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Chungbuk National University Lectures

Spectral Analysis of Large Networks:

Quantum Probabilistic Approach and Applications

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March–May 2010

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Preface

The so-called *network science* has grown to be a vast research area, creating a new paradigm to understand various complex networks appearing in physics, chemistry, biology, epidemiology, ecology, sociology, engneering, etc. For example, proteomics, one of the current big issues in system biology, needs a new mathematical approach to exploring the structure of protein-protein interaction. To describe and understand the nature of complex networks is a present issue, however, our goal is to establish a methodology of controlling its dynamics. These lectures, keeping our ambitious goal in mind, aims at mathematical foundation of complex networks with special emphasis of their spectral properties. Moreover, we will see how the quantum probabilistic ideas are useful in spectral analysis.

In the real world one finds networks in their basic form as interrelations among objects. Such networks are described in terms of graph theory, namely, objects under consderation being set as points in a plane and two objects in interrelation being connected by an arc therein, we obtain a geometric description of the network called a graph (in fact, the mathematical definition of a graph makes us to abandon even such a geometric image).

The graph theory, tracing back to Euler's famous problem on seven bridges in Königsberg, has become one of the main subjects in discrete mathematics. From mathematical point of view most attention has been paid to "beautiful" graphs, e.g., reasonable size for handling and/or possessing nice symmetry, but little to very large graphs in the real world because of being "dirty" or "complex." Examples of such dirty graphs are telephone networks, the internet (physical connections among PC's), the WWW (hyperlinks of webpages), Hollywood costars, coauthors of articles, human or social relations, biological networks, etc.

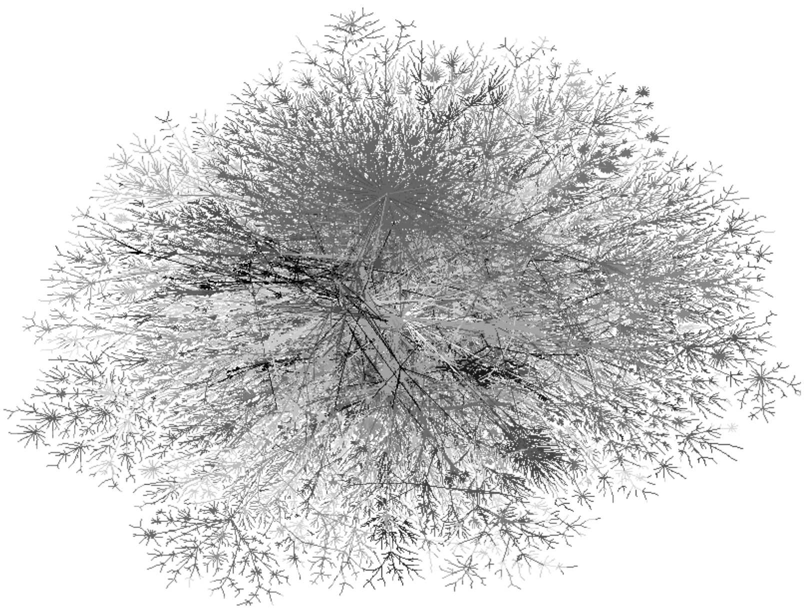


Figure 1: The internet

During the last decade as the development of computer technology, some characteristics became computable for very large networks in the real world. As a few physical quantities are

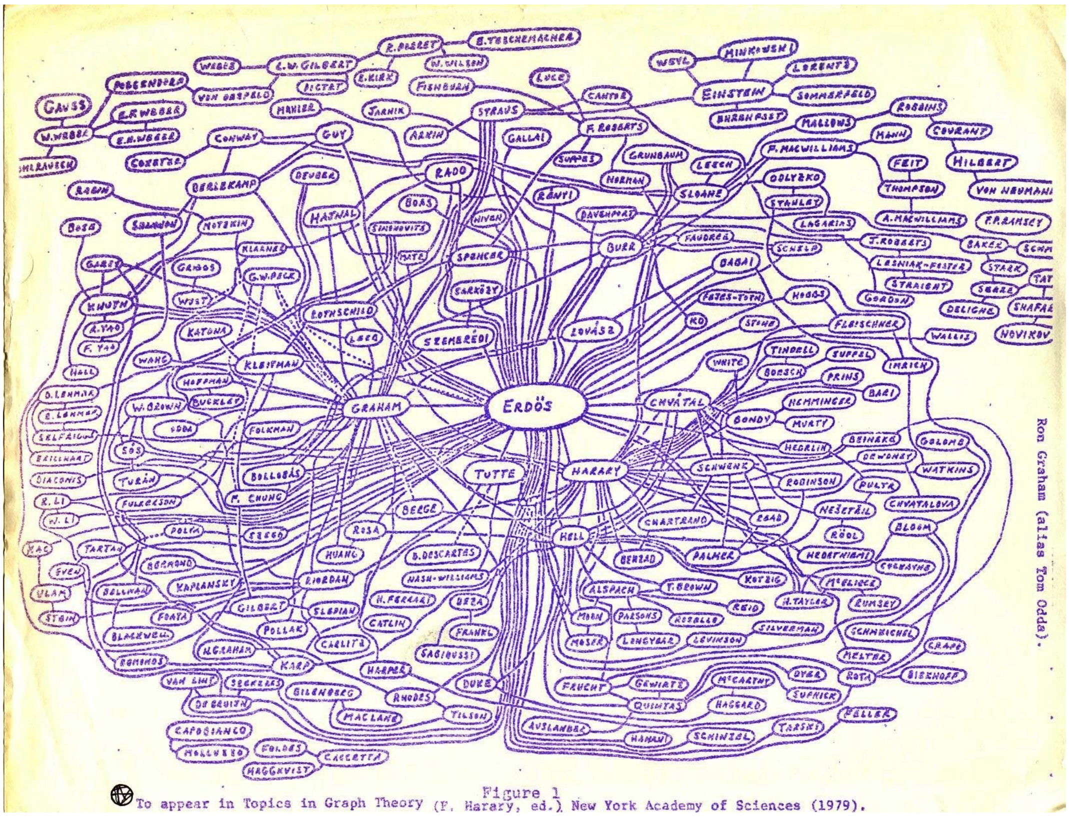


Figure 2: Paul Erdös' coauthors

used efficiently for description of gas in stead of the set of huge number of Newton equations, we believe reasonably that such large networks can be captured in terms of a small number of statistical characteristics carefully chosen. Up to now the prevailing characteristics of large complex networks in the real worlds are:

1. Small world phenomenon dating back to Stanley Milgram's small world experiment (1967), saying that the mean distance of two vertices is small $O(\log n)$ relative to the large number n of vertices.
2. Large cluster coefficient ($C \geq 0.7$), i.e., locally most vertices are connected each other.
3. Existence of hubs, as indicated by the long tail of the power law degree distribution $p(k) \propto k^{-\gamma}$ ($\gamma > 1$).

Mathematical models for complex networks were proposed in the following epoch-making papers:

[1] D. J. Watts and S. H. Strogatz: *Collective dynamics of 'small-world' networks*, Nature **393** (1998), 440–442.

[2] A.-L. Barabási and R. Albert: *Emergence of scaling in random networks*, Science **286** (1999), 509–512.

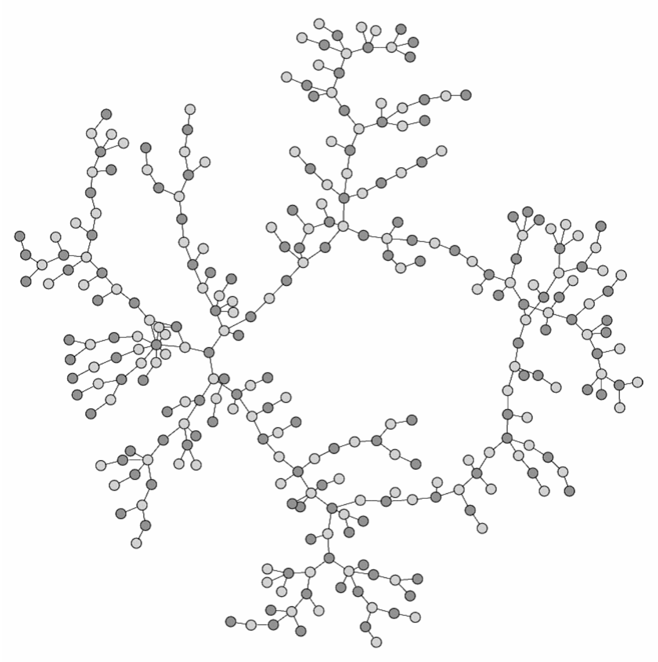


Figure 3: High school dating

Since then up to now many papers have been published with only few mathematical rigorous results. Our intention is to develop mathematical study of those models as well as to propose new models. For a mathematical model of a large complex network, a single graph seems to be not suitable. In order to capture characteristics of their large size we reasobnably take a growing graph and study its asymptotic behavior. And for characteristics of its complexity it is natural to consider statistical quantities of a random ensemble of graphs. In these lectures, therefore, one should keep in mind that a graph is intended to grow and/or to be random.

0.1 Quantum Probability = Noncommutative Probability

Quantum probability theory provides a framework of extending the measure-theoretical (Kolmogorovian) probability theory. The idea traces back to von Neumann (1932), who, aiming at the mathematical foundation for the statistical questions in quantum mechanics, initiated a parallel theory by making a selfadjoint operator and a trace play the roles of a random variable and a probability measure, respectively.

One of the main purposes of these lectures is to test the quantum probabilistic techniques in the study of large complex networks, in particular, their spectral properties.

0.2 From Coin-toss to Graph Spectrum

0.2.1 Classical probabilistic model

The toss of a fair coin is modelled by a random variable X on a probability space $(\Omega, \mathcal{F}, P)$ satisfying the property:

$$P(X = +1) = P(X = -1) = \frac{1}{2}$$

Rather than the random variable itself more essential is the probability distribution of X defined by

$$\mu = \frac{1}{2}\,\delta_{-1} + \frac{1}{2}\,\delta_{+1} \tag{0.1}$$

The moment sequence is one of the most fundamental characteristics of a probability measure. For μ in (0.1) the moment sequence is calculated with no difficulty as

$$M_m(\mu) = \int_{-\infty}^{+\infty} x^m \mu(dx) = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 0, & \text{otherwise.} \end{cases} \tag{0.2}$$

When we wish to recover a probability measure from the moment sequence, we meet in general a delicate problem called *determinate moment problem.* For the coin-toss there is no such an obstacle and we can recover the Bernoulli distribution from the moment sequence.

0.2.2 Quantum probabilistic (matrix) model

We set

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad e_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{0.3}$$

Then $\{e_0, e_1\}$ is an orthonormal basis of the two-dimensional Hilbert space $\mathbf{C}^2$ and A is a selfadjoint operator acting on it. It is straightforward to see that

$$\langle e_0, A^m e_0 \rangle = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 0, & \text{otherwise,} \end{cases} \tag{0.4}$$

which coincides with (0.2). In other words, the coin-toss is modeled also by using the two-dimensional Hilbert space $\mathbf{C}^2$ and the matrix A. In our terminology, letting $\mathcal{A}$ be the $*$-algebra generated by A, the coin-toss is modeled by an algebraic random variable A in an algebraic probability space $(\mathcal{A}, e_0)$. We call A an *algebraic realization* of the random variable X.

0.2.3 Noncommutative Structure

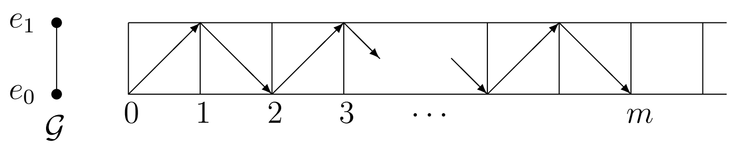
Once we come to an algebraic realization of a classical random variable, we are naturally led to the non-commutative paradigm. Let us consider the decomposition

$$A = A^+ + A^- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{0.5}$$

which yields a simple proof of (0.4). In fact, note first that

$$\langle e_0, A^m e_0 \rangle = \langle e_0, (A^+ + A^-)^m e_0 \rangle = \sum_{\epsilon_1,\dots,\epsilon_m \in \{\pm\}} \langle e_0, A^{\epsilon_m} \cdots A^{\epsilon_1} e_0 \rangle. \tag{0.6}$$

Let $\mathcal{G}$ be a connected graph consisting of two vertices e_0, e_1. Observing the obvious fact that (0.6) coincides with the number of m-step walks starting at and terminating at e_0 (see the figure below), we obtain (0.4).



Thus, computation of the mth moment of A is reduced to counting the number of certain walks in a graph through (0.5). This decomposition is in some sense canonical and is called the *quantum decomposition* of A.

0.2.4 Relation to Graph

We now note that A in (0.3) is the adjacency matrix of the graph $\mathcal{G}$. Having established the identity

$$\langle e_0, A^m e_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots, \tag{0.7}$$

we say that μ is the spectral distribution of A in the state e_0. In other words, we obtain an integral expression for the number of returning walks in the graph by means of such a spectral distribution. A key role in deriving (0.7) is again played by the quantum decomposition.

0.3 Quantum Probabilistic Approach

For (in particular, asymptotic) spectral analysis some techniques peculiar to quantum probability seem to be useful. They are

(a) quantum decomposition (using noncommutative structure behind)

(b) various concepts of independence and corresponding quantum central limit theorems

(c) partition statistics for computing the moments of spectral distributions

A basic reference throughout these lectures is:

[3] A. Hora and N. Obata: Quantum Probability and Spectral Analysis of Graphs, Springer, 2007.

References

[1] D. J. Watts and S. H. Strogatz: *Collective dynamics of 'small-world' networks*, Nature **393** (1998), 440–442.

[2] A.-L. Barabási and R. Albert: *Emergence of scaling in random networks*, Science **286** (1999), 509–512.

[3] A. Hora and N. Obata: Quantum Probability and Spectral Analysis of Graphs, Springer, 2007.

[4] B. Bollobás: Modern Graph Theory, Graduate Texts in Mathematics Vol. 184, Springer-Verlag, New York, 1998.

[5] N. Biggs: Algebraic Graph Theory (2nd Edition), Cambridge University Press, Cambridge, 1993.

[6] C. Godsil and G. Royle: Algebraic Graph Theory, Springer, 2001.

[7] D. M. Cvetković, M. Doob and H. Sachs: Spectra of Graphs: Theory and Applications (3rd rev. enl. ed.), New York, Wiley, 1998.

[8] L. Collatz and U. Sinogowitz: *Spektren endlicher Grafen*, Abh. Math. Sem. Univ. Hamburg **21** (1957), 63–77.

[9] C. D. Godsil and B. D. McKay: *Constructing cospectral graphs*, Aeq. Math. **25** (1982), 257–268.

[1–2] are epoch-making papers on mathematical models for complex networks. [3] is our basic textbook. [4] is a standard textbook on graph theory. [5–6] is a standard textbook on algebraic graph theory. [7] is a comprehensive book about graph spectra. [8–9] are for original references.

1 Graphs and Matrices

1.1 Graphs

Definition 1.1.1 Let V be a non-empty set and E a subset of $\{\{x,y\}\,;\,x,y\in V,\,x\neq y\}$. Then the pair $G=(V,E)$ is called a *graph* with vertes set V and the edge set E. An element of V is called a *vertex* and an element of E an *edge*. We say that two vertices $x,y\in V$ are adjacent, denoted by $x\sim y$, if $\{x,y\}\in E$.

A *geometric representation* of a graph $G=(V,E)$ is a figure obtained by assigning each $x\in V$ to a point in a plane and drawing a line (or an arc) between two planer points if they are adjacent in G. Appearance of the geometric representation of a graph varies widely. For example, the following two figures represent the same graph.

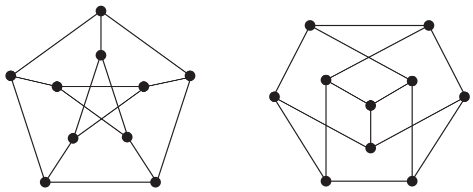


Figure 1.1: Two geometric representation of the Petersen graph

Definition 1.1.2 A graph $G=(V,E)$ is called *finite* if V is a finite set, i.e., $|V|<\infty$.

Definition 1.1.3 For a vertex $x\in V$ of a graph G we set

$$\deg(x)=\deg_G(x)=|\{y\in V\,;\,y\sim x\}|,$$

which is called the *degree* of x.

Definition 1.1.4 A graph $G=(V,E)$ is called *localy finite* if $\deg(x)<\infty$ for all $x\in V$.

Definition 1.1.5 A graph $G=(V,E)$ is called *regular* if every vertex has a constant finite degree, i.e., if there exists a constant number κ such that $\deg(x)=\kappa$ for all $x\in V$. To be more precise, such a graph is called κ-regular.

Definition 1.1.6 A finite sequence of vertices $x_0,x_1,\dots,x_n\in V$ is called a *walk* of length n if

$$x_0\sim x_1\sim\cdots\sim x_n\,, \tag{1.1}$$

where some of $x_0,x_1,\dots,x_n$ may coincide. A walk (1.1) is called a *path* of length n if $x_0,x_1,\dots,x_n$ are distinct from each other. A walk (1.1) is called a *cycle* of length $n\geq 3$ if $x_0,x_1,\dots,x_{n-1}$ are distinct from each other and $x_n=x_0$.

In usual we do not consider an orientation of a path. Namely, if (1.1) is a path,

$$x_n \sim x_{n-1} \sim \cdots \sim x_0$$

is the same path. For a cycle, we do not consider the initial vertex either. Namely, if $x_0 \sim x_1 \sim \cdots \sim x_{n-1} \sim x_0$ is a cycle, then $x_1 \sim x_2 \sim \cdots \sim x_{n-1} \sim x_0 \sim x_1$ stands for the same cycle.

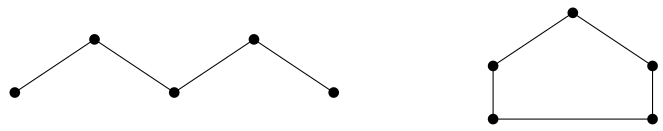


Figure 1.2: P_5: path of length 4 (left). C_5: cycle of length 5 (right)

Definition 1.1.7 A graph $G = (V, E)$ is *connected* if every pair of distinct vertices $x, y \in V$ $(x \neq y)$ are connected by a walk (or equivalently by a path).

