ is defined by a transition matrix T, a matrix indexed by $\mathbb{Z} \times \mathbb{Z}$ satisfying

$$(T)_{ij} \geq 0, \qquad \sum_{i \in \mathbb{Z}} (T)_{ij} = 1 \quad \text{for all } j \in \mathbb{Z}.$$



An isotropic random walk is defined by $T = \frac{1}{2} A$, where A is the adjacency matrix.



Definition 14.1.1 Let μ be a probability distribution on $\mathbb{Z}$. Then,

$$\mu_t = e^{t(T-E)} \mu \tag{14.1}$$

where μ is identifined with a vector indexed by $\mathbb{Z}$, is called the distribution of the *continuous time random walk* Z_t on $\mathbb{Z}$ with an initial distribution μ. Namey,

$$P(Z_t = x) = \mu_t(x).$$

Let us examine tht right-hand side in (14.1) defines a probability distribution on $\mathbb{Z}$. By definition,

$$e^{tT} \mu(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} T^n \mu(x).$$

Since T is a transition matrix, $T^n \mu$ is a prpbability distribution so that

$$\sum_{x \in \mathbb{Z}} e^{tT} \mu(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{x \in \mathbb{Z}} T^n \mu(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t.$$

Hence,

$$e^{-t} \sum_{x \in \mathbb{Z}} e^{tT} \mu(x) = 1,$$

which shows that $e^{t(T-E)} \mu$ is a probability distribution on $\mathbb{Z}$.

Differentiating both sides of (14.1) by t, we obtain

$$\frac{d}{dt} \mu_t = (T-E) e^{t(T-E)} \mu = (T-E) \mu_t \, .$$

Therefore, μ_t is the solution of the initial value problem:

$$\frac{d}{dt}\mu_t = (T-E)\mu_t\,, \qquad \mu_0 = \mu. \tag{14.2}$$

In particular, for $T = \frac{1}{2}A$ we see that

$$(T-E)\mu_t = \frac{1}{2}(A-2E)\mu_t\,.$$

We note that

$$\frac{1}{2}(A-2E)\mu_t(x) = \frac{1}{2}(\mu_t(x+1)+\mu_t(x-1)-2\mu(x)) = \frac{1}{2}\Delta\mu_t(x),$$

where $\Delta = A-2E$ is the discrete Laplacian. Thus, (14.2) becomes

$$\frac{\partial}{\partial t}\mu_t = \frac{1}{2}\Delta\mu_t\,, \tag{14.3}$$

which is known as the *heat equation.*

What is a quantum counter part? A plausible candidate is obtained by transferring (14.3) into the Schrödinger equation:

$$i\hbar\frac{\partial\varphi_t}{\partial t} = H\varphi_t.$$

Let us consider the Hamiltonian $H = -\Delta$ in view of (14.3) and set $\hbar = 1$. Then the Schrödinger equation becomes

$$i\frac{\partial\varphi_t}{\partial t} = -\Delta\varphi_t\,,$$

or equivalently,

$$\frac{\partial\varphi_t}{\partial t} = i\Delta\varphi_t = i(A-2E)\varphi_t. \tag{14.4}$$

Given an initial state φ, the solution to (14.4) is given by

$$\varphi_t = e^{it\Delta}\varphi = e^{-2it}e^{itA}\varphi. \tag{14.5}$$

In the classical case, we have

$$P(Z_t = x) = \mu_t(x) = e^{\frac{t}{2}\Delta}\mu(x) = e^{\frac{t}{2}(A-2E)}\mu(x).$$

In the quantum case, from (14.5) we set

$$P(X_t = x) = |\varphi_t(x)|^2 = |e^{it\Delta}\varphi(x)|^2 = |e^{itA}\varphi(x)|^2. \tag{14.6}$$

Since e^{itA} is unitary (this is an informal statement, we need to check the selfadjointness of A), we have $\|e^{itA}\varphi\|^2 = \|\varphi\|^2 = 1$. Hence

$$\sum_{x\in\mathbb{Z}} |\varphi_t(x)|^2 = \|\varphi_t\|^2 = \|e^{-2it}e^{itA}\varphi\|^2 = \|\varphi\|^2 = 1,$$

which means that (14.6) defines a probability distribution on $\mathbb{Z}$.

Now we give the following

Definition 14.1.2 Let A be the adjacency matrix of a graph $G = (V, E)$ and $\varphi \in \ell^2(V)$ with $\|\varphi\| = 1$. A *(cotinuous-time) quantum walk* on G with an initial state φ is a "stochastic process" $\{X_t\}$ such that

$$P(X_t = x) = |e^{itA}\varphi(x)|^2 = |\langle \delta_x, e^{itA}\varphi\rangle|^2, \qquad x \in V.$$

Here $e^{itA}\varphi$ is called the *amplitute wave function.*

Remark 14.1.3 The quantum walk $\{X_t\}$ is not defined as a stochastic process but only time evolution of probability measures is specified.