Definition 1.1.8 Two graphs $G = (V, E)$ and $G' = (V', E')$ are called *isomorphic* if there exists a bijection $f : V \to V'$ satisfying

$$x \sim y \quad \Longleftrightarrow \quad f(x) \sim f(y).$$

In that case we write $G \cong G'$.

Definition 1.1.9 Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We say that G' is a *subgraph* of G if $V' \subset V$ and $E' \subset E$.

In fact, a path and a cycle defined in Definition 1.1.6 are subgraphs. We denote by P_n and C_n a path and a cycle with n vertices, respectively.

1.2 Adjacency Matrices

1.2.1 Definition

Let V and V' be arbitrary non-empty set. A function $a : V \times V' \to \mathbf{R}$ is regarded as a *matrix A indexed by $V \times V'$* in the sense that the matrix element of A is defined by $(A)_{xy} = a(x, y)$. In this case we write $A = (a_{xy})$ too.

Definition 1.2.1 Let $G = (V, E)$ be a graph. A matrix $A = (a_{xy})$ indexed by $V \times V$ is called the *adjacency matrix* of G if

$$a_{xy} = \begin{cases} 1, & \text{if } x \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1.2.2 *Let $G = (V, E)$ be a graph and A its adjacency matrix. Then, A is a matrix indexed by $V \times V$ satisfying the following conditions:*

(i) $(A)_{xy} \in \{0, 1\}$;

(ii) $(A)_{xy} = (A)_{yx}$;

(iii) $(A)_{xx} = 0$.

Conversely, if a matrix $A = (a_{xy})$ indexed by $V \times V$, V being a non-empty set, satisfies the above three conditions, then A is the adjacency matrix of a graph G with V being the vertex set.

PROOF. Obvious. ∎

A matrix S indexed by $V \times V'$ is called a *permutation matrix* if

(i) $(S)_{xy'} \in \{0, 1\}$;

(ii) $\sum_{y' \in V'} (S)_{xy'} = 1$ for all $x \in V$;

(iii) $\sum_{x \in V} (S)_{xy'} = 1$ for all $y' \in V'$.

If S is a permutation matrix, it is necessary that $|V| = |V'|$.

The transposed matrix S^T is defined in a usual manner: $(S^T)_{y'x} = S_{xy'}$ for $x \in V$ and $y' \in V'$. Then $S^T = S^{-1}$ in the sense that SS^T is the identity matrix indexed by $V \times V$ and $S^T S$ is the identity matrix indexed by $V' \times V'$.

Lemma 1.2.3 *Let A and A' be the adjacency matrices of graphs $G = (V, E)$ and $G' = (V', E')$, respectively. Then $G \cong G'$ if and only if there exists a permutation matrix S indexed by $V \times V'$ such that $A' = S^{-1}AS$*

PROOF. Suppose that $G \cong G'$. We choose an isomorphism $f : V \to V'$ and define a matrix S indexed by $V \times V'$ by

$$(S)_{xy'} = \begin{cases} 1, & \text{if } y' = f(x), \\ 0, & \text{otherwise.} \end{cases}$$

We see easily that S is a permutation matrix satisfying $SA' = AS$.

Conversely, suppose that a permutation matrix S indexed by $V \times V'$ verifies $A' = S^{-1}AS$. Then a bijection $f : V \to V'$ is defined by the condition that

$$(A)_{xy} = \begin{cases} 1, & \text{if } y = f(x), \\ 0, & \text{otherwise.} \end{cases}$$

It is then easy to see that f becomes an isomorphism betwen G and G'. ∎

1.2.2 Representing the Adjacency Matrix in a Usual Form

In order to represent the adjacency matrix A of a graph $G = (V, E)$ in a usual form of $n \times n$ square matrix, where $n = |V|$, we need numbering the vertices. This is performed by taking a bijection $f : V \to \{1, 2, \dots, n\} = V'$. Then we obtain a graph $G' = (V', E')$ in such a way that $\{i, j\} \in E'$ if and only if $\{f^{-1}(i), f^{-1}(j)\} \in E$. By definition we have $G \cong G'$. The adjacency matrix A' of G' is indexed by $V' \times V'$ and admits a usual expression of a square matrix. It follows from Lemma 1.2.3 that A and A' are related as $A = SA'S^{-1}$.

Consider another numbering, that is, another bijection $f_1 : V \to \{1, 2, \dots, n\} = V'$. Then we obtain another square matrix A'_1 as the adjacency matrix of G'_1, which is related to A as $A = S_1 A'_1 S_1^{-1}$. Then we have

$$S_1 A'_1 S_1^{-1} = SA'S^{-1}$$

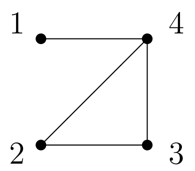
so that

$$A'_1 = S_1 S A' (S_1 S)^{-1}.$$

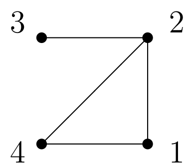
Note that $S_1 S$ is a usual permutation matrix on $\{1, 2, \dots, n\}$. Consequently,

Lemma 1.2.4 *Let A, A' be the adjacency matrices of a graph G obtained from two ways of numbering the vertices. Then there exists a permutation matrix on $\{1, 2, \dots, n\}$, $n = |V|$, such that $A' = S^{-1}AS$.*

Example 1.2.5 We obtain "different" adjacency matrices by different numbering the vertices of the same graph.



$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

1.2.3 Some Properties in Terms of Adjacency Matrices

All the information of a graph (up to isomorphism) are obtained from its adjacency matrix.

(1) A graph $G = (V, E)$ is not connected if and only if there exists a numbering the vertices such that the adjacency matrix admits a block diagonal expression of the form:

$$A = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix} \qquad (A_1,\ A_2 \text{ are square matrices})$$

In this case A_1 and A_2 are the adjacency matrices of subgraphs of G which are not connected.

(2) A graph is called *complete* if every pair of vertices are connected by an edge. A comlete graph with n vertices is denoted by K_n. A graph is complete if and only if the adjacency matrix is of the form:

$$A = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & \cdots & & 0 & 1 \\ 1 & \cdots & & 1 & 0 \end{bmatrix}$$

(3) A graph $G = (V, E)$ is called *bipartite* if V admits a partition $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, such that any pair of vertices in a common V_i does not constitute an edge. A graph is bipartite if and only if the adjacency matrix admits a block diagonal expression of the form:

$$A = \begin{bmatrix} O & B \\ B^T & O \end{bmatrix} \qquad \text{(two zero matrices are square matrices).}$$

(4) A graph $G = (V, E)$ is called *complete bipartite* if it is bipartite and every pair of vertices $x \in V_1$, $y \in V2$ constitute an edge. In that case we write $G = K_{m,n}$ with $m = |V_1|$ and $n = |V_2|$. In particular, $K_{1,n}$ is called a *star*.

A graph is complete bipartite if and only if the adjacency matrix is of the form:

$$A = \begin{bmatrix} O & B \\ B^T & O \end{bmatrix} \qquad \text{(all elements of } B \text{ are 1).}$$

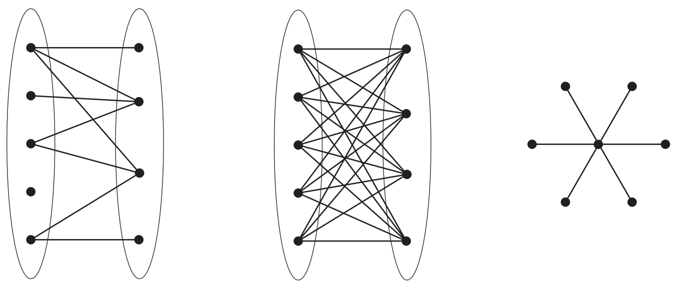


Figure 1.3: Bipartite graph, complete bipartite graph $K_{4,5}$, star $K_{1,6}$

1.3 Characteristic Polynomials

Let $G = (V, E)$ be a finite graph with $|V| = n$. Numbering the vertices, we write down its adjacency matrix in the usual form of an $n \times n$ matrix, say A. The characteristic polynomial of A is defined by

$$\varphi_A(x) = |xE - A| \, (= \det(xE - A)).$$

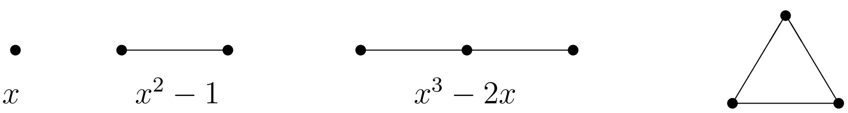
It is noted that $\varphi_A(x)$ is determined independently of the numbering. In fact, let A' be the adjacenct matrix obtained by a different numbering. From Lemma 1.2.4 we know that $A' = S^{-1}AS$ with a permutation matrix S. Then,

$$\varphi_{A'}(x) = |xE - A'| = |xE - S^{-1}AS| = |S^{-1}(xE - A)S| = |S^{-1}||xE - A||S| = \varphi_A(x).$$

We call $\varphi_A(x)$ the *characteristic polynomial* of G and denote it by $\varphi_G(x)$. Obviously, $\varphi_G(x)$ is a polynomial of degree n of the form:

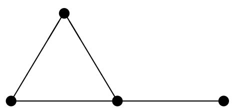
$$\varphi_G(x) = x^n + c_1x^{n-1} + c_2x^{n-2} + c_3x^{n-3} + \cdots . \tag{1.2}$$

Example 1.3.1 Simple examples are:



x $\qquad x^2 - 1$ $\qquad x^3 - 2x$ $\qquad x^3 - 3x - 2$

Example 1.3.2 One more example. The characteristic polynomial of the following graph is $\varphi(x) = x^4 - 4x^2 - 2x + 1$.



Theorem 1.3.3 *Let the characteristic polynomial of a finite graph G be given as in* (1.2)*. Then,*

(1) $c_1 = 0$.

(2) $-c_2 = |E|$.

(3) $-c_3 = 2\triangle$, *where $\triangle$ is the number of triangles in G.*

Proof. Let $A = [a_{ij}]$ be the adjacency matrix of G written down in the usual form of $n \times n$ matrix after numbering the vertices. Noting that the diagonal elements of A vanish, we see that the characteristic polynomial of G is given by

$$\varphi_G(x) = |xE - A| = \begin{vmatrix} x & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & x & \cdots & -a_{2n} \\ \vdots & & \ddots & \vdots \\ -a_{n1} & \cdots & \cdots & x \end{vmatrix}.$$

For simplicity, the matrix in the right-hand side is denoted by $B = [b_{ij}]$. We then have

$$\varphi_G(x) = |B| = \sum_{\sigma \in \mathcal{S}(n)} \operatorname{sgn}(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)}. \tag{1.3}$$

For $\sigma \in \mathcal{S}_n$ we set

$$\operatorname{supp}\sigma = \{i \,|\, \sigma(i) \neq i\}.$$

Then (1.3) becomes

$$\varphi_G(x) = \sum_{k=0}^{n} \sum_{\substack{\sigma\in\mathcal{S}(n)\\ |\operatorname{supp}\sigma|=k}} \operatorname{sgn}(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)} \equiv \sum_{k=0}^{n} f_n(x) \tag{1.4}$$

Since the indeterminat x appears only in the diagonal of B, we see that $f_n(x) = c_k x^{n-k}$.

(1) $k = 1$. Since there is no permutation σ such that $|\operatorname{supp}\sigma| = 1$, we have $c_1 = 0$.

(2) $k = 2$. The permutations σ satisfying $|\operatorname{supp}\sigma| = 2$ are parametrized as $\sigma = (i\ j)$ $(1 \le i < j \le n)$. For such a permutation we have $\operatorname{sgn}(\sigma) = -1$. Hence we have

$$f_2(x) = \sum_{1\le i<j\le n} (-1)(-a_{ij})(-a_{ji})x^{n-2} = -\sum_{1\le i<j\le n} a_{ij}x^{n-2}$$

where we used $a_{ij}a_{ji} = a_{ij}^2 = a_{ij}$. Therefore, $c_2 = -|E|$.

(3) $k = 3$. The permutations σ satisfying $|\operatorname{supp}\sigma| = 3$ are parametrized as

$$\sigma = (i\ j\ k), \quad \sigma = (i\ k\ j), \quad 1 \le i < j < k \le n.$$

Noting that $\operatorname{sgn}(\sigma) = 1$ for such cyclic permutations, we have

$$f_3(x) = -\sum_{1\le i<j<k\le n} (a_{ij}a_{jk}a_{ki} + a_{ik}a_{kj}a_{ji})x^{n-3}.$$

We see that $a_{ij}a_{jk}a_{ki}$ takes values 1 or 0 according as three vertices i, j, k forms a triangle or not. The same situation occuring for the second term, we conclude that $-c_3 = 2\triangle$. ∎

1.4 The Path Graph P_n and Chebyshev Polynomials

Let $V = \{1, 2, \dots, n\}$ and $E = \{\{i, i+1\}\,;\ i = 1, 2, \dots, n-1\}$. The graph (V, E) is called a *path* with n vertices and is denoted by P_n.

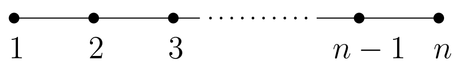


Figure 1.4: Path P_n

Lemma 1.4.1 *Let $\varphi_n(x) = \varphi_{P_n}(x)$ be the characteristic polynomial of the path P_n. The it holds that*

$$\begin{aligned} &\varphi_1(x) = x, \\ &\varphi_2(x) = x^2 - 1, \\ &\varphi_n(x) = x\varphi_{n-1}(x) - \varphi_{n-2}(x), \quad n \ge 3 \end{aligned} \tag{1.5}$$

PROOF. We have already seen in Example 1.3.1 that

$$\varphi_1(x) = x, \quad \varphi_2(x) = x^2 - 1.$$

Let us compute $\varphi_n(x)$ for $n \geq 3$. By definition we have

$$\varphi_n(x) = \begin{vmatrix} x & -1 & & & & \\ -1 & x & -1 & & & \\ & -1 & x & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & x & -1 \\ & & & & -1 & x \end{vmatrix}.$$

By cofactor expansion with respect to the first column, we get

$$\begin{aligned} \varphi_n(x) &= \lambda\varphi_{n-1}(x) + \begin{vmatrix} -1 & -1 & & & & \\ & x & -1 & & & \\ & -1 & x & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & x & -1 \\ & & & & -1 & x \end{vmatrix} \\ &= x\varphi_{n-1}(x) - \varphi_{n-2}(x), \end{aligned}$$

as desired. ∎

Setting $\varphi_0(x) = 1$, we may understand that the reccurence relation in (1.5) holds for $n \geq 2$.

Lemma 1.4.2 *For $n = 0, 1, 2, \dots$ there exists a polynomial $U_n(x)$ such that*

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{1.6}$$

Moreover, $U_n(x)$ satisfies the following reccurence relation:

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0. \tag{1.7}$$

PROOF. By elementary knowledge of trigonometric functions. ∎

Definition 1.4.3 The series of polynomials $U_n(x)$ is called the *Chebyshev polynomial of the second kind.*

Theorem 1.4.4 *The characteristic polynomial of the path P_n is given by $U_n(x/2)$.*

PROOF. Let $\varphi_n(x)$ be the characteristic polynomial of P_n. We see easily from (1.5) and (1.7) that the reccurence relations of $\varphi_n(x)$ and $U_n(x/2)$ coincide together with the initial conditions. ∎

1.5 Laplacians, Transition Matrices, Q-matrices

With a given graph $G = (V, E)$ we associate various matrices in addition to the adjacency matrices.

Definition 1.5.1 The *Laplacian* of a locally finite graph $G = (V, E)$ is a matrix L defined by

$$(L)_{xy} = (A)_{xy} - \delta_{xy}\deg(x), \qquad x, y \in V.$$

Or equivalently,

$$L = A - D$$

where D is the diagonal matrix defined by

$$(D)_{xy} = \begin{cases} \deg(x), & x = y, \\ 0, & \text{otherwise.} \end{cases}$$

In some literatures, the Laplacian is defined to be $-L = D - A$.

Definition 1.5.2 *A function $f : V \to \mathbf{C}$ is called harmonic if $Lf = 0$.*

Theorem 1.5.3 *$Lf = 0$ if and only if*

$$f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y), \qquad x \in V, \quad , \deg(x) \geq 1.$$

PROOF. By definition $Lf = 0$ if and only if $Df = Af$. On the other hand, we know that

$$Df(x) = \deg(x)\, f(x),$$
$$Af(x) = \sum_{y \in V} (A)_{xy} f(y) = \sum_{y \sim x} f(y).$$

Hence the assertion follows. ∎

Remark 1.5.4 Let $G = (V, E)$ be a graph. We may give an orientation to each edges. In other words, we may define a pair of maps $i, t : E \to V$ such that $e = \{i(e), t(e)\}$. We call $i(e)$ and $t(e)$ the initial vertex of e and the terminal vertex of e, respectively. Fix such an orientation. Now define the coboundary operator $d : C(V) \to C(E)$ by

$$df(e) = f(t(e)) - f(o(e)).$$

Then we have

$$\langle df, dg \rangle = -\langle f, Lg \rangle, \qquad f, g \in C(V).$$

In other words, $-L = d^* d$ holds.

Definition 1.5.5 The *transition matrix* of a locally finite graph $G = (V, E)$ is a matrix T defined by

$$(T)_{xy} = \begin{cases} \dfrac{1}{\deg(x)}, & y \sim x, \\ 0, & \text{otherwise.} \end{cases}$$

The transition matrix T is nothing else the transition matrix of the isotropic random walk on the graph G, namely, the (time homogeneous) Markov chain $\{X_n\}$ on the state space V with transition probability

$$(T)_{xy} = P(X_n = y | X_{n-1} = x).$$

In this context, $I - T$ is called the *Laplacian* of the random walk.

Definition 1.5.6 The *Q-matrix* of a connected graph $G = (V, E)$ is defined by

$$(Q)_{xy} = q^{\partial(x,y)}, \qquad x, y \in V,$$

where q is a parameter and $\partial(x, y)$ the graph distance.

1.6 Generalization of Graphs

(1) Directed graph. One may consider naturally the case where every edge of a graph is given a direction. Such an object is called a *directed graph*. In terms of the adjacency matrix A, a directed graph is characterized by the following properties:

(i) $(A)_{xy} \in \{0, 1\}$;

(ii) $(A)_{xy} = 1$ implies $(A)_{yx} = 0$;

(iii) $(A)_{xx} = 0$.

(2) Multigraph. In its geometric representation one may allow to draw two or more edges connecting two vertices (multi-edge) and one or more arcs connecting a vertex with itself (loop). In terms of the adjacency matrix A, a directed graph is characterized by the following properties:

(i) $(A)_{xy} \in \{0, 1, 2, \dots\}$;

(ii) $(A)_{xy} = (A)_{yx}$.

Moreover, each edge may be given a direction to obtain a directed multigraph.

(3) Network. An arbitrary matrix gives rise to a graph where each directed edge $\overrightarrow{xy}$ is associated with the value A_{xy}. Such an object is called generally a *network*. A transition diagram of a Markov chain is an example.

In regard to (1) and (2), a graph in these lectures is sometimes called a *undirected simple graph*.

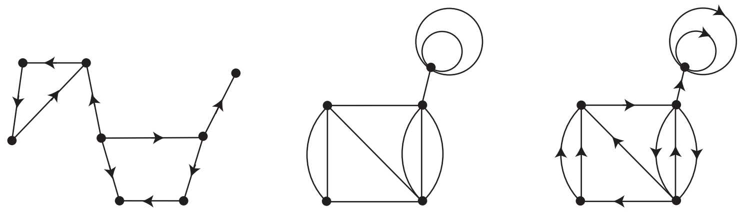
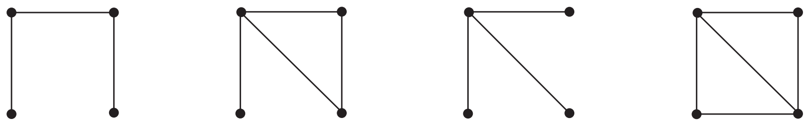


Figure 1.5: Directed graph, multigraph, directed multigraph.

Exercises 1

1.1. Find the adjacency matrices and the characteristic polynomials of the following graphs.



1.2. Examine the numbers of vertices, edges, and triangles of the above graphs in terms of characteristic polynomals.

1.3. Compute the characteristic polynomial of the complete graph K_n. Ans. $\varphi(x) = (x - (n-1))(x+1)^{n-1}$.

1.4*. Let $G = (V, E)$ be a graph with a vertex a of degree one. Let $b \in V$ be a unique vertex adjacent to a. Let $G' = G[V \backslash \{a\}]$, $G'' = G[V \backslash \{a, b\}]$ be induced subgraphs obtained by deleting $\{a\}$ and $\{a, b\}$, respectively. Prove that

$$\varphi_G(x) = x\varphi_{G'}(x) - \varphi_{G''}(x).$$

Examine this formula by examples.

References

[4] N. Biggs: Algebraic Graph Theory (2nd Edition), Cambridge University Press, Cambridge, 1993.

[5] B. Bollobás: Modern Graph Theory, Graduate Texts in Mathematics Vol. 184, Springer-Verlag, New York, 1998.

2 Spectra of Graphs

2.1 Spectra

Let $G = (V, E)$ be a finite graph with $|V| = n$ and let A be the adjacency matrix represented in a usual form of $n \times n$ matrix after numbering the vertices. Since A becomes a real symmetric matrix, its eigenvalues are all real, say, $\lambda_1 < \lambda_2 < \cdots < \lambda_s$. Then, the characteristic polynomial of G is factorized as

$$\varphi_G(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_s)^{m_s}, \tag{2.1}$$

where $m_i \geq 1$ (called the multiplicity of λ_i) and $\sum_i m_i = n$.

Definition 2.1.1 Let $G = (V, E)$ be a finite graph and let $\varphi_G(x)$ its characteristic polynomial in the form (2.1). The the array

$$\mathrm{Spec}\,(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ m_1 & m_2 & \cdots & m_s \end{pmatrix} \tag{2.2}$$

is called the *spectrum* of G. Each λ_i is called an *eigenvalue* of G and m_i its *multiplicity.*

In fact, (2.2) is nothing else the spectrum of the adjacency matrix A. Obviously, (2.2) does not depend on the choice of numbering vertices. Moreover,

Lemma 2.1.2 *If* $G \cong G'$, *then* $\mathrm{Spec}\,(G) = \mathrm{Spec}\,(G')$.

Remark 2.1.3 The converse assertion of Lemma 2.1.2 is not valid, however, it is known that the converse is true for graphs with four or less vertices. In Section 2.6 we show examples of two non-isomorphic graphs whose spectra coincide.

Example 2.1.4 Here are some simple examples.

$$\mathrm{Spec}\,(\,\bullet\,) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \mathrm{Spec}\,(\,\bullet\!\!-\!\!\!-\!\!\bullet\,) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\mathrm{Spec}\,(\,\bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet\,) = \begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathrm{Spec}\left(\triangle\right) = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Theorem 2.1.5 *The spectrum of the path* P_n *is given by*

$$\mathrm{Spec}\,(P_n) = \begin{pmatrix} 2\cos\dfrac{\pi}{n+1} & \cdots & 2\cos\dfrac{k\pi}{n+1} & \cdots & 2\cos\dfrac{n\pi}{n+1} \\ 1 & \cdots & 1 & \cdots & 1 \end{pmatrix}$$

PROOF. First we find the zeroes of the Chebyshev polynomial of the second kind. By definition,

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta.$$

In view of the right-hand side we see easily that $U_n(x) = 0$ if

$$\theta = \frac{k\pi}{n+1}, \qquad k = 1, 2, \dots, n.$$

For these θ, $\cos\theta$ are mutually distinct. Thus

$$x_k = \cos\frac{k\pi}{n+1}, \qquad k = 1, 2, \dots, n, \tag{2.3}$$

form n different zeroes of $U_n(x)$. Since $U_n(x)$ is a polynomial of degree n, (2.3) exhaust the zeroes of $U_n(x)$ and each x_k has multiplicity one.

By Theorem 1.4.4 the characteristic polynomial of P_n is given by $U_n(x/2)$. For the spectrum of P_n it is sufficient to find its zeroes. From the above argument we see that the zeroes of $U_n(x/2)$ are

$$\lambda_k = 2\cos\frac{k\pi}{n+1}, \qquad k = 1, 2, \dots, n,$$

each of which is of multiplicity one. This shows the assertion. ∎

2.2 Number of Walks

Let A be the adjacency matrix of a locally finite graph $G = (V, E)$. Then for any $m = 1, 2, \dots$ and $x, y \in V$ the matrix element $(A^m)_{xy}$ is defined as usual by

$$(A^m)_{xy} = \sum_{z_1,\dots,z_{m-1}\in V} (A)_{xz_1}(A)_{z_1z_2}\cdots(A)_{z_{m-1}y}\,.$$

Note that

$$(A)_{xz_1}(A)_{z_1z_2}\cdots(A)_{z_{m-1}y} = \begin{cases} 1, & \text{if } x \sim z_1 \sim \cdots \sim z_{m-1} \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $(A^m)_{xy}$ is the number of walks of length m connecting x and y. If the graph G is locally finite, $(A^m)_{xy} < \infty$. Therefore, the powers of A is well-defined.

We record the above result in the following

Lemma 2.2.1 *Let $G = (V, E)$ be a locally finite graph and A its adjacency matrix. Then, for any $m = 1, 2, \dots$ and $x, y \in V$, the matrix element $(A^m)_{xy}$ coincides with the number of walks of length m connecting x and y.*

Theorem 2.2.2 *Let $G = (V, E)$ be a finite graph and $\lambda_1 < \lambda_2 < \cdots < \lambda_s$ exhaust its eigenvalues. For $x, y \in V$ there exist constant numbers $c_i = c_i(x, y)$ $(i = 1, 2, \dots, s)$ such that*

$$(A^m)_{xy} = \sum_{i=1}^{s} c_i(x, y)\lambda_i^m\,.$$

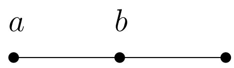
Here we tacitly understand that $0^0 = 1$ when $\lambda_i = 0$.

PROOF. The first equality is due to Lemma 2.2.1. For the second equality we consider the diagonalization of A. In fact, since A is real symmetric, taking a suitable orthogonal matrix U we have

$$A = U \begin{bmatrix} \lambda_1 E_{m_1} & & \\ & \ddots & \\ & & \lambda_s E_{m_s} \end{bmatrix} U^{-1}.$$

It is then obvious that evey element of (A^m) is a linear combination of $\lambda_1^m, \dots, \lambda_s^m$. ∎

Example 2.2.3 Let us compute the number of m-step walks connecting a and b:



We know the spectrum of the graph:

$$\begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}.$$

Hence

$$N_m(a,b) = c_1(-\sqrt{2})^m + c_2 0^m + c_3(\sqrt{2})^m$$

with some constants c_1, c_2, c_3. For small m's we see easily that

$$N_0(a,b) = 0, \qquad N_1(a,b) = 1, \qquad N_2(a,b) = 0.$$

Hence

$$\begin{aligned} &c_1 + c_2 + c_3 = 0 \\ &-\sqrt{2}\, c_1 + \sqrt{2}\, c_3 = 1 \\ &2c_1 + 2c_3 = 0 \end{aligned}$$

Solving these equations we obtain

$$N_m(a,b) = \begin{cases} 0, & m \ge 0 \text{ is even}, \\ 2^{(m-1)/2}, & m \ge 1 \text{ is odd}. \end{cases}$$

2.3 Maximal Eigenvalue

It is important to know a bound of $\mathrm{Spec}\,(G)$. Let $\lambda_{\max}(G)$ and $\lambda_{\min}(G)$ denote the maximal and minimal eigenvalues of G, respectively. We shall show a simple estimate of $\lambda_{\max}(G)$.

Some statistics concerning the degrees of vertices play an interesting role. We set

$$\begin{aligned} d_{\max}(G) &= \max\{\deg(x) \mid x \in V\}, \\ d_{\min}(G) &= \min\{\deg(x) \mid x \in V\}, \\ \bar{d}(G) &= \frac{1}{|V|} \sum_{x \in V} \deg(x). \end{aligned}$$

Obviously,

$$d_{\min}(G) \le \bar{d}(G) \le d_{\max}(G).$$

Theorem 2.3.1 *For a finite graph $G = (V, E)$ it holds that*

$$d_{\min}(G) \le \bar{d}(G) \le \lambda_{\max}(G) \le d_{\max}(G).$$

PROOF. We regard the adjacency matrix A as a linear transformation on $\mathbf{C}^n$.

1° We prove $\bar{d}(G) \le \lambda_{\max}(G)$. Let $\boldsymbol{v} = [v_i] \in \mathbf{C}^n$ be the vector whose elements are all one. Then,

$$\langle \boldsymbol{v}, A\boldsymbol{v} \rangle = \sum_{i=1}^{n} \overline{v_i}\,(A\boldsymbol{v})_i = \sum_{i,j=1}^{n} \overline{v_i}\,(A)_{ij} v_j = \sum_{i,j=1}^{n} (A)_{ij} = \sum_{i \in V} d(i).$$

Since $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = n = |V|$, we have

$$\frac{\langle \boldsymbol{v}, A\boldsymbol{v} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v} \rangle} = \frac{1}{|V|} \sum_{i \in V} d(i) = \bar{d}(G). \tag{2.4}$$

On the other hand, it is known from knowledge of linear algebra that

$$\lambda_{\min}(A) \le \frac{\langle \boldsymbol{u}, A\boldsymbol{u} \rangle}{\langle \boldsymbol{u}, \boldsymbol{u} \rangle} \le \lambda_{\max}(A) \quad \text{for all } \boldsymbol{u} \neq \mathbf{0}. \tag{2.5}$$

Combining (2.4) and (2.5), we come to

$$\bar{d}(G) \le \lambda_{\max}(A) = \lambda_{\max}(G).$$

2° We show $\lambda_{\max}(G) \le d_{\max}(G)$. Since $\lambda_{\max}(G)$ is real, we may choose its eigenvector $\boldsymbol{u} = [u_i]$ whose elements are all real. Then, for any i we have $(A\boldsymbol{u})_i = \lambda_{\max} u_i$. Multiplying a constant, we may assume that

$$\alpha \equiv \max\{u_i\,;\, i = 1, 2, \dots, n\} > 0$$

and choose i_0 such that $u_{i_0} = \alpha$. Then,

$$\begin{aligned} \lambda_{\max}(G)\alpha = \lambda_{\max}(G) u_{i_0} &= (A\boldsymbol{u})_{i_0} = \sum_{i \sim i_0} u_i \\ &\le \alpha |\{i \in V \mid i \sim i_0\}| = \alpha d(i_0) \le \alpha d_{\max}(G), \end{aligned}$$

which implies that $\lambda_{\max}(G) \le d_{\max}(G)$. ∎

Corollary 2.3.2 *If G is a regular graph with degree κ, we have $\lambda_{\max}(G) = \kappa$.*

PROOF. For a regular graph we have $\bar{d}(G) = d_{\max}(G) = \kappa$. ∎

2.4 Spectral Distribution of a Graph

Definition 2.4.1 Let G be a finite graph with

$$\mathrm{Spec}\,(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_s \\ m_1 & m_2 & \dots & m_s \end{pmatrix}.$$

The *spectral (eigenvalue) distribution* of G is a probability measure on $\mathbf{R}$ defined by

$$\mu = \frac{1}{|V|} \sum_{i=1}^{s} m_i \delta_{\lambda_i}\,,$$

where δ_λ stands for the delta-measure.

It is sometimes convenient to use the list of eigenvalues of A with multiplicities, say, $\lambda_1, \lambda_2, \dots, \lambda_n$, $n = |V|$. Then the spectral distribution is

$$\mu = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k}\,.$$

Example 2.4.2 The spectral distribution of the path P_n is given by

$$\mu = \frac{1}{n} \sum_{k=1}^{n} \delta_{2\cos\frac{k\pi}{n+1}}$$

Remark 2.4.3 The delta measure δ_λ is a Borel probability measure on $\mathbf{R}$. For a Borel set $E \subset \mathbf{R}$ we have

$$\delta_\lambda(E) = \begin{cases} 1, & \text{if } \lambda \in E, \\ 0, & \text{otherwise} \end{cases}$$

Hence for a continuous function $f(x)$ on $\mathbf{R}$ we have

$$\int_{-\infty}^{+\infty} f(x)\delta_\lambda(dx) = f(\lambda).$$

Definition 2.4.4 Let μ be a probability measure on $\mathbf{R}$. The integral, if exists,

$$M_m(\mu) = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots \tag{2.6}$$

is called the *m-th moment* of μ.

Theorem 2.4.5 *Let μ be the spectral distribution of a finite graph $G = (V, E)$. Then,*

$$M_m(\mu) = \frac{1}{|V|} \mathrm{Tr}\, A^m, \qquad m = 1, 2, \dots. \tag{2.7}$$

PROOF. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A, listed with multiplicities. Then by definition,

$$M_m(\mu) = \int_{-\infty}^{+\infty} x^m \mu(dx) = \frac{1}{n}\sum_{k=1}^{n} \lambda_k^m.$$

Since $\lambda_1^m, \dots, \lambda_n^m$ is the eigenvalues of A^m with multiplicities, their sum coincides with the trace of A^m. Hence, (2.7) follows. ∎

Lemma 2.4.6 *Let A be the adjacency matrix of a finite graph $G = (V, E)$.*

(1) $\mathrm{Tr}\, A = 0$.

(2) $\mathrm{Tr}\,(A^2) = 2|E|$.

(3) $\mathrm{Tr}\,(A^3) = 6\triangle$.

PROOF. We show only (3). By definition

$$\mathrm{Tr}\,(A^3) = \sum_{x,y,z \in V} (A)_{xy}(A)_{yz}(A)_{zx} = |\{(x,y,z) \in V^3\,;\, x \sim y \sim z \sim x\}| = 6\triangle.$$

∎

The most basic characteristics of a spectral distribution are the mean and the variance, which are defined by

$$\mathrm{mean}\,(\mu) = M_1(\mu) = \int_{-\infty}^{+\infty} x\mu(dx),$$

$$\mathrm{var}\,(\mu) = M_2(\mu) - M_1(\mu)^2 = \int_{-\infty}^{+\infty} (x - \mathrm{mean}\,(\mu))^2 \mu(dx).$$

Proposition 2.4.7 *Let μ be the spectral distribution of a finite graph $G = (V, E)$. Then,*

$$\mathrm{mean}\,(\mu) = 0, \qquad \mathrm{var}\,(\mu) = 2\,\frac{|E|}{|V|}\,.$$

Proposition 2.4.8 *Let $\lambda_1, \dots, \lambda_n$ be the eugenvalues of a graph $G = (V, E)$, $|V| = n$. Then*

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^2$$

2.5 Asymptotic Spectral Distributions of P_n and K_n

2.5.1 P_n as $n \to \infty$

The spectral distribution of P_n is

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{2\cos\frac{k\pi}{n+1}}\,,$$

see Example 2.4.2. Let $f(x)$ be a bounded continuous function. The we have

$$\int_{-\infty}^{+\infty} f(x)\mu_n(dx) = \frac{1}{n}\sum_{k=1}^{n} f\left(2\cos\frac{k\pi}{n+1}\right) \to \int_0^1 f(2\cos\pi t)dt, \quad \text{as } n \to \infty,$$

which follows by the definition of Riemann integral. By change of variable, one gets

$$\int_0^1 f(2\cos\pi t)dt = \int_{-2}^{2} f(x)\,\frac{dx}{\pi\sqrt{4-x^2}}\,.$$

Consequently,

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty} f(x)\mu_n(dx) = \int_{-2}^{2} f(x)\,\frac{dx}{\pi\sqrt{4-x^2}}\,, \qquad f \in C_b(\mathbf{R}), \tag{2.8}$$

where $C_b(\mathbf{R})$ denotes the space of bounded continuous function on $\mathbf{R}$.

It is easy to see that

$$\frac{dx}{\pi\sqrt{4-x^2}}\,\chi_{(-2,2)}(x)dx$$

is a probability measure on $\mathbf{R}$. We call it the *arcsine law* with variance 2. Then from the limit formula (2.8) we state the following

Proposition 2.5.1 *The spectral distribution of P_n converges weakly to the arcsine law with variance* 2.