We shall study the case when the initial state is the vacuum state: $\varphi = \delta_o$, where $o \in V$ is a fixed origin. In that case

$$P(Y_t = n) = \sum_{x \in V_n} |\langle \delta_x, e^{itA}\delta_o\rangle|^2, \qquad V_n = \{x \in V\,;\, \partial(x, o) = n\},$$

is also interesting.

14.2 Method of Quantum Decomposition

Let μ be the spectral distribution of A in the vacuum state δ_o, namey,

$$\langle \delta_o, A^m \delta_o\rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots. \tag{14.7}$$

According to the stratification:

$$V = \bigcup_{n=0}^{\infty} V_n$$

we define a unit vector by

$$\Phi_n = \frac{1}{\sqrt{|V_n|}} \sum_{x \in V_n} \delta_x\,, \qquad n = 0, 1, 2, \dots,$$

and let $\Gamma(G) \subset \ell^2(V)$ be the subspace spanned by them. Moreover, the adjacency matrix admits a quantum decomposition:

$$A = A^+ + A^- + A^\circ.$$

We assume

(H1) $\Gamma(G)$ is invariant under the actions of the quantum components A^+, A^-, A°.

This is a condition for a graph and the choice of an origin. Under (H1) we find a Jacobi coefficient $(\{\omega_n\}, \{\alpha_n\})$ in such a way that

$$\begin{aligned} A^+\Phi_n &= \sqrt{\omega_{n+1}}\,\Phi_{n+1}\,, \\ A^-\Phi_0 &= 0, \quad A^-\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1}\,, \\ A^\circ\Phi_n &= \alpha_{n+1}\Phi_n. \end{aligned}$$

The spectral distribution μ is also determined by the Jacobi coefficient $(\{\omega_n\}, \{\alpha_n\})$ appearing in the famous three-term recurrence relation satisfied by the orthogonal polynomials $\{P_n(x)\}$.

We pose the second condition for the graph G under consideration.

(H2) For any $n = 0, 1, 2, \dots$ and $k = 1, 2, \dots$ the number of k-step walks from o to $x \in V_n$ does not depend on the choice of x but depends only on n and k.

For example, (H2) is satisfied if G is rotationally symmetric around o, i.e., for any pair $x, y \in V$ with $\partial(x, o) = \partial(y, o)$ there exists an automorphism $\alpha \in \mathrm{Aut}\,(G)$ such that $\alpha(o) = o$ and $\alpha(x) = \alpha(y)$. In particular, (H2) is satisfied by a distance-transitive graph.

Theorem 14.2.1 *Notations and assumptions being as above, the amplitude wave function is given by*

$$\langle \delta_x, e^{itA}\delta_0 \rangle = \frac{1}{\|P_n\| \sqrt{|V_n|}} \int_{-\infty}^{+\infty} P_n(s) e^{its} \mu(ds), \qquad x \in V_n\,. \tag{14.8}$$

Moreover,

$$\langle \Phi_n, e^{itA}\Phi_0 \rangle = \frac{1}{\|P_n\|} \int_{-\infty}^{+\infty} P_n(s) e^{its} \mu(ds), \qquad n = 0, 1, 2, \dots. \tag{14.9}$$

PROOF. We note first that

$$\langle \delta_x, e^{itA}\delta_o \rangle = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \langle \delta_x, A^k \delta_o \rangle = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} |\{o \to x\,;\, k\text{-step walk}\}|$$

does not depend on the choice of $x \in V_n$ by (H2). Hence we have

$$\langle \delta_x, e^{itA}\delta_o \rangle = \frac{1}{|V_n|} \sum_{y \in V_n} \langle \delta_y, e^{itA}\delta_o \rangle = \frac{1}{\sqrt{|V_n|}} \langle \Phi_n, e^{itA}\Phi_0 \rangle. \tag{14.10}$$

Using the isometry $\Gamma(G) \to L^2(\mathbb{R}, \mu)$ defined by $\Phi_n \mapsto \|P_n\|^{-1} P_n$, the last expression becomes

$$= \frac{1}{\sqrt{|V_n|}} \|P_n\|^{-1} \langle P_n, e^{its} P_0 \rangle_\mu = \frac{1}{\|P_n\| \sqrt{|V_n|}} \int_{-\infty}^{+\infty} P_n(s) e^{its} \mu(dx).$$