2.5.2 K_n as $n \to \infty$

The spectral distribution of K_n is

$$\mu_n = \frac{1}{n}\,\delta_{n-1} + \frac{n-1}{n}\,\delta_{-1}\,.$$

In a similar manner as in Section 2.5.1 we have

$$\int_{-\infty}^{+\infty} f(x)\mu_n(dx) = \frac{1}{n}\,f(n-1) + \frac{n-1}{n}\,f(-1) \to f(-1), \quad \text{as } n \to \infty.$$

Since

$$f(-1) = \int_{-\infty}^{+\infty} f(x)\delta_{-1}(dx)$$

and δ_{-1} is a probability measure, we may state that the spectral distribution of K_n converges weakly to δ_{-1}. However, notice that

$$\text{mean}\,(\mu_n) = 0, \qquad \text{var}\,(\mu_n) = 2\,\frac{|E|}{|V|} = n-1,$$

and

$$\text{mean}\,(\delta_{-1}) = -1, \qquad \text{var}\,(\delta_{-1}) = 0.$$

Thus, it is hardly to say that the limit measure δ_{-1} reflects basic statistical properties of μ_n for a large n.

The above unconfort was caused by $\mathrm{var}(\mu_n) \to \infty$ as $n \to \infty$. In order to capture a reasonable limit measure it is necessary to handle a normalized measure. In general, for a probability measure μ with mean $\mathrm{mean}(\mu) = m$ and variance $\mathrm{var}(\mu) = \sigma^2$, the normalization is defined by

$$\int_{-\infty}^{+\infty} f(x)\bar{\mu}(dx) = \int_{-\infty}^{+\infty} f\left(\frac{x-m}{\sigma}\right)\mu(dx).$$

Then $\mathrm{mean}(\bar{\mu}) = 0$ and $\mathrm{var}(\bar{\mu}) = 1$.

Proposition 2.5.2 *The normalized spectral distribution of K_n converges weakly to δ_0.*

PROOF. Let $f(x)$ be a bounded continuous function on $\mathbf{R}$. We have

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)\bar{\mu}_n(dx) &= \int_{-\infty}^{+\infty} f\left(\frac{x}{\sqrt{n-1}}\right)\mu_n(dx) \\ &= \frac{1}{n} f\left(\frac{n-1}{\sqrt{n-1}}\right) + \frac{n-1}{n} f\left(\frac{-1}{\sqrt{n-1}}\right) \\ &\to f(0), \quad \text{as } n \to \infty.\end{aligned}$$

This completes the proof. ∎

In Section 2.5.1, for the asymptotic spectral distribution of P_n we did not take the normalization. The normalization yields essentially nothing new thanks to the fact that

$$\mathrm{mean}(\mu_n) = 0, \qquad \mathrm{var}(\mu_n) = 2\,\frac{|E|}{|V|} = \frac{2(n-1)}{n}.$$

Namely, the variance of μ_n stays bounded by 2 as $n \to \infty$.

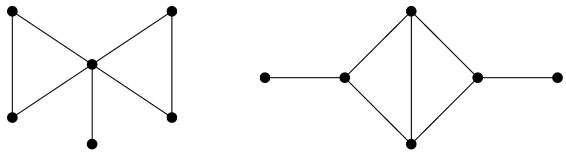
2.6 Isospectral (Cospectral) Graphs

We show a pair of non-isomorphic graphs that have the same spectra.

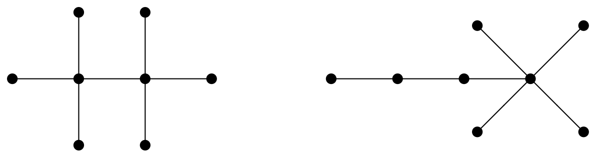
Example 2.6.1 $\varphi(x) = x^5 - 4x^3 = x^3(x-2)(x+2)$.

Example 2.6.2 (Baker)

$$\begin{aligned}\varphi(x) &= x^6 - 7x^4 - 4x^3 + 7x^2 + 4x - 1 \\ &= (x-1)(x+1)^2(x^3 - x^2 - 5x + 1)\end{aligned}$$



Example 2.6.3 (Collatz–Sinogowitz) $\varphi(x) = x^8 - 7x^6 + 9x^4$



For more information see e.g.,

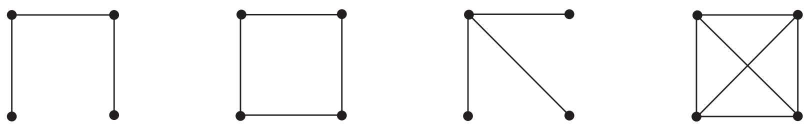
[6] D. M. Cvetković, M. Doob and H. Sachs: Spectra of Graphs: Theory and Applications (3rd rev. enl. ed.), New York, Wiley, 1998.

[7] L. Collatz and U. Sinogowitz: *Spektren endlicher Grafen*, Abh. Math. Sem. Univ. Hamburg **21** (1957), 63–77.

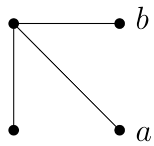
[8] C. D. Godsil and B. D. McKay: *Constructing cospectral graphs*, Aeq. Math. **25** (1982), 257–268.

Exercises 2

2.1. Find the spectra and spectral distributions of the following graphs.



2.2. Find the number of m-step walks connecting a and b.



2.3. Examine Example 2.6.1.

2.4*. Let C_n be a cycle of n vertices. Find Spec$\,(C_n)$.

2.5*. Let μ_n be the spectral distribution of C_n. Study the asymptotics of μ_n as $n \to \infty$.

2.6* Prove the formula:

$$\prod_{k=1}^{m} 2\cos\frac{k\pi}{2m+1} = 1.$$

[Hint: Use Spec (P_n)]

3 Adjacency Algebras

3.1 Adjacency Algebras

Let A be the adjacency matrix of a locally finite graph $G = (V, E)$. In Section 2.2 we showed that every matrix element of A^m $(m = 1, 2, \dots)$ is defined and finite, so we may form their linear conbination. Let $\mathcal{A}(G)$ denote the set of linear combinations of $E, A, A^2, \dots$ with complex coefficients.

Equipped with the usual operations, $\mathcal{A}(G)$ becomes a commutative algebra over $\mathbf{C}$ with the multiplication identity E. Moreover, we define the involution by

$$(c_0 E + c_1 A + c_2 A^2 + \dots + c_m A^m)^* = \bar{c_0} E + \bar{c_1} A + \bar{c_2} A^2 + \dots + \bar{c_m} A^m$$

so that $\mathcal{A}(G)$ becomes a $*$-algebra.

Definition 3.1.1 The above $\mathcal{A}(G)$ is called the *adjacency algebra* of G.

Proposition 3.1.2 *If G is a finite graph,* $\dim \mathcal{A}(G)$ *coincides with the number of different eigenvalues of A.*

Proof. Let $\lambda_1 < \dots < \lambda_s$ be the different eigenvalues of A. Then, by a suitable orthogonal matrix U we have

$$U^{-1} A U = \begin{bmatrix} \lambda_1 E_{m_1} & & \\ & \ddots & \\ & & \lambda_s E_{m_s} \end{bmatrix} \equiv D.$$

We see that $\{E, D, D^2, \dots, D^{s-1}\}$ is linearly independent, but $\{E, D, D^2, \dots, D^{s-1}, D^s\}$ is not. In fact,

$$(D - \lambda_1 E) \cdots (D - \lambda_s E) = O.$$

Therefore, the algebra $U^{-1} \mathcal{A} U$ is of dimension s, so is $\mathcal{A}(G)$. ∎

Proposition 3.1.3 *For a connected finite graph $G = (V, E)$ we have*

$$\dim \mathcal{A}(G) \geq \operatorname{diam}(G) + 1.$$

Proof. For simplicity put $\operatorname{diam}(G) = d$. If $d = 0$, we have $|V| = 1$ and $\dim \mathcal{A}(G) = 1$ so the assertion is clear. Assume that $d \geq 1$. By definition of the diameter there exists a pair of verices $x, y \in V$ such that $\partial(x, y) = d$. Choose one path of length d connecting x, y, say,

$$x = x_0 \sim x_1 \sim \dots \sim x_k \sim x_{k+1} \sim \dots \sim x_d = y.$$

In this case, $x_0, x_1, \dots, x_d$ are all distinct and $\partial(x, x_k) = k$ $(0 \leq k \leq d)$. In particular, there is no walk of length $\leq k - 1$ connecting x and x_k. Hence

$$(A^m)_{x x_k} = 0, \quad 0 \leq m \leq k - 1; \qquad (A^k)_{x x_k} \geq 1.$$

Now suppose that

$$\alpha_0 E + \alpha_1 A + \cdots + \alpha_d A^d = 0, \qquad \alpha_i \in \mathbf{C}. \tag{3.1}$$

Taking the xx_d-element of (3.1), since

$$(A^m)_{xx_d} = 0, \quad 0 \le m \le d-1; \qquad (A^d)_{xx_d} \ge 1,$$

we have $\alpha_d (A^d)_{xx_d} = 0$ so $\alpha_d = 0$. Next taking the xx_{d-1}-element of (3.1), we have $\alpha_{d-1} = 0$. We can continue this argument to have $\alpha_0 = \cdots = \alpha_{d-1} = \alpha_d = 0$. Namely, $\{E, A, \dots, A^d\}$ is linearly independent. So $\dim \mathcal{A}(G) \ge d+1$. ∎

Corollary 3.1.4 *A connected finite graph $G = (V, E)$ has at least* $\operatorname{diam}(G) + 1$ *different eigenvalues.*

PROOF. By combining Propositions 3.1.2 and 3.1.3. ∎

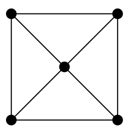
Example 3.1.5 (1) K_n $(n \ge 2)$.

$$(\text{number of different eigenvalues}) = 2, \qquad \operatorname{diam}(K_n) = 1.$$

(2) P_n $(n \ge 1)$.

$$(\text{number of different eigenvalues}) = n, \qquad \operatorname{diam}(P_n) = n-1.$$

(3) G as below. $\varphi_G(x) = x^2(x+2)(x^2 - 2x - 4)$



$$(\text{number of different eigenvalues}) = 4, \quad \operatorname{diam}(G) = 2.$$

3.2 Distance-Regular Graphs (DRGs)

Let $G = (V, E)$ be a connected graph and fix a vertex $o \in V$ as an origin (root). We set

$$V_n = \{x \in V \,;\, \partial(x, o) = n\}, \qquad n = 0, 1, 2, \dots.$$

Obviously,

$$V_0 = \{o\}, \qquad V_1 = \{x \in V \,;\, x \sim o\}.$$

If G is a finite graph, there exists $m_0 \ge 1$ such that $V_{m_0 - 1} \ne \emptyset$ and $V_{m_0} = \emptyset$. If G is an infinite, locally finite graph, $V_n \ne \emptyset$ for all $n \ge 0$. In any case we have a partition of the vertices:

$$V = \bigcup_{n=0}^{\infty} V_n \tag{3.2}$$

which is called the *stratification* of the graph G with respect to the origin $o \in V$.

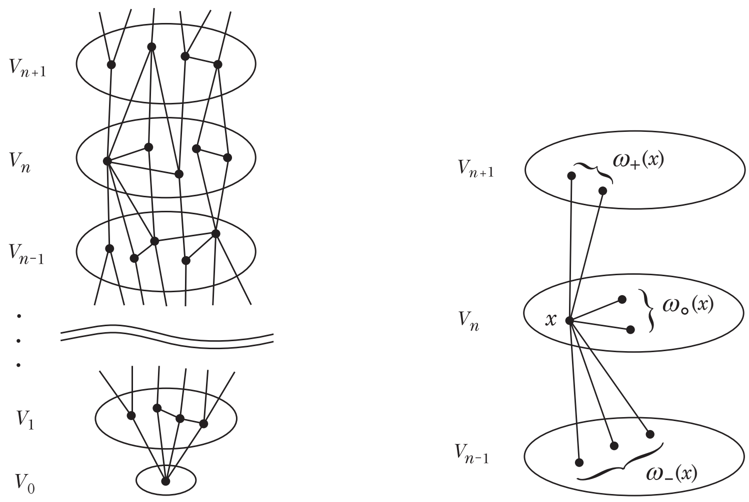


Figure 3.1: Stratification and $\omega_\epsilon(x)$

Lemma 3.2.1 *Let G be a connected, locally finite graph and let* (3.2) *be a stratification. If $x \in V_n$ and $y \sim x$, we have $y \in V_{n+1} \cup V_n \cup V_{n-1}$.*

Proof. Obvious. ∎

Given a stratification, for $x \in V_n$ we define

$$\omega_+(x) = \{y \in V_{n+1}\,;\, y \sim x\},$$
$$\omega_\circ(x) = \{y \in V_n\,;\, y \sim x\},$$
$$\omega_-(x) = \{y \in V_{n-1}\,;\, y \sim x\}$$

It is convenient to write

$$\omega_\epsilon(x) = \{y \in V_{n+\epsilon}\,;\, y \sim x\}, \qquad \epsilon \in \{+,-,\circ\},$$

where ϵ takes the values $+1, -1, 0$ according to $\epsilon = +, -, \circ$. Note also that

$$\deg(x) = \omega_+(x) + \omega_\circ(x) + \omega_-(x), \qquad x \in V.$$

Definition 3.2.2 A connected graph $G = (V, E)$ is called *distance-regular* if, for any stratification of G, the functions $\omega_\epsilon : V \to \{0, 1, 2, \dots\}$ ($\epsilon \in \{+,-,\circ\}$) are constant on V_n, and the constants are independent of the choice of stratification. In that case we put

$$b_n = \omega_+(x), \qquad c_n = \omega_-(x), \qquad a_n = \omega_\circ(x),$$

by taking $x \in V_n$.

It is obvious that

$$a_0 = c_0 = 0, \qquad b_0 = \deg(x), \quad x \in V_0\,.$$

Since any vertex x may be chosen as an origin for stratification, $\deg(x) = b_0$ for all $x \in V$. That is, a distace-regular graph is regular with degree b_0. Therefore,

$$a_n + b_n + c_n = b_0, \qquad n = 1, 2, \dots.$$

Lemma 3.2.3 *If G is a finite DRG, letting $d = \mathrm{diam}\,(G)$, we have*

$$V = \bigcup_{n=0}^{d} V_n\,, \qquad V_0, V_1, \dots, V_d \neq \emptyset. \tag{3.3}$$

If G is an infinite DRG, $V_n \neq \emptyset$ for all $n = 0, 1, 2, \dots$.

PROOF. By definition, there is a path of length d. Taking one of the end vertex as an origin, the associated stratification satisfies conditions in (3.3). Then, we have

$$b_0 > 0, \quad \dots, \quad b_{d-1} > 0, \qquad b_d = 0. \tag{3.4}$$

Let $o \in V$ be an aritrary vertex and take $v \in V$ such that

$$\partial(o, v) = \max\{\partial(o, x)\,;\, x \in V\} \equiv p.$$

Then $p \leq d$ and the associated stratification is

$$V = \bigcup_{k=0}^{p} V'_k\,, \qquad V'_0, V'_1, \dots, V'_p \neq \emptyset.$$

Then,

$$b_0 > 0, \quad \dots, \quad b_{p-1} > 0, \qquad b_p = 0. \tag{3.5}$$

In order that (3.4) and (3.5) are consistent, we have $p = d$. ∎

Corollary 3.2.4 *In a finite distance-regular graph, every vertex is an end vertex of a diameter.*

Definition 3.2.5 For a finite distance-regular graph G, the table of associated constant numbers

$$\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_d \\ a_0 & a_1 & a_2 & \cdots & a_d \\ b_0 & b_1 & b_2 & \cdots & b_d \end{pmatrix}$$

is called the *intersection array* of G. If G is infinite, the array becomes infinite.

Since $a_n + b_n + c_n = b_0$ is constant, the row of $a_0, a_1, \dots$ may be omitted. Note that

$$\begin{aligned} &c_0 = 0, \quad c_1 > 0, \quad \cdots, \quad c_{d-1} > 0, \quad c_d > 0, \\ &b_0 > 0, \quad b_1 > 0, \quad \cdots, \quad b_{d-1} > 0, \quad b_d = 0. \end{aligned}$$

Example 3.2.6 (1) The cheapest examples are C_n $(n \geq 3)$ and K_n $(n \geq 1)$.
(2) Let $K_{n,m}$ be the complete bipartite graph. It is distance-regular if and only if $n = m$.
(3) The Petersen graph is distance-regular.
(4) A homogeneous tree of degree κ, T_κ, is distance-regular.
(5) P_n $(n \geq 3)$ is not distance-regular (since it is not regular).
(6) $\mathbf{Z}^2$ is not distance-regular.

Definition 3.2.7 A connected graph is called *distance-transitive* if, for any $x, x', y, y' \in V$ with $\partial(x, y) = \partial(x', y')$ there exists $\alpha \in \mathrm{Aut}\,(G)$ such that $\alpha(x) = x'$ and $\alpha(y) = y'$.

Proposition 3.2.8 *A distance-transitive graph is distance-regular.*

In fact, (1)–(4) in Example 3.2.6 are all distance-transitive. The converse of Proposition 3.2.8 is not valid, for examples see Godsil–Royle [9: p.69], Brouwer et al. [10: p.136].

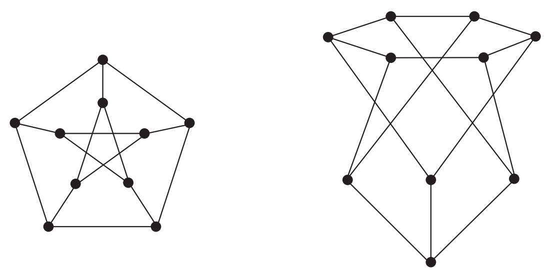


Figure 3.2: Petersen graph

3.3 Adjacency Algebras of Distance-Regular Graphs

Definition 3.3.1 Let $G = (V, E)$ be a connected graph. For $k = 0, 1, 2, \dots$ we define a matrix $A^{(k)}$ indexed by $V \times V$ by

$$(A^{(k)})_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = k, \\ 0, & \text{otherwise} \end{cases}$$

This matrix is called the *k-th distance matrix*.

Obviously,

$$A^{(0)} = E \quad \text{(identity)}, \qquad A^{(1)} = A \quad \text{(adjacency matrix)}$$

and we have

$$\sum_{k=0}^{\infty} A^{(k)} = J, \qquad J \text{ is the matrix whose elements are all one.}$$

Lemma 3.3.2 *Let G be a distance-regular graph with the intersection array*

$$\begin{pmatrix} c_0 & c_1 & c_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ b_0 & b_1 & b_2 & \cdots \end{pmatrix}.$$

Then,

$$AA^{(k)} = c_{k+1}A^{(k+1)} + a_k A^{(k)} + b_{k-1}A^{(k-1)}, \qquad k = 0, 1, 2, \dots. \tag{3.6}$$

Here we understand that $A^{(-1)} = O$ and $A^{(d+1)} = O$ for $d = \mathrm{diam}\,(G) < \infty$.

Proof. For $k = 0$ the equality (3.6) is obvious. Let $k \geq 1$. Let $x, y \in V$ and set $n = \partial(x, y)$. Then, by definition

$$(AA^{(k)})_{xy} = \sum_{z \in V} (A)_{xz}(A^{(k)})_{zy} = |\{z \in V\,;\, \partial(z, x) = 1,\ \partial(z, y) = k\}|.$$

It is obvious by the triangle inequality,

$$\{z \in V\,;\, \partial(z, x) = 1,\ \partial(z, y) = k\} = \emptyset$$

unless $k-1 \le n \le k+1$. Namely,

$$(AA^{(k)})_{xy} = 0 \qquad \text{unless } k-1 \le n \le k+1.$$

Asuume that $k-1 \le n \le k+1$. Then, by definition of the intersection array, we have

$$|\{z \in V\,;\, \partial(z,x)=1,\, \partial(z,y)=k\}| = \begin{cases} c_n, & k=n-1, \\ a_n, & k=n, \\ b_n, & k=n+1. \end{cases}$$

Thus,

$$(AA^{(k)})_{xy} = \begin{cases} c_{k+1}, & \partial(x,y)=k+1, \\ a_k, & \partial(x,y)=k, \\ b_{k-1}, & \partial(x,y)=k-1. \end{cases}$$

This completes the proof. ∎

Lemma 3.3.3 *For $k=0,1,2,\dots,d$, $A^{(k)}$ is a polynomial in A with degree k.*

Proof. For $k=0,1$ the assertion is apparently true. In fact,

$$A^{(0)} = f_0(A), \qquad f_0(x)=1,$$
$$A^{(1)} = f_1(A), \qquad f_1(x)=x.$$

It follows from Lemma 3.3.2 that

$$A^{(k)} = f_k(A), \qquad f_k(x) = \frac{1}{c_k}\,(x-a_{k-1})f_{k-1}(x) - \frac{b_{k-2}}{c_k}\,f_{k-2}(x).$$

for $k=2,3,\dots,d$. Note that $c_1>0,\cdots,c_d>0$. ∎

Theorem 3.3.4 *Let G be a distance-regular graph. Then the adjacency algebra $\mathcal{A}(G)$ coincides with the linear span of $\{A^{(0)},A^{(1)},\dots\}$. Moreover, $\{A^{(0)},A^{(1)},\dots\}$ are linearly independent so they form a linear basis of $\mathcal{A}(G)$.*

Proof. It follows from Lemma 3.3.3 that the adjacency algebra $\mathcal{A}(G)$ contains the linear span of $\{A^{(0)},A^{(1)},\dots\}$. On the other hand, since

$$A^{(k)} = f_k(A) = \beta_k A^k + \dots, \qquad \beta_k > 0,$$

we see that A^k is a linear combination of $A^{(0)},A^{(1)},\dots,A^{(k)}$. Therefore, $\mathcal{A}(G)$ is contained in the linear span of $\{A^{(0)},A^{(1)},\dots\}$. ∎

Theorem 3.3.5 *If G is a finite distance-regular graph,* $\dim\mathcal{A}(G) = \operatorname{diam}(G)+1$ *and A has* $\operatorname{diam}(G)+1$ *distinct eigenvalues.*

Proof. Immediate from Theorem 3.3.4. ∎

Theorem 3.3.6 (Linearization formula) *For $i, j, k \in \{0, 1, 2, \dots, d\}$ there exists a unique constant p_{ij}^k such that*

$$A^{(i)}A^{(j)} = \sum_{k=0}^{d} p_{ij}^k A^{(k)} \qquad i, j \in \{0, 1, 2, \dots, d\}. \tag{3.7}$$

Moreover, for $x, y \in V$ with $\partial(x, y) = k$,

$$|\{z \in V\,;\, \partial(z, x) = i,\ \partial(z, y) = j\}|$$

does not depend on the choice of x, y but depends on k, and coincides with p_{ij}^k.

Proof. The first half is obvious by Theorem 3.3.4. Let $x, y \in V$ with $\partial(x, y) = l$. Let us observe the matrix element of (3.7). From the left-hand side we get

$$(A^{(i)}A^{(j)})_{xy} = \sum_{z \in V} (A^{(i)})_{xz}(A^{(j)})_{zy} = |\{z \in V\,;\, \partial(z, x) = i,\ \partial(y, z) = j\}|$$

On the other hand,

$$\left(\sum_{k=0}^{d} p_{ij}^k A^{(k)} \right)_{xy} = p_{ij}^l\,,$$

which is constant for all $x, y \in V$ with $\partial(x, y) = l$. Therefore, for such a pair x, y we have

$$|\{z \in V\,;\, \partial(z, x) = i,\ \partial(y, z) = j\}| = p_{ij}^l$$

as desired. ∎

Definition 3.3.7 The constant numbers $\{p_{ij}^k\}$ are called the *intersection numbers* of a distance-regular graph G.

The intersection numbers satisfies:

(1) $p_{1n}^{n-1} = b_{n-1}, \quad p_{1n}^{n} = a_n, \quad p_{1n}^{n+1} = c_{n+1}$.

(2) $p_{ij}^k = 0$ unless $|i - j| \le k \le i + j$.

(3) $p_{ij}^k = p_{ji}^k$.

(4) $p_{00}^0 = 1, \quad p_{0i}^0 = p_{i0}^0 = 0$ for $i \ge 1$.

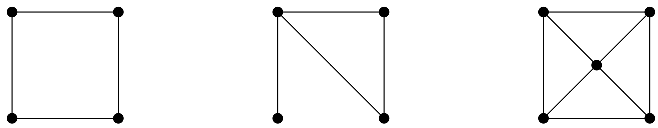
Remark 3.3.8 In some of the literature, a distance-regular graph is defined to be a connected graph for which the set of conatants $\{p_{ij}^k\}$, where $i, j, k \in \{0, 1, 2, \dots\}$,

$$p_{ij}^k = |\{z \in V\,;\, \partial(z, x) = i,\ \partial(y, z) = j\}|$$

is independent of the choice of $x, y \in V$ with $\partial(x, y) = k$. This condition is seemingly stronger than that of our definition (Definition 3.2.2) as is seen in (1) above; however, they are equivalent.

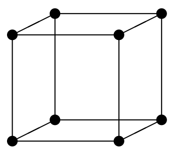
Exercises 3

3.1 For each of the following graphs find the adjacency matrix A and distance matrix $A^{(k)}$. Then find the relations between the powers of A and $A^{(0)}, A^{(1)}, A^{(2)}, \dots$. Finally compare the dimensions of the adjacency algebras and the diameters of the graphs.

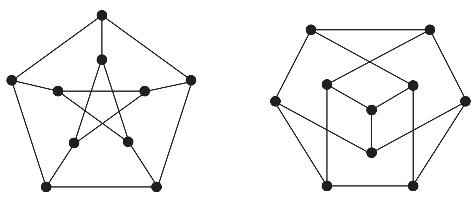


3.2 Is the 2-dimensional integer lattice $\mathbb{Z}^2$ distance-regular?

3.3 Is the cube distance-regular?



3.4* Verify that the Petersen graph is distance-regular and find its intersection array.



3.5* Let n, d be natural numbers. Set $F = \{1, 2, \dots, n\}$ and $V = \{x = (\xi_1, \xi_2, \dots, \xi_d)\,;\, \xi_i \in F\}$. For $x = (\xi_i), y = (\eta_i) \in V$ define

$$\partial(x, y) = |\{1 \le i \le d\,;\, \xi_i \ne \eta_i\}|,$$

and draw an edge between x, y if $\partial(x, y) = 1$. Thus we obtain a graph $G = (V, E)$, called a *Hamming graph.* Show that the Hamming graph is distance-regular and find the intersection array.

3.6* Define a polynomial $T_n(x)$ by $T_n(\cos\theta) = \cos n\theta$ and set

$$\tilde{T}_0(x) = T_0(x) = 1, \qquad \tilde{T}_n(x) = 2T_n\left(\frac{x}{2}\right), \quad n \ge 1.$$

Let A and $A^{(k)}$ be the adjacency matrix and the k-th distance matrix of $\mathbb{Z}$, respectively. Show that $A^{(k)} = \tilde{T}_k(A)$. ($\{T_n(x)\}$ are calle the *Chebyshev polynomial of the first kind.*)

4 Quantum Probability

4.1 Algebraic Probability Spaces

Definition 4.1.1 Let $\mathcal{A}$ be a $*$-algebra over $\mathbf{C}$ with multiplication unit $1_{\mathcal{A}}$. A function $\varphi : \mathcal{A} \to \mathbf{C}$ is called a *state* on $\mathcal{A}$ is

(i) φ is linear;

(ii) $\varphi(a^*a) \ge 0$;

(iii) $\varphi(1_{\mathcal{A}}) = 1$.

Then, the pair $(\mathcal{A}, \varphi)$ is called an *algebraic probability space.*

Example 4.1.2 Let $M(n, \mathbf{C})$ be the set of $n \times n$ complex matrices. Equipped with the usual operations, $M(n, \mathbf{C})$ becomes a $*$-algebra. Typical states are listed below:

(i) (trace)

$$\varphi_{\mathrm{tr}}(a) = \frac{1}{n}\mathrm{tr}\, a.$$

(ii) (vector state) Let $\xi \in \mathbf{C}^n$ with $\|\xi\| = 1$.

$$\varphi_\xi(a) = \langle \xi, a\xi \rangle.$$

(iii) (density matrix) Let $\rho \in M(n, \mathbf{C})$ satisfying $\rho = \rho^* \ge 0$ and $\mathrm{Tr}\, \rho = 1$. Then

$$\varphi_\rho(a) = \mathrm{Tr}\,(\rho a).$$

Every state on $M(n, \mathbf{C})$ is of this form and the density matrix is determined uniquely.

Example 4.1.3 (Classical probability space) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let

$$L^{\infty-} = \bigcap_{1 \le p < \infty} L^p(\Omega, \mathcal{F}, P)$$

be the set of all random variables having finite moments of all orders. Equipped with the pointwise operations, $L^{\infty-}$ is a commutative $*$-algebra.

$$\varphi(a) = \mathbf{E}[a] = \int_\Omega a(\omega) P(d\omega), \qquad a \in L^{\infty-}$$

is a state on $L^{\infty-}$.

Example 4.1.4 Let $\mathbf{C}[X]$ be the set of polynomials in the indeterminant X with complex coefficients. Equipped with the usual addition, scalar multiplication and product, $\mathbf{C}[X]$ becomes a commutative algebra. Moreover, we define the involution ($*$-operation) by

$$(c_0 + c_1 X + \cdots + c_n X^n)^* = \overline{c_0} + \overline{c_1} X + \cdots + \overline{c_n} X^n\,.$$

Thus, $\mathbf{C}[X]$ becomes a $*$-algebra. Let $\mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ be the set of probability measures on $\mathbf{R}$ that admit finite moments of all orders, i.e.,

$$\int_{-\infty}^{+\infty} |x|^m \mu(dx) < \infty.$$

Let $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$. Then

$$\varphi(a) = \mu(a) = \int_{-\infty}^{+\infty} a(x)\mu(dx), \qquad a \in \mathbf{C}[X],$$

is a state on $\mathbf{C}[X]$. Thus, $(\mathbf{C}[X], \mu)$ is an algebraic probability space. For $m = 1, 2, \dots$

$$M_m(\mu) = \int_{-\infty}^{+\infty} x^m \mu(dx)$$

is called the *m-th moment* of μ, and $\{M_0(\mu) = 1, M_1(\mu), M_2(\mu), \dots\}$ the *moment sequence* of μ.

Definition 4.1.5 Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. An element $a \in \mathcal{A}$ is called an *algebraic random variable* or a *random variable* for short. If $a = a^*$, we call it *real*.

Theorem 4.1.6 *Let $(\mathcal{A}, \varphi)$ be an algebraic probability space and let $a = a^* \in \mathcal{A}$ be a real random variable. Then, there exists a probability measure $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ such that*

$$\varphi(a^m) = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots. \tag{4.1}$$

Definition 4.1.7 A probability measure μ satisfying (4.1) is called the *distribution* of a in φ. As discussed later, μ is not uniquely determined in general.

PROOF. Set $M_m = \varphi(a^m)$ and consider the Hanckel determinant:

$$\Delta_m = |H_m|, \qquad H_m = \begin{bmatrix} M_0 & M_1 & \cdots & M_m \\ M_1 & M_2 & \cdots & M_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ M_m & M_{m+1} & \cdots & M_{2m} \end{bmatrix}. \tag{4.2}$$

It follows from Hamburger's theorem (1920) that there exists a probability measure $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ such that

$$M_m = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots,$$

if and only if

(M1) $\Delta_m > 0$ for all m; or

(M2) there exists $m_0 \geq 1$ such that $\Delta_1 > 0, \dots, \Delta_{m_0 - 1} > 0$ and $\Delta_{m_0} = \cdots = 0$.

We shall check this condition for our Δ_m defined in (4.2). For

$$\boldsymbol{u} = \begin{bmatrix} u_0 \\ \vdots \\ u_m \end{bmatrix} \in \mathbf{C}^{m+1}$$

we have

$$\begin{aligned}
\langle \boldsymbol{u}, H_m \boldsymbol{u} \rangle &= \sum_{i,j=0}^{m} \overline{u_i} M_{ij} u_j = \sum_{i,j=0}^{m} \overline{u_i}\, u_j \varphi(a^{i+j}) \\
&= \varphi\left(\sum_{i,j=0}^{m} \overline{u_i}\, u_j a^{i+j} \right) = \varphi\left(\left(\sum_{i=0}^{m} u_i a_i \right)^* \left(\sum_{j=0}^{m} u_j a^j \right) \right) \geq 0,
\end{aligned}$$

which shows that H_m is positive definite. Hence its eigenvalues are all non-negative real numbers and $\Delta_m \geq 0$.

We next show that $\Delta_m = 0$ implies $\Delta_{m+1} = 0$. Suppose that $\Delta_m = 0$. Then there exists $\boldsymbol{u} \neq \mathbf{0}$ such that $H_m \boldsymbol{u} = \mathbf{0}$. Set

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{u} \\ 0 \end{bmatrix} \in \mathbf{C}^{m+2}.$$

Apparently, $\boldsymbol{v} \neq \mathbf{0}$. Since

$$H_{m+1} \boldsymbol{v} = \begin{bmatrix} H_m & * \\ * & M_{2m} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ 0 \end{bmatrix} = \begin{bmatrix} H_m \boldsymbol{u} \\ * \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ * \end{bmatrix},$$

we have

$$\langle \boldsymbol{v}, H_{m+1} \boldsymbol{v} \rangle = 0.$$

Having shown that H_{m+1} is positive definite, we see that $\Delta_{m+1} = 0$. ∎

Remark 4.1.8 In Theorem 4.1.6 the probability distribution μ is not uniquely determined in general (determinate moment problem).

Towards application to graphs we mention two basic states on the adjacency algebra $\mathcal{A}(G)$ of a graph G.

(1) Assume that $|V| < \infty$. We define $\varphi_{\mathrm{tr}} : \mathcal{A} \to \mathbf{C}$ by

$$\varphi_{\mathrm{tr}}(a) = \frac{1}{|V|} \mathrm{Tr}\,(a) = \frac{1}{|V|} \sum_{x \in V} (a)_{xx}\,, \qquad a \in \mathcal{A}(G).$$

One can check easily that φ_{tr} is a state on $\mathcal{A}(G)$. We call it the normalized trace. The distribution of A in φ_{tr} coincides with the spectral distribution of G. Namely,

$$\varphi_{\mathrm{tr}}(A^m) = \int_{-\infty}^{+\infty} x^m \mu(dx),$$

where

$$\mu = \frac{1}{|V|}\sum_{i=1}^{s} m_i \delta_{\lambda_i}\,.$$

(2) We put

$$C_0(V) = \{f : V \to \mathbf{C}\,;\, f(x) = 0 \text{ except finitely many } x \in V\}.$$

Equipped with the usual operation, $C_0(V)$ becomes a complex vector space. We define the inner product by

$$\langle f, g\rangle = \sum_{x\in V} \overline{f(x)}\, g(x).$$

With each $x \in V$ we associate a function $e_x \in C(V)$ defined by

$$e_x(y) = \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\{e_x\}$ becomes a basis of $C_0(V)$ sastisfying $\langle e_x, e_y\rangle = \delta_{xy}$.

The adjacency algebra acts on $C_0(V)$ from the left as usual:

$$bf(x) = \sum_{y\in V} (b)_{xy} f(y), \qquad b \in \mathcal{A}(G), \quad f \in C_0(V).$$

Let us choose and fix an origin (root) of the graph, say, $o \in V$. Then,

$$\varphi_o(a) = (a)_{oo} = \langle e_o, ae_o\rangle, \qquad a \in \mathcal{A}(G),$$

is a state on $\mathcal{A}(G)$. Thus, $(\mathcal{A}(G), \varphi_o) = (\mathcal{A}(G), e_o)$ is an algebraic probability space. We sometimes call φ_o the *vacuum state* at $o \in V$.