Thus, we obtain (14.8). (14.9) is already clear. ∎

Corollary 14.2.2 *Under the same assumptions as in Theorem 14.2.1 we have*

$$P(Y(t) = n) = |\langle \Phi_n, e^{itA}\Phi_0 \rangle|^2.$$

PROOF. In fact, by (14.10) we obtain

$$P(Y(t) = n) = \sum_{x \in V_n} |\langle \delta_x, e^{itA}\delta_o \rangle|^2 = |\langle \Phi_n, e^{itA}\Phi_0 \rangle|^2.$$

∎

14.3 Growing Regular Graphs

Let $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph. As in Section 11.3 we assume that $\Gamma(G^{(\nu)})$ is asymptotic invariant, i.e., the following three conditions:

(A1) $\lim_\nu \kappa(\nu) = \infty$, where $\kappa(\nu)$ is the degree of $G^{(\nu)}$.

(A2) For each $n = 1, 2, \dots$ the limit

$$\omega_n = \lim_\nu M(\omega_-|V_n^{(\nu)}) < \infty$$

exists and

$$\lim_\nu \Sigma^2(\omega_-|V_n^{(\nu)}) = 0, \qquad \sup_\nu \max(\omega_-|V_n^{(\nu)}) < \infty.$$

(A3) For each $n = 0, 1, 2, \dots$ the limit

$$\alpha_{n+1} = \lim_\nu \frac{M(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty$$

exists and

$$\lim_\nu \frac{\Sigma^2(\omega_\circ|V_n^{(\nu)})}{\kappa(\nu)} = 0, \qquad \sup_\nu \frac{\max(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty.$$

Theorem 14.3.1 *Notations and assumptions being as above, let* $(\Gamma, \{\Psi_n\}, B^+, B^-)$ *be the interacting Fock space associated with* $\{\omega_n\}$ *and* B° *the diagonal operator defined by* $\{\alpha_n\}$*, where* $\{\omega_n\}$ *and* $\{\alpha_n\}$ *are given in conditions (A1)–(A3). Then we have*

$$\lim_\nu \left\langle \Phi_n^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_j^{(\nu)} \right\rangle = \langle \Psi_n, B^{\epsilon_m} \cdots B^{\epsilon_1} \Psi_j \rangle, \tag{14.11}$$

for any $\epsilon_1, \dots, \epsilon_m \in \{+, -, \circ\}$, $m = 1, 2, \dots,$ *and* $j, n = 0, 1, 2, \dots$.

Let μ be a probability distribution on $\mathbb{R}$ whose Jacobi coefficient is given by $(\{\omega_n\}, \{\alpha_n\})$, or equivalently the vacuum spectral distribution of $B^+ + B^- + B^\circ$. Let $\{P_n(s)\}$ be the orthogonal polynomials associated with μ. Note that $(\{\omega_n\}, \{\alpha_n\})$ does not necessarily determine μ uniquely, but $\{P_n(s)\}$ uniquely. Then (14.11) implies that

$$\lim_\nu \left\langle \Phi_n^{(\nu)}, \left(\frac{A_\nu}{\sqrt{\kappa(\nu)}} \right)^m \Phi_0^{(\nu)} \right\rangle = \frac{1}{\|P_n\|} \int_{-\infty}^{+\infty} P_n(s) s^m \mu(ds), \tag{14.12}$$

for any $m = 1, 2, \dots$ and $n = 0, 1, 2, \dots$. Therefore, for any polynomial $f(s)$ we have

$$\lim_\nu \left\langle \Phi_n^{(\nu)}, f\left(\frac{A_\nu}{\sqrt{\kappa(\nu)}} \right) \Phi_0^{(\nu)} \right\rangle = \frac{1}{\|P_n\|} \int_{-\infty}^{+\infty} P_n(s) f(s) \mu(ds). \tag{14.13}$$

We should like to replace $f(s)$ with e^{its}. Among others, a simple possible case is stated in the following

Theorem 14.3.2 *Notations and assumptions being as above, we assume in addition that* μ *is supported by a compact interval* $[-L, L]$. *If* $\|A_\nu\| \le \sqrt{\kappa(\nu)}\, L$ *for all* ν, *then* (14.13) *holds for any continuous function* $f \in C(\mathbb{R})$. *In particular,*

$$\lim_{\nu} \left\langle \Phi_n^{(\nu)}, \exp\left(it\,\frac{A_\nu}{\sqrt{\kappa(\nu)}}\right) \Phi_0^{(\nu)} \right\rangle = \frac{1}{\|P_n\|} \int_{-\infty}^{+\infty} P_n(s) e^{its} \mu(ds). \tag{14.14}$$