Let μ be the distribution of A in φ_o. Then we have

$$\varphi_o(A^m) = |\{m\text{-step walks from } o \text{ to itself}\}| = \int_{-\infty}^{+\infty} x^m \mu(dx).$$

Theorem 4.1.9 *If G is a finite distance-regular graph, we have*

$$\varphi_{\rm tr} = \varphi_o \qquad \textit{(as a state on the adjacency algebra } \mathcal{A}(G)\textit{)}.$$

PROOF. Let $a \in \mathcal{A}(G)$. We see from Theorem 3.3.4 that a is a linear combination of distance matrices:

$$a = \sum_{k=0}^{d} c_k A^{(k)}.$$

Then, $(a)_{xx} = c_0$ for all $x \in V$, and $(a)_{xx} = (a)_{oo}$ Therefore,

$$\varphi_{\rm tr}(a) = \frac{1}{|V|}\sum_{x\in V} (a)_{xx} = (a)_{oo} = \varphi_o(a).$$

This proves the assertion. ∎

4.2 Interacting Fock Spaces (IFS's)

Definition 4.2.1 A real sequence $\{\omega_n\}_{n=1}^{\infty}$ is called a *Jacobi sequence* if

(i) [infinite type] $\omega_n > 0$ for all $n \geq 1$; or

(ii) [finite type] there exists $m_0 \geq 1$ such that $\omega_1 > 0$, $\omega_2 > 0$, $\dots$, $\omega_{m_0-1} > 0$, $\omega_{m_0} = \omega_{m_0+1} = \cdots = 0$.

By definition $(0, 0, \dots)$ is a Jacobi sequence ($m_0 = 1$).

Given a Jacobi sequence $\{\omega_n\}$, we consider a Hilbert space Γ as follows: If $\{\omega_n\}$ is of infinite type, let Γ be an infinite dimensional Hilbert space with an orthonormal basis $\{\Phi_0, \Phi_1, \dots\}$. If $\{\omega_n\}$ is of finite type, let Γ be an m_0-dimensional Hilbert space with an orthonormal basis $\{\Phi_0, \Phi_1, \dots, \Phi_{m_0-1}\}$. We call Φ_0 the *vacuum vector*.

We next define linear operators $B^{\pm}$ on Γ by

$$B^+\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1}, \quad n = 0, 1, \dots, \tag{4.3}$$

$$B^-\Phi_0 = 0, \quad B^-\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1}, \quad n = 1, 2, \dots, \tag{4.4}$$

where we understand $B^+\Phi_{m_0-1} = 0$ when $\{\omega_n\}$ is of finite type. We call B^- the *annihilation operator* and B^+ the *creation operator*.

Definition 4.2.2 A pair of sequences $(\{\omega_n\}, \{\alpha_n\})$ is called a *Jacobi parameter* or *Jacobi coefficients* if

(i) $\{\omega_n\}$ is a Jacobi sequence of infinite type and $\{\alpha_n\}$ is an infinite real sequence; or

(ii) $\{\omega_n\}$ is a Jacobi sequence of finite type with length m_0 and $\{\alpha_1, \alpha_2, \dots, \alpha_{m_0+1}\}$ is a finite real sequence with $m_0 + 1$ terms.

Given a Jacobi parameter $(\{\omega_n\}, \{\alpha_n\})$ we define the Hilbert space Γ with an orthonormal basis $\{\Phi_n\}$, the annihilation oprtator B^- and the creation operator B^+ as above. In addition we define the *conservation operator* by

$$B^\circ\Phi_n = \alpha_{n+1}\Phi_n, \quad n = 0, 1, 2, \dots. \tag{4.5}$$

Definition 4.2.3 The quintuple $(\Gamma, \{\omega_n\}, B^+, B^-, B^\circ)$ obtained as above is called an *interaction Fock space* associated with a Jacobi parameter $(\{\omega_n\}, \{\alpha_n\})$. When $\{\alpha_n = 0\}$ is a null sequence, we omit B° and $\{\alpha_n\}$.

We note that

$$(B^+)^* = B^-, \qquad (B^-)^* = B^+, \qquad (B^\circ)^* = B^\circ.$$

Let $\mathcal{A}$ be the $*$-algebra generated by B^+, B^-, B°, i.e., the set of all (noncommutative) polynomials in B^+, B^-, B°. Then the function φ_0 defined by

$$\varphi_0(a) = \langle \Phi_0, a\Phi_0 \rangle, \qquad a \in \mathcal{A}$$

is a state on $\mathcal{A}$. We call $(\mathcal{A}, \varphi_0) = (\mathcal{A}, \Phi_0)$ an interacting Fock probability space with vacuum state.

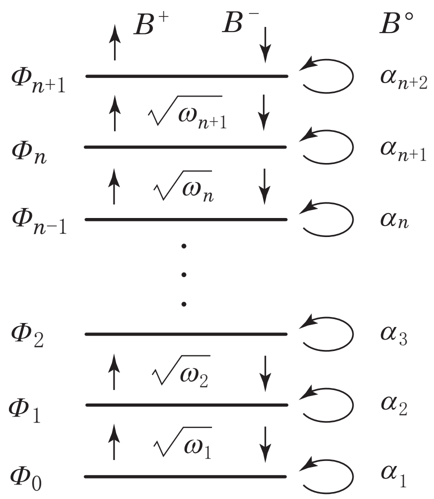


Figure 4.1: Interaction Fock space

4.3 Orthogonal Polynomials

We denote the inner product of $L^2(\mathbf{R}, \mu)$ by

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} \overline{f(x)}\, g(x)\, \mu(dx).$$

Now we define a sequence of polynomials $P_0(x), P_1(x), \dots$ by the following reccursive formula:

$$P_0 = 1, \qquad P_1 = x - \frac{\langle P_0, x \rangle}{\langle P_0, P_0 \rangle} P_0, \qquad P_2 = x^2 - \frac{\langle P_0, x^2 \rangle}{\langle P_0, P_0 \rangle} P_0 - \frac{\langle P_1, x^2 \rangle}{\langle P_1, P_1 \rangle} P_1, \quad \dots,$$

$$P_n = x^n - \sum_{k=0}^{n-1} \frac{\langle P_k, x^n \rangle}{\langle P_k, P_k \rangle} P_k \,.$$

This is the co-called Gram-Schmidt orthogonalization. Then,

$$P_n(x) = x^n + \cdots, \qquad \langle P_m, P_n \rangle = 0 \quad \text{for } m \neq n.$$

We call $\{P_n\}$ the *orthogonal polynomials* associated with μ.

The procedure of forming the orthogonal polynomials stops at the m_0 step if

$$\langle P_0, P_0 \rangle > 0, \quad \dots, \quad \langle P_{m_0-1}, P_{m_0-1} \rangle > 0, \quad \langle P_{m_0}, P_{m_0} \rangle = 0$$

happens. In that case the orthogonal polynomials consists of $P_0(x), P_1(x), \dots, P_{m_0-1}(x)$. This happens if and only if $\mathrm{supp}\,(\mu)$ consists of exactly m_0 points, i.e., μ is a sum of delta measures at different m_0 points with positive coefficients.

Theorem 4.3.1 (The three-term recurrence relation) *Let $\{P_n(x)\}_{n=0}^{\infty}$ be the orthogonal polynomials associated with $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$. Then there exist a pair of sequences $\{\alpha_n\}_{n=1}^{\infty}$*

and $\{\omega_n\}_{n=1}^{\infty}$ *with* $\alpha_n \in \mathbf{R}$, $\omega_n > 0$, *such that*

$$\begin{aligned} &P_0(x) = 1, \\ &P_1(x) = x - \alpha_1, && (4.6) \\ &xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x), \quad n = 1, 2, \dots. && (4.7) \end{aligned}$$

Moreover,

$$\|P_0\| = 1, \quad \|P_n\|^2 = \omega_1\omega_2\cdots\omega_n, \quad n \geq 1, \tag{4.8}$$

$$\alpha_1 = M_1(\mu) = \text{mean}\,(\mu) = \int_{-\infty}^{+\infty} x\mu(dx), \tag{4.9}$$

$$\omega_1 = \text{var}\,(\mu) = \int_{-\infty}^{+\infty} (x - \alpha_1)^2\mu(dx). \tag{4.10}$$

Proof. Well known and omitted. ∎

Definition 4.3.2 We call the pair of sequences $(\{\alpha_n\}_{n=1}^{\infty}, \{\omega_n\}_{n=1}^{\infty})$ the *Jacobi coefficients* of the orthogonal polynomial associated with μ (or simply of μ).

Remark 4.3.3 Setting $P_{-1} = 0$ and understanding $\omega_0 P_{-1} = 0$, we regard (4.7) is valid also for $n = 0$. Remind that ω_0 is not defined.

Remark 4.3.4 If the orthogonal polynomials consists of m_0 polynomials, we understand the Jacobi coefficients are given by $(\{\alpha_1, \alpha_2, \dots, \alpha_{m_0}\}, \{\omega_1, \omega_2, \dots, \omega_{m_0-1}\})$.

Example 4.3.5 Let $\tilde{T}_n(x)$ be the polynomial defined in Exercise 3.6. They are orthogonal polynomials associated with the arcsine law

$$\frac{1}{\pi}\frac{dx}{\sqrt{4 - x^2}}, \quad -2 < x < 2.$$

The Jacobi parameters are $\{\omega_n\} = \{2, 1, 1, \dots\}$ and $\{\alpha_n\} = \{0, 0, 0, \dots\}$.

Theorem 4.3.6 *Let* $\mu \in \mathfrak{P}_{\text{fm}}(\mathbf{R})$ *and* $(\{\omega_n\}, \{\alpha_n\})$ *its Jacobi coefficients. Let* $(\Gamma, \{\Phi_n\}, B^+, B^-, B^\circ)$ *be the interacting Fock space associated with* $(\{\omega_n\}, \{\alpha_n\})$. *Then it holds that*

$$M_m(\mu) = \int_{-\infty}^{+\infty} x^m\mu(dx) = \langle \Phi_0, (B^+ + B^\circ + B^-)^m\Phi_0 \rangle \tag{4.11}$$

Proof. Using

$$\|P_n\| = \sqrt{\omega_n \cdots \omega_1}\,,$$

we obtain from (4.7)

$$x\,\frac{P_n(x)}{\sqrt{\omega_n\cdots\omega_1}} = \sqrt{\omega_{n+1}}\,\frac{P_{n+1}(x)}{\sqrt{\omega_{n+1}\cdots\omega_1}} + \alpha_{n+1}\frac{P_n(x)}{\sqrt{\omega_n\cdots\omega_1}} + \sqrt{\omega_n}\,\frac{P_{n-1}(x)}{\sqrt{\omega_{n-1}\cdots\omega_1}}\,. \tag{4.12}$$

We define an isometry $U : \Gamma \to L^2(\mathbf{R}, \mu)$ by

$$U\Phi_n = \frac{P_n(x)}{\sqrt{\omega_n \cdots \omega_1}}, \qquad n = 0, 1, 2, \dots .$$

Then, we have

$$x\, U\Phi_n = \sqrt{\omega_{n+1}}\, U\Phi_{n+1} + \alpha_{n+1}\, U\Phi_n + \sqrt{\omega_n}\, U\Phi_{n-1}\,,$$

so

$$\begin{aligned} U^* x\, U\Phi_n &= \sqrt{\omega_{n+1}}\, \Phi_{n+1} + \alpha_{n+1}\, \Phi_n + \sqrt{\omega_n}\, \Phi_{n-1} \\ &= (B^+ + B^\circ + B^-)\Phi_n\,. \end{aligned}$$

Therefore,

$$U^* x\, U = B^+ + B^\circ + B^-.$$

Then we have

$$\begin{aligned} \langle \Phi_0, (B^+ + B^\circ + B^-)^m \Phi_0 \rangle &= \langle U\Phi_0, U(B^+ + B^\circ + B^-)^m \Phi_0 \rangle = \langle U\Phi_0, x^m U\Phi_0 \rangle \\ &= \langle P_0, x^m P_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx) = M_m(\mu). \end{aligned}$$

This proves the assertion. ∎

Remark 4.3.7 U is not necessarily unitary, i.e, surjective.

4.4 Applications to Distance-Regular Graphs

Let $G = (V, E)$ be a connected graph. Fix an origin $o \in V$ we consider the stratification:

$$V = \bigcup_{n=0}^{\infty} V_n\,, \qquad V_n = \{x \in V\,;\, \partial(x, o) = n\}.$$

Let A be the adjacency matrix.

We define three matrices A^ϵ as follows: Let $x \in V_n$.

$$(A^+)_{yx} = \begin{cases} 1, & \text{if } y \sim x \text{ and } y \in V_{n+1}, \\ 0, & \text{otherwise,} \end{cases}$$

$$(A^\circ)_{yx} = \begin{cases} 1, & \text{if } y \sim x \text{ and } y \in V_{n}, \\ 0, & \text{otherwise,} \end{cases}$$

$$(A^-)_{yx} = \begin{cases} 1, & \text{if } y \sim x \text{ and } y \in V_{n-1}, \\ 0, & \text{otherwise,} \end{cases}$$

It is convenient to unify the above in the following form:

$$(A^\epsilon)_{yx} = \begin{cases} 1, & \text{if } y \sim x \text{ and } y \in V_{n+\epsilon}, \\ 0, & \text{otherwise,} \end{cases} \qquad \epsilon \in \{+, -, \circ\}.$$

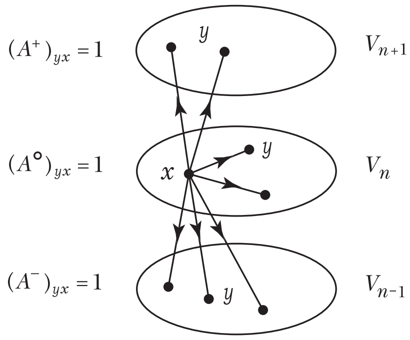


Figure 4.2: Quantum decomposition of the adjacency matrix

Lemma 4.4.1 (1) $A = A^+ + A^- + A^\circ$.

(2) $(A^+)^* = A^-$ *and* $(A^-)^* = A^+$.

(3) $(A^\circ)^* = A$.

PROOF. Easy. ∎

Definition 4.4.2 We call $A = A^+ + A^- + A^\circ$ the *quantum decomposition* of the adjacency matrix with respect to the origin $o \in V$. Each A^ϵ is called a *quantum component*.

We define

$$\Phi_n = \frac{1}{\sqrt{|V_n|}} \sum_{x \in V_n} e_x \,.$$

By definition, $\Phi_0 = e_o$. We note that

$$\langle \Phi_m, \Phi_n \rangle = \delta_{mn}.$$

Let $\Gamma = \Gamma(G, o)$ denote the subspace of $C(V)$ spanned by $\Phi_0, \Phi_1, \dots$.

Lemma 4.4.3 *For* $x \in V_n$,

$$A^\epsilon e_x = \sum_{y \in V_{n+\epsilon},\, y \sim x} e_y \,, \qquad \epsilon \in \{+, -, \circ\}.$$

Lemma 4.4.4

$$A^\epsilon \Phi_n = \frac{1}{\sqrt{|V_n|}} \sum_{y \in V_{n+\epsilon}} |\omega_{-\epsilon}(y)| e_y \tag{4.13}$$

PROOF. Let us consider A^+. By definition

$$\sqrt{|V_n|} A^+ \Phi_n = \sum_{x \in V_n} A^+ e_x = \sum_{y \in V_{n+1}} |\omega_-(y)| e_y \,,$$

which proves the assertion. ∎

We see from (4.13) that $A^\epsilon \Phi_n$ is not necesarily a constant multiple of $\Phi_{n+\epsilon}$, in other words, Γ is not necessarily closed under the actions of the quantum components. The quqnatum probabilistic approach is useful in the case where

(i) Γ is closed under the actions of the quantum components;

(ii) Γ is asymptotically closed under the actions of the quantum components.

Here we discuss typical examples for (i).

Theorem 4.4.5 *Let G be a distance-regular graph with the intersection array:*

$$\begin{pmatrix} c_0 & c_1 & c_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ b_0 & b_1 & b_2 & \cdots \end{pmatrix}.$$

Fix an origin $o \in V$, we consider the stratification of G, the unit vectors $\Phi_0 = e_o, \Phi_1, \Phi_2, \dots$, the linear space $\Gamma = \Gamma(G, o)$, and the quantum decomposition of the adjacency matrix $A = A^+ + A^- + A^\circ$. Then, Γ is invariant under the actions of the quantum components A^ϵ. Moreover,

$$A^+\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1}, \quad n = 0, 1, \dots, \tag{4.14}$$

$$A^-\Phi_0 = 0, \quad A^-\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1}, \quad n = 1, 2, \dots, \tag{4.15}$$

$$A^\circ\Phi_n = \alpha_{n+1}\Phi_n, \quad n = 0, 1, 2, \dots, \tag{4.16}$$

where

$$\omega_n = b_{n-1}c_n\,, \qquad \alpha_n = a_{n-1}, \qquad n = 1, 2, \dots.$$

Proof. We continue the calculation of (4.13). Since G is distance-regular, we know that for $y \in V_{n+\epsilon}$,

$$|\omega_{-\epsilon}(y)| = \begin{cases} c_{n+1}\,, & \text{if } \epsilon = +, \\ a_n\,, & \text{if } \epsilon = \circ, \\ b_{n-1}\,, & \text{if } \epsilon = -. \end{cases}$$

Then, for $\epsilon = +$ we have

$$A^+\Phi_n = \frac{1}{\sqrt{|V_n|}} \sum_{y \in V_{n+1}} c_{n+1} e_y = c_{n+1} \frac{\sqrt{|V_{n+1}|}}{\sqrt{|V_n|}}\,\Phi_{n+1}\,. \tag{4.17}$$

Similarly,

$$A^-\Phi_n = \frac{1}{\sqrt{|V_n|}} \sum_{y \in V_{n-1}} b_{n-1} e_y = b_{n-1} \frac{\sqrt{|V_{n-1}|}}{\sqrt{|V_n|}}\,\Phi_{n-1} \tag{4.18}$$

and

$$A^\circ\Phi_n = \frac{1}{\sqrt{|V_n|}} \sum_{y \in V_n} a_n e_y = a_n \Phi_n\,. \tag{4.19}$$

Now (4.16) is obvious from (4.18). We note that

$$b_n|V_n| = c_{n+1}|V_{n+1}|,$$

wich counts the number of edges betwen two strata V_n and V_{n+1}. Then, the coefficient on the right-hand side of (4.17) becomes

$$c_{n+1}\,\frac{\sqrt{|V_{n+1}|}}{\sqrt{|V_n|}} = c_{n+1}\sqrt{\frac{b_n}{c_{n+1}}} = \sqrt{b_n c_{n+1}} = \sqrt{\omega_{n+1}}\,.$$

Similarly, for (4.18) we have

$$b_{n-1}\,\frac{\sqrt{|V_{n-1}|}}{\sqrt{|V_n|}} = b_{n-1}\sqrt{\frac{c_n}{b_{n-1}}} = \sqrt{b_{n-1}c_n} = \sqrt{\omega_n}\,.$$

These show that (4.14) and (4.15). ∎

The main point is that, accroding to the quantum decomposition of the adjacency matrix $A = A^+ + A^- + A^\circ$, we found an interacting Fock space structure. Thus,

$$A\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1} + \alpha_{n+1}\Phi_n + \sqrt{\omega_n}\,\Phi_{n-1}, \qquad n = 0, 1, 2, \dots, \tag{4.20}$$

where

$$\omega_n = b_{n-1}c_n\,, \qquad \alpha_n = a_{n-1}, \qquad n = 1, 2, \dots.$$

Theorem 4.4.6 *Let G be a distance-regular graph with adjacency matrix A. Let $(\{\omega_n\}, \{\alpha_n\})$ be defined by*

$$\omega_n = b_{n-1}c_n\,, \qquad \alpha_n = a_{n-1}\,, \qquad n = 1, 2, \dots,$$

where a_n, b_n, c_n come from the intersection array of G. A probability measure μ satisfies

$$\varphi_o(A^m) = (A^m)_{oo} = M_m(\mu) = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots,$$

if and only if the Jacobi coefficients of μ coincide with $(\{\omega_n\}, \{\alpha_n\})$.

Exercises 4

4.1 Let $T_n(x)$ be a polynomial of degree n determined by

$$T_n(\cos\theta) = \cos n\theta.$$

Show that

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

and

$$\int_{-1}^{1} T_m(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} \pi, & m = n = 0, \\ \pi/2, & m = n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

4.2 Let G is a finite distance-regular graph. Then two states φ_{tr} and φ_o on the adjacency algebra $\mathcal{A}(G)$ coincide. [Hint: Any $a \in \mathcal{A}(G)$ is a linear combination of distance matrices: $a = \sum_{k=0}^{d} c_k A^{(k)}$.]

4.3 Let $(\Gamma(\mathbf{C}), \{\Phi_n\}, B^+, B^-)$ be an interacting Fock space associated with $\{\omega_n\}$. Examine the action of the commutator $[B^-, B^+] = B^-B^+ - B^+B^-$. In particular, the cases when $\{\omega_n = n\}$ (Boson Fock space), $\{\omega_n \equiv 1\}$ (free Fock space), and $\{\omega_1 = 1, \omega_2 = \cdots = 0\}$ (Fermion Fock space).

4.4* Find the Jacobi coefficients associated with the one-dimensional integer lattice $\mathbb{Z}$.

4.5* Find the Jacobi coefficients associated with the homogeneous tree of degree κ. ($\mathbb{Z}$ is the case of $\kappa = 2$)

4.6* Prove that every state φ on $M(n, \mathbf{C})$ is expressible in terms of a density matrix $\rho \in M(n, \mathbf{C})$ in such a way that

$$\varphi(a) = \mathrm{Tr}\,(\rho a), \qquad a \in M(n, \mathbf{C}).$$

Moreover, ρ is uniquely determined.

4.7* Let us study the cube in detail (Exercise 3.3).

(1) Find the spectrum.

(2) Find the Jacobi coefficients.

(3) Find the associated polynomials $\{P_n(x)\}$ determined by the three-term recurrence relation.

(4) Examine that $\{P_n(x)\}$ is orthogonal polynomials associated with the spectral distribution.

5 Stieltjes Transform and Continued Fraction

5.1 Overview

With each $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ we associated two sequences, the moment sequence $\{M_m\}$ and the Jacobi parameter $(\{\omega_n\}, \{\alpha_n\})$.

$$\begin{tikzcd} & \mathfrak{P}_{\mathrm{fm}}(\mathbf{R}) \arrow[dl, "M"'] \arrow[dr, "J"] & \\ \mathfrak{M} \arrow[rr, "\mathrm{bij}"'] & & \mathfrak{J} \end{tikzcd}$$

Here we repeat the definitions of $\mathfrak{M}$. For an infinite sequence of real numbers $\{M_0 = 1, M_1, M_2, \dots\}$ we define the *Hankel determinants* by

$$\Delta_m = \det \begin{bmatrix} M_0 & M_1 & \dots & M_m \\ M_1 & M_2 & \dots & M_{m+1} \\ \vdots & \vdots & & \vdots \\ M_m & M_{m+1} & \dots & M_{2m} \end{bmatrix}, \qquad m = 0, 1, 2, \dots. \tag{5.1}$$

Let $\mathfrak{M}$ be the set of infinite sequences of real numbers $\{M_0 = 1, M_1, M_2, \dots\}$ satisfying one of the following two conditions:

(i) [infinite type] $\Delta_m > 0$ for all $m = 0, 1, 2, \dots$;

(ii) [finite type] there exists $m_0 \geq 1$ such that $\Delta_0 > 0, \Delta_1 > 0, \dots, \Delta_{m_0-1} > 0$ and $\Delta_{m_0} = \Delta_{m_0+1} = \cdots = 0$.

Let $\mathfrak{J}$ be the set of pairs of sequences $(\{\omega_n\}, \{\alpha_n\})$ satisfying one of the following conditions:

(i) [infinite type] $\{\omega_n\}$ is a Jacobi sequence of infinite type and $\{\alpha_n\}$ is an infinite sequence of real numbers;

(ii) [finite type] $\{\omega_n\}$ is a Jacobi sequence of finite type and $\{\alpha_n\}$ is a finite real sequence $\{\alpha_1, \dots, \alpha_{m_0}\}$, where $m_0 \geq 1$ is the smallest number such that $\omega_{m_0} = 0$.

The map $\mathfrak{P}_{\mathrm{fm}}(\mathbf{R}) \to \mathfrak{M}$ is surjective. In fact, it follows from Hamburger's theorem that for any $\{M_m\}$ satisfying condition (M1) or (M2) the exists $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ whose moment sequence coincides with $\{M_m\}$. But the map $\mathfrak{P}_{\mathrm{fm}}(\mathbf{R}) \to \mathfrak{M}$ is not injective.

Definition 5.1.1 A probability measure $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ is called the *solution of a determinate moment problem* if $M^{-1}(M(\mu)) = \{\mu\}$.

Theorem 5.1.2 (Carlemen's moment test) *Let $\{M_m\} \in \mathfrak{M}$. If*

$$\sum_{m=1}^{\infty} M_{2m}^{-\frac{1}{2m}} = +\infty,$$

then there exists a unique $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ such that $M_m(\mu) = M_m$ for all $m = 1, 2, \dots$.

The proof is omitted, see e.g., Shohat–Tamarkin [11].

Example 5.1.3 (1) If $\mathrm{supp}\,(\mu)$ is compact, then μ is the solution of a determinate moment problem.

(2) A classical Gaussian measure $N(m, \sigma^2)$ is the solution of a determinate moment problem. The density of the standard Gaussian measure $N(0, 1)$ is given by

$$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\,.$$

In fact, by the Stirling formula we have

$$M_{2m} = \frac{(2m)!}{2^m m!} \sim \sqrt{2}\left(\frac{2m}{e}\right)^m.$$

(3) The classical Poisson measure with parameter $\lambda > 0$ is defined by

$$p_\lambda = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\, \delta_k\,.$$

The Poisson measure is the solution of a determinate moment problem. It is easily verified that $M_m \le (\lambda + m)^m$.

Recall that, given $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$, we obtain the Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$ from the three-term recurrence relation (Theorem 4.3.1) satisfied by the orthogonal polynomials $\{P_n\}$ associated with μ. Since the Gram-Schmidt orthogonalization is performed by using the moments of μ, the Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$ depend only on $\{M_m(\mu)\}$. Therefore, the map $\mathfrak{M} \to \mathfrak{J}$ is well defined.

Theorem 5.1.4 *The map $F : \mathfrak{M} \to \mathfrak{J}$ is bijective.*

The proof is omitted, see e.g., Hora–Obata [3].

Remark 5.1.5 $F^{-1} : \mathfrak{J} \to \mathfrak{M}$ is expressed explicitly by the Accardi–Bożejko formula [12].

Theorem 5.1.6 (Carleman) *Let $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ and $(\{\omega_n\}, \{\alpha_n\})$ be its Jacobi coefficients. If*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{\omega_n}} = +\infty,$$

then μ is the solution of a determinate moment problem. (If $\{\omega_n\}$ contains 0, we understand the above condition is satisfied.)

The main topic in this chapter is how to recover $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ from $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$ when the uniqueness holds. For that purpose we need the Stieltjes transform.

$$\begin{tikzcd}
 & \mu \arrow[r] \arrow[d] \arrow[ddl] & \{P_n(x)\} \arrow[dd] \\
 & G(z) \arrow[dl] \arrow[dr] & \\
\{M_m\} & & (\{\omega_n\}, \{\alpha_n\}) \arrow[ll]
\end{tikzcd}$$

5.2 Stieltjes Transform

For a probability measure $\mu \in \mathfrak{P}(\mathbf{R})$ (not necessarily having finite moments) the *Stieltjes transform* or the *Cauchy transform* is defined by

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x}\,. \tag{5.2}$$

The integral exists for all $z \in \mathbf{C} \setminus \operatorname{supp}\mu$ since the distance between such a z and $\operatorname{supp}\mu$ is positive. We list some fundamental properties, the proofs of which are straightforward.

Proposition 5.2.1 *Let $G(z) = G_\mu(z)$ be the Stieltjes transform of a probability measure $\mu \in \mathfrak{P}(\mathbf{R})$.*

(1) *$G(z)$ is analytic on $\mathbf{C} \setminus \operatorname{supp}\mu$.*

(2) *$\operatorname{Im} G(z) < 0$ for $\operatorname{Im} z > 0$ and $\operatorname{Im} G(z) > 0$ for $\operatorname{Im} z < 0$.*

(3) *$|G(z)| \le |\operatorname{Im} z|^{-1}$ for $\operatorname{Im} z \neq 0$.*

(4) *$G(\bar{z}) = \overline{G(z)}$. In particular, $G(z)$ is completely determined by its values on the upper half plane $\{\operatorname{Im} z > 0\}$.*

Example 5.2.2 For $\mu = \sum_{j=1}^{s} p_j \delta_{\lambda_j}$ we have

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \sum_{j=1}^{s} \frac{p_j}{z-\lambda_j}\,.$$

In contrast with the moment sequence, we have the following

Theorem 5.2.3 *For two probability measure $\mu_1, \mu_2 \in \mathfrak{P}(\mathbf{R})$, $G_{\mu_1} = G_{\mu_2}$ implies $\mu_1 = \mu_2$.*

The proof is direct from the inversion formula mentioned below.

Theorem 5.2.4 (Stieltjes inversion formula) *Let $G(z)$ be the Stieltjes transform of $\mu \in \mathcal{P}(R)$. Then for any pair of real numbers $s < t$, we have*

$$-\frac{2}{\pi} \lim_{y \to +0} \int_s^t \operatorname{Im} G(x+iy)dx = F(t) + F(t-0) - F(s) - F(s-0),$$

where F is the distribution function defined by $F(x) = \mu((-\infty, x])$.

Theorem 5.2.5 *Let $G(z)$ be the Stieltjes transform of $\mu \in \mathcal{P}(R)$. Then*

$$\rho(x) = -\frac{1}{\pi} \lim_{y \to +0} \operatorname{Im} G(x+iy)$$

exists $x \in \mathbf{R}$ a.e. and $\rho(x)dx$ is the absolutely continuous part of μ.

The discrete or singular continuous part of μ is more complicated to obtain from its Stieltjes transform. For our later application we only need the following

Proposition 5.2.6 *Let $\mu \in \mathfrak{P}(\mathbf{R})$. Then its Stieltjes transform $G(z)$ has a simple pole at $z = a \in \mathbf{R}$ if and only if a is an isolated point of $\operatorname{supp}\mu$, i.e., μ is a convex combination of δ_a and a probability measure $\nu \in \mathfrak{P}(\mathbf{R})$ such that $\operatorname{supp}\nu \cap \{a\} = \varnothing$ in such a way that*

$$\mu = c\delta_a + (1-c)\nu, \qquad 0 < c \le 1.$$

In that case, $c = \mathrm{Res}_{z=a} G(z)$.

5.3 Continued Fraction

First we recall the notion of a continued fraction. In general, expressions of the forms

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n}}}} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n} \tag{5.3}$$

and

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \tag{5.4}$$

are called *continued fractions*. Since the expressions in the left hand sides are space-consuming, we hereafter adopt the ones in the right hand sides. We only need to consider complex numbers $\{a_k\}$ and $\{b_k\}$. For the infinite continued fraction (5.4), if

$$\tau_n = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n}$$

exists (namely, denominator is not zero) except finitely many n and $\lim_{n\to\infty} \tau_n$ exists, we say that the infinite fraction converges and define

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots = \lim_{n\to\infty} \tau_n .$$

In other words, the value of the infinite continued fraction (5.4) is defined as the limit of the *nth approximant*:

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots = \lim_{n\to\infty} \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n} .$$

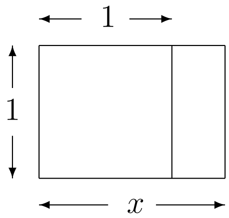
Example 5.3.1 (Euclidean algorithym) Every rational number q/p, $0 \le q \le p$, $p = 1, 2, \dots$, admits a continuous fraction expansion of the form:

$$\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \dots + \frac{1}{b_n}.$$

For example,

$$\frac{5}{13} = \cfrac{1}{2 + \cfrac{3}{5}} = \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{2}{3}}} = \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}$$

Example 5.3.2 (Golden number) The *golden number* x is defined in such a way that the big and small rectangles in the following picture are similar.



In fact, x satisfies that $x^2 - x - 1 = 0$ so that

$$x = \frac{1 + \sqrt{5}}{2} = 1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \cdots$$

This may be derived by successive application of rationalization of numerators. But, formally the following derivation is much simpler:

$$x = 1 + \frac{1}{x} = 1 + \cfrac{1}{1 + \cfrac{1}{x}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{x}}} = \dots = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

5.4 Finite Jacobi Matrices

Let $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$ and set

$$T = T_n = \begin{bmatrix} \alpha_1 & \sqrt{\omega_1} & & & & & \\ \sqrt{\omega_1} & \alpha_2 & \sqrt{\omega_2} & & & & \\ & \sqrt{\omega_2} & \alpha_3 & \sqrt{\omega_3} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \sqrt{\omega_{n-2}} & \alpha_{n-1} & \sqrt{\omega_{n-1}} \\ & & & & & \sqrt{\omega_{n-1}} & \alpha_n \end{bmatrix}, \tag{5.5}$$

whenever $\omega_{n-1} > 0$. A matrix of the form (5.5) is called a *Jacobi matrix (of finite type)*.

We set

$$e_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Proposition 5.4.1

$$\langle e_0, (z-T)^{-1} e_0 \rangle = \frac{1}{z-\alpha_1} \, \underset{-}{} \, \frac{\omega_1}{z-\alpha_2} \, \underset{-}{} \, \frac{\omega_2}{z-\alpha_3} \, \underset{-}{} \cdots \underset{-}{} \, \frac{\omega_{n-1}}{z-\alpha_n}. \tag{5.6}$$

PROOF. We set

$$(z-T)^{-1} e_0 = f = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix}.$$

First note that

$$\langle e_0, (z-T)^{-1} e_0 \rangle = \langle e_0, f \rangle = f_0 .$$

On the other hand, we see from $(z-T)f = e_0$ that

$$\begin{cases} (z-\alpha_1) f_0 - \sqrt{\omega_1}\, f_1 = 1, \\ -\sqrt{\omega_i}\, f_{i-1} + (z-\alpha_{i+1}) f_i - \sqrt{\omega_{i+1}}\, f_{i+1} = 0, \qquad i = 1, 2, \ldots, n-2, \\ -\sqrt{\omega_{n-1}}\, f_{n-2} + (z-\alpha_n) f_{n-1} = 0. \end{cases} \tag{5.7}$$

From the first relation in (5.7) we obtain

$$f_0 \left\{ (z-\alpha_1) - \sqrt{\omega_1}\, \frac{f_1}{f_0} \right\} = 1,$$

and hence

$$f_0 = \frac{1}{z - \alpha_1 - \sqrt{\omega_1}\, \dfrac{f_1}{f_0}}. \tag{5.8}$$

Similarly, from (5.7) we obtain

$$-\sqrt{\omega_i}\, f_{i-1} + f_i \left\{ (z-\alpha_{i+1}) - \sqrt{\omega_{i+1}}\, \frac{f_{i+1}}{f_i} \right\} = 0,$$

and therefore

$$\sqrt{\omega_i}\, \frac{f_i}{f_{i-1}} = \frac{\omega_i}{z - \alpha_{i+1} - \sqrt{\omega_{i+1}}\, \dfrac{f_{i+1}}{f_i}}. \tag{5.9}$$

Finally, from (5.7) we have

$$\sqrt{\omega_{n-1}}\, \frac{f_{n-1}}{f_{n-2}} = \frac{\omega_{n-1}}{z-\alpha_n}. \tag{5.10}$$

Combining (5.8)–(5.10), we come to

$$f_0 = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \cdots - \frac{\omega_{n-1}}{z-\alpha_n}\,,$$

from which (5.6) follows. ∎

Proposition 5.4.2 *For $k = 1, 2, \dots, n$ we define monic polynomials $P_k(z) = z^k + \cdots$ and $Q_{k-1}(z) = z^{k-1} + \cdots$ by*

$$\frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \cdots - \frac{\omega_{k-1}}{z-\alpha_k} = \frac{Q_{k-1}(z)}{P_k(z)}\,. \tag{5.11}$$

Then, the following recurrence relations are satisfied:

$$\begin{cases} P_0(z) = 1, \quad P_1(z) = z - \alpha_1\,, \\ P_k(z) = (z-\alpha_k)P_{k-1}(z) - \omega_{k-1}P_{k-2}(z), \quad k = 2, 3, \dots, n, \end{cases} \tag{5.12}$$

$$\begin{cases} Q_0(z) = 1, \quad Q_1(z) = z - \alpha_2\,, \\ Q_k(z) = (z-\alpha_{k+1})Q_{k-1}(z) - \omega_k Q_{k-2}(z), \quad k = 2, 3, \dots, n-1. \end{cases} \tag{5.13}$$

PROOF. By induction, see also Exercise 1. ∎

Proposition 5.4.3 (Determinantal formula) *For $k = 1, 2, \dots, n$ it holds that*

$$P_k(z) = \det \begin{bmatrix} z-\alpha_1 & -\sqrt{\omega_1} & & & & \\ -\sqrt{\omega_1} & z-\alpha_2 & -\sqrt{\omega_2} & & & \\ & -\sqrt{\omega_2} & z-\alpha_3 & -\sqrt{\omega_3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\sqrt{\omega_{k-2}} & z-\alpha_{k-1} & -\sqrt{\omega_{k-1}} \\ & & & & -\sqrt{\omega_{k-1}} & z-\alpha_k \end{bmatrix} = \det(z - T_k).$$

For $k = 2, 3, \dots, n$ it holds that

$$Q_{k-1}(z) = \det \begin{bmatrix} z-\alpha_2 & -\sqrt{\omega_2} & & & \\ -\sqrt{\omega_2} & z-\alpha_3 & -\sqrt{\omega_3} & & \\ & \ddots & \ddots & \ddots & \\ & & -\sqrt{\omega_{k-2}} & z-\alpha_{k-1} & -\sqrt{\omega_{k-1}} \\ & & & -\sqrt{\omega_{k-1}} & z-\alpha_k \end{bmatrix}.$$

PROOF. By expanding the determinants in the last column one can check easily that these determinants satisfy the recurrence relations in (5.12) and (5.13). ∎

We now need spectral properties of the Jacobi matrix T.