Proof. Given $f \in C(\mathbb{R})$, we may choose a polynomial $g(s)$ which approximates $f(s)$ uniformly on the interval $[-L, L]$, say $|f(s) - g(s)| < \epsilon$ for $|s| \le L$. Then

$$\left| \left\langle \Phi_n^{(\nu)}, f\left(\frac{A_\nu}{\sqrt{\kappa(\nu)}}\right) \Phi_0^{(\nu)} \right\rangle - \left\langle \Phi_n^{(\nu)}, g\left(\frac{A_\nu}{\sqrt{\kappa(\nu)}}\right) \Phi_0^{(\nu)} \right\rangle \right|$$
$$\le \left\| f\left(\frac{A_\nu}{\sqrt{\kappa(\nu)}}\right) - g\left(\frac{A_\nu}{\sqrt{\kappa(\nu)}}\right) \right\| \le \sup\left\{ |f(s) - g(s)|\,;\, |s| \le \frac{\|A_\nu\|}{\sqrt{\kappa(\nu)}} \right\}.$$

By assumption the last quantity is bounded by ϵ independently of ν. Then, by an obvious application of triangle inequality we get (14.13). ∎

Corollary 14.3.3 *Notations and assumptions being as in Theorem 14.3.2, we have*

$$\lim_{\nu} P\left(Y^{(\nu)}\left(\frac{t}{\sqrt{\kappa}}\right) = n \right) = \frac{1}{\|P_n\|^2} \left| \int_{-\infty}^{+\infty} P_n(s) e^{its} \mu(ds) \right|^2.$$

Example 14.3.4 A growing homogeneous tree T_κ satisfies conditions (A1)–(A3) with $\omega_n \equiv 1$ and $\alpha_n \equiv 0$. The corresponding distribution μ is the famous Wigner semicircle law, supported by $[-2, 2]$ and having the density $\sqrt{4 - s^2}/2\pi$. The corresponding orthogonal polynomilas $\{\tilde{U}_n(s)\}$ are given by

$$\tilde{U}_n(s) = U_n\left(\frac{s}{2}\right), \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$

where $\{U_n(x)\}$ are called the Chebychev polynomials of the second kind. Note that $\|\tilde{U}_n\| = 1$. Then (14.14) becomes

$$\lim_{\kappa\to\infty} \left\langle \Phi_n^{(\kappa)}, \exp\left(it\,\frac{A_\kappa}{\sqrt{\kappa}}\right) \Phi_0^{(\kappa)} \right\rangle = \frac{1}{2\pi} \int_{-2}^{2} \tilde{U}_n(s)\, e^{its} \sqrt{4 - s^2}\, ds.$$

The last integral is computed, by direct application of Gegenbauer's integral formula for the Bessel function:

$$\frac{1}{2\pi} \int_{-2}^{2} \tilde{U}_n(s)\, e^{its} \sqrt{4 - s^2}\, ds = (n+1)\, i^n \frac{J_{n+1}(2t)}{t}.$$

Consequently,

$$\lim_{\kappa\to\infty} P(Y(t) = n) = (n+1)^2\, \frac{J_{n+1}^2(2t)}{t^2}, \tag{14.15}$$

where informally,

$$Y(t) = \lim_{\kappa\to\infty} Y^{(\kappa)}\left(\frac{t}{\sqrt{\kappa}}\right).$$

Remark 14.3.5 For the distribution obtained in (14.15) we have further the asymptotics for a large t. The distribution of $Y(t)/t$ as $t \to \infty$ is given by

$$\frac{x^2}{\pi\sqrt{4-x^2}}\,\chi_{[0,2)}(x)dx.$$

The proof due to Konno is by characteristic function, i.e., based on the identity:

$$\lim_{t\to\infty} \mathbb{E}\left(\exp\left(iz\,\frac{Y(t)}{t}\right)\right) = \int_0^2 e^{izx}\,\frac{x^2}{\pi\sqrt{4-x^2}}\,dx.$$

14.4 Open Questions (Proposals)

1. Modify the argument in Example 14.3.4 to get concrete results for deformed vacuum states (Haagerup states).

2. Formulate the argument in Section 14.3 for a deformed vacuum states for a general growing regular graphs.

3. Concrete computation for Hamming graphs, Johnson graphs, and other distance-regular graphs. Some results are known for $H(d,2)$ (d-dimensional hypercube) as $d \to \infty$, C_N and K_N as $N \to \infty$, see Konno"s lecture notes.

4. Limit theorems for discrete-time quantum walks on homogeneous trees T_κ.

5. Relationship between discrete-time and continuous-time quantum walks.

6. Relation to a quantum random walk introduced by Biane.

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