Proposition 5.4.4 *Every eigenvalue of $T = T_n$ is real and simple. Moreover,*

$$\operatorname{Spec} T_n = \{\lambda \in \mathbf{C}\,;\, P_n(\lambda) = 0\}. \tag{5.14}$$

PROOF. Since T is an $n \times n$ real symmetric matrix, it has n real eigenvalues. (5.14) is obvious from $\det(z - T_n) = P_n(z)$, see Proposition 5.4.3.

We prove that every eigenspace of T is of one dimension. Let λ be an eigenvalue of T and f a correswponding eigenvector. We write

$$f = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix}.$$

Then $(\lambda - T)f = 0$ is equivalent to the following

$$\begin{cases} (\lambda - \alpha_1) f_0 - \sqrt{\omega_1}\, f_1 = 0, \\ -\sqrt{\omega_i}\, f_{i-1} + (\lambda - \alpha_{i+1}) f_i - \sqrt{\omega_{i+1}}\, f_{i+1} = 0, \quad i = 1, 2, \dots, n-2, \\ -\sqrt{\omega_{n-1}}\, f_{n-2} + (\lambda - \alpha_n) f_{n-1} = 0. \end{cases} \tag{5.15}$$

Now let h, g be two eigenvectors corresponding to λ. Choose $(\alpha, \beta) \in \mathbf{R}^2$, $(\alpha, \beta) \neq (0, 0)$, such that $\alpha g_0 + \beta h_0 = 0$. Since $f = \alpha g + \beta h$ satisfies $(\lambda - T)f = 0$, we have (5.15). Note that $f_0 = 0$. Then, succesive application of (5.15) implies $f_1 = \cdots = f_{n-1} = 0$. Thus we have $f = 0$, which means that g and h are linearly dependent. Consequently, the eigenspace corresponding to λ is of one dimension. ∎

Proposition 5.4.5 *For $\lambda \in \operatorname{Spec} T$ we put*

$$f(\lambda) = \begin{bmatrix} P_0(\lambda) \\ P_1(\lambda)/\sqrt{\omega_1} \\ \vdots \\ P_{n-1}(\lambda)/\sqrt{\omega_{n-1} \cdots \omega_1} \end{bmatrix}. \tag{5.16}$$

Then $f(\lambda) \neq 0$ and $Tf(\lambda) = \lambda f(\lambda)$. Namely, $f(\lambda)$ is an eigenvector associated with λ.

PROOF. $f(\lambda) \neq 0$ is obvious since $P_0(\lambda) = 1$. In view of (5.12) we obtain

$$\begin{aligned} &P_0(\lambda) = 1, \\ &P_1(\lambda) = \lambda - \alpha_1\,, \\ &P_k(\lambda) = (\lambda - \alpha_k) P_{k-1}(\lambda) - \omega_{k-1} P_{k-2}(\lambda), \quad k = 2, 3, \dots, n-1, \\ &0 = (\lambda - \alpha_n) P_{n-1}(\lambda) - \omega_{n-1} P_{n-2}(\lambda). \end{aligned}$$

The last identity comes from $P_n(\lambda) = \det(\lambda - T) = 0$. Then a simple computation yields

$$\sqrt{\omega_1}\,\frac{P_1(\lambda)}{\sqrt{\omega_1}} = \lambda - \alpha_1 = (\lambda - \alpha_1)P_0(\lambda),$$
$$\sqrt{\omega_k}\,\frac{P_k(\lambda)}{\sqrt{\omega_k\cdots\omega_1}} = (\lambda - \alpha_k)\frac{P_{k-1}(\lambda)}{\sqrt{\omega_{k-1}\cdots\omega_1}} - \sqrt{\omega_{k-1}}\,\frac{P_{k-2}(\lambda)}{\sqrt{\omega_{k-2}\cdots\omega_1}}\,,$$

for $k = 2, 3, \dots, n-1$, and

$$0 = (\lambda - \alpha_n)\frac{P_{n-1}(\lambda)}{\sqrt{\omega_{n-1}\cdots\omega_1}} - \sqrt{\omega_{n-1}}\,\frac{P_{n-2}(\lambda)}{\sqrt{\omega_{n-2}\cdots\omega_1}}\,.$$

The above relations are combined into a single identity: $(\lambda - T)f(\lambda) = 0$. ∎

Proposition 5.4.6 *Define a measure μ on $\mathbf{R}$ by*

$$\mu = \sum_{\lambda\in\operatorname{Spec}T} \|f(\lambda)\|^{-2}\delta_\lambda\,, \tag{5.17}$$

where $f(\lambda) \in \mathbf{R}^n$ is given by (5.16). *Then, $\mu \in \mathfrak{P}_{\rm fm}(\mathbf{R})$ and*

$$\langle e_0, (z - T)^{-1}e_0\rangle = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z - x}\,. \tag{5.18}$$

Proof. Since every eigenvalue of T is simple (Proposition 5.4.4), we see from Proposition 5.4.5 that $\{\|f(\lambda)\|^{-1}f(\lambda)\,;\,\lambda \in \operatorname{Spec}T\}$ becomes a complete orthonormal basis of $\mathbf{C}^n$. Hence

$$\begin{aligned}\langle e_0, (z - T)^{-1}e_0\rangle &= \sum_{\lambda\in\operatorname{Spec}T} \langle e_0, \|f(\lambda)\|^{-1}f(\lambda)\rangle\langle \|f(\lambda)\|^{-1}f(\lambda), (z - T)^{-1}e_0\rangle \\ &= \sum_{\lambda\in\operatorname{Spec}T} \|f(\lambda)\|^{-2}\langle e_0, f(\lambda)\rangle\langle(\bar z - T)^{-1}f(\lambda), e_0\rangle \\ &= \sum_{\lambda\in\operatorname{Spec}T} \|f(\lambda)\|^{-2}(z - \lambda)^{-1}.\end{aligned}$$

where we used $\langle e_0, f(\lambda)\rangle = P_0(\lambda) = 1$ and $(\bar z - T)^{-1}f(\lambda) = (\bar z - \lambda)^{-1}f(\lambda)$. Then, in view of (5.17) we obtain

$$\langle e_0, (z - T)^{-1}e_0\rangle = \sum_{\lambda\in\operatorname{Spec}T} \frac{\|f(\lambda)\|^{-2}}{z - \lambda} = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z - x}\,,$$

which proves (5.18).

We need to show that $\mu(\mathbf{R}) = 1$. This may be proved by observing asymptotics of both sides of (5.18). In fact, with the help of Propositions 5.4.1 and 5.4.2 we see that

$$\lim_{\substack{z\to\infty\\ \operatorname{Re}z=0}} z\langle e_0, (z - T)^{-1}e_0\rangle = \lim_{\substack{z\to\infty\\ \operatorname{Re}z=0}} \frac{zQ_{n-1}(z)}{P_n(z)} = 1, \tag{5.19}$$

where we applied the fact that both $zQ_{n-1}(z)$ and $P_n(z)$ are monic polynomials of degree n. On the other hand,

$$\lim_{\substack{z\to\infty \\ \operatorname{Re} z=0}} z\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \int_{-\infty}^{+\infty} \mu(dx) = \mu(\mathbf{R}) \tag{5.20}$$

by the dominated convergence theorem. We see from (5.19) and (5.20) that $\mu(\mathbf{R}) = 1$. ∎

Definition 5.4.7 For any probability measure μ (not necessarily having moments) the integral

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x}, \qquad \operatorname{Im} z \neq 0$$

converges and $G_\mu(z)$ becomes a holomorphic function in $\{\operatorname{Im} z \neq 0\} = \mathbf{C}\backslash\mathbf{R}$. We call $G_\mu(z)$ the *(Cauchy-) Stieltjes transform* of μ.

Theorem 5.4.8 *Let $\alpha_1, \dots, \alpha_n \in \mathbf{R}$ and $\omega_1 > 0, \dots, \omega_{n-1} > 0$. Then the polynomials $P_0(z), P_1(z), \dots, P_{n-1}(z)$ defined by the recurrence relation* (5.12) *are the orthogonal polynomials associated with μ defined in* (5.17)*. Therefore, the Jacobi coefficients of μ is $(\{\alpha_1, \dots, \alpha_n\}, \{\omega_1, \dots, \omega_{n-1}\})$. Moreover, the Stieltjies transform $G_\mu(z)$ admits a continued fraction expansion:*

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \cdots - \frac{\omega_{n-1}}{z-\alpha_n}.$$

Proof. By using the recurrence formula (5.12) we may see easily that

$$P_0(T)e_0 = e_0\,, \quad P_k(T)e_0 = \sqrt{\omega_k\cdots\omega_1}\,e_k\,, \qquad k = 1, 2, \dots, n-1. \tag{5.21}$$

On the other hand, for any polynomials p, q with real coefficients we have

$$\begin{aligned}
\langle p(T)e_0, q(T)e_0\rangle &= \sum_{\lambda\in\operatorname{Spec} T} \langle p(T)e_0, \|f(\lambda)\|^{-1}f(\lambda)\rangle\langle \|f(\lambda)\|^{-1}f(\lambda), q(T)e_0\rangle \\
&= \sum_{\lambda\in\operatorname{Spec} T} \|f(\lambda)\|^{-2}\langle e_0, p(T)f(\lambda)\rangle\langle q(T)f(\lambda), e_0\rangle \\
&= \sum_{\lambda\in\operatorname{Spec} T} \|f(\lambda)\|^{-2}p(\lambda)q(\lambda)\langle e_0, f(\lambda)\rangle\langle f(\lambda), e_0\rangle \\
&= \sum_{\lambda\in\operatorname{Spec} T} \|f(\lambda)\|^{-2}p(\lambda)q(\lambda) \\
&= \int_{-\infty}^{+\infty} p(x)q(x)\mu(dx).
\end{aligned}$$

Hence, in particular,

$$\int_{-\infty}^{+\infty} P_j(x)P_k(x)\mu(dx) = \langle P_j(T)e_0, P_k(T)e_0\rangle = \omega_j\cdots\omega_1\langle e_j, e_k\rangle$$

so that $P_0(z), P_1(z), \dots, P_{n-1}(z)$ are the orthogonal polynomials associated with μ. ∎

5.5 General Case

Let $(\{\omega_n\},\{\alpha_n\}) \in \mathfrak{J}$ be of infinite type. Then for any n, defining a Jacobi matrix T_n as in (5.5), we obtain a probability measure μ_n and the polynomials $\{P_0(x), P_1(x), \dots, P_n(x)\}$ as in the previous section. Since these polynomials are defined by the recurrence relation with $(\{\omega_n\},\{\alpha_n\})$, $\{P_0(x), P_1(x), \dots, P_n(x)\}$ are common for all μ_m for $m \ge n$. Consequently, given $(\{\omega_n\},\{\alpha_n\})$, we have an infinite sequence of probability measures mu_n, and an infinite sequence polynomials

$$P_0(x) = 1, \quad P_1(x), \dots, P_n(x) = x^n + \cdots, \quad , \dots .$$

Lemma 5.5.1 *Let $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ be a probability measure whose Jacobi coefficients are $(\{\omega_n\},\{\alpha_n\}) \in \mathfrak{J}$. Then, for any $m = 1, 2, \dots$ we have*

$$\lim_{n\to\infty} M_m(\mu_n) = M_m(\mu).$$

Proof. In general, $M_m(\nu)$ is described by the first m terms of the Jacobi coefficients of ν. Suppose that $n \ge m$. Then we see that

$$M_m(\mu_n) = M_m(\mu_{n+1}) = \cdots = M_m(\mu),$$

from which the assertion is clear. ∎

Theorem 5.5.2 *Let $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ be the solution of a determinate moment problem and $(\{\omega_n\},\{\alpha_n\})$ be the Jacobi coefficients. Then the Stieltjes transform $G_\mu(z)$ admits a continued fraction expansion:*

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \cdots - \frac{\omega_{n-1}}{z-\alpha_n},$$

where the right-hand side converges in $\{\mathrm{Im}\, z \neq 0\}$.

Proof. By Theorem 5.4.8 we have

$$\int_{-\infty}^{+\infty} \frac{\mu_n(dx)}{z-x} = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \cdots - \frac{\omega_{n-1}}{z-\alpha_n}.$$

On the other hand, it follows from Lemma 5.5.1 and the assumption that μ_n converges to μ weakly. Since $x \mapsto 1/(z-x)$ is a bounded continuous function on $\mathbf{R}$, we have

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \frac{\mu_n(dx)}{z-x} = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x}.$$

This completes the proof. ∎

Exercises 5

1. Compute the following continued fractions:

(1) $\dfrac{1}{2}\underset{+}{}\dfrac{3}{5}\underset{+}{}\dfrac{2}{3}$

(2) $\dfrac{1}{z-1}\underset{-}{}\dfrac{3}{z-2}\underset{-}{}\dfrac{1}{z}$

2. Find the continued fraction expansion.

(1) $\dfrac{7}{45}$

(2) $\dfrac{z+1}{z^2+2}$

3. Compute the following continued fractions:

(1) [silver number] $2+\dfrac{1}{2}\underset{+}{}\dfrac{1}{2}\underset{+}{}\dfrac{1}{2}\underset{+}{}\cdots$

(2) $\dfrac{1}{z}\underset{+}{}\dfrac{a}{z}\underset{+}{}\dfrac{a}{z}\underset{+}{}\cdots \qquad (a>0)$

4. Let $\mu = \dfrac{1}{4}\,\delta_{-2} + \dfrac{1}{2}\,\delta_0 + \dfrac{1}{4}\,\delta_{+1}$. Compute the Stieltjes transform $G(z)$. Then find its poles and residues.

5. Let $\sqrt{z}$ be defined by taking a branch of $\sqrt{1}=1$. Find the following limits:

$$\lim_{y\to+0}\sqrt{z} \qquad \lim_{y\to-0}\sqrt{z}$$

Similarly, define $\sqrt{z^2-4}$ by taking a branch in such a way that $\sqrt{z^2-4}>0$ for $z>2$. Compurte the following

$$\lim_{y\to+0}\sqrt{z^2-4} \qquad \lim_{y\to-0}\sqrt{z^2-4}$$

where $z=x+iy$.

6 Kesten Distributions

6.1 Homogeneous Trees

Definition 6.1.1 A connected graph is called a *tree* if it has no cycles. A tree is called *homogeneous* if it is regular.

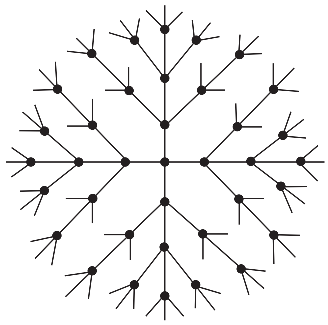


Figure 6.1: Homogeneous tree of degree 4

Let T_κ be the homoeeous tree of deree $\kappa \geq 2$ and $A = A_\kappa$ the adjacency matrix. We choose and fix a vertex $o \in T_\kappa$ as an origin (root). Our interests are:

(i) Find the vacuum spectral distribution of A, namely, a probability measure satisfying

$$\langle e_o, A^m e_o \rangle = |\{m\text{-step walks from } o \text{ to itself}\}|. = \int_{-\infty}^{+\infty} x^m \mu_\kappa(dx), \qquad m = 1, 2, \dots,$$

(ii) Asymptotic behavior of μ_κ for a large κ.

6.2 Vacuum Spectral Distribution

Recall that T_κ is a distance-regular graph with intersection array:

$$\begin{pmatrix} 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ \kappa & \kappa-1 & \kappa-1 & \dots \end{pmatrix}$$

We see from Theorem 4.4.6 that the vacuum spectral distribution $\mu = \mu_\kappa$ has the Jacobi parameter

$$\omega_n = b_{n-1}c_n : \kappa, \kappa-1, \kappa-1, \dots; \qquad \alpha_n = a_{n-1} \equiv 0.$$

Namely,

Lemma 6.2.1 *The vacuum spectral distribution μ_κ is a probability measure whose Jacobi coefficients are*

$$\omega_1 = \kappa, \quad \omega_2 = \omega_3 = \dots = \kappa - 1, \qquad \alpha_1 = \alpha_2 = \dots = 0.$$

Therefore, μ_κ is determined by

$$\int_{-\infty}^{+\infty} \frac{\mu_\kappa(dx)}{z-x} = \frac{1}{z-}\,\frac{\kappa}{z-}\,\frac{\kappa-1}{z\;\;-}\,\frac{\kappa-1}{z\;\;-}\cdots .$$

We now introduce the following

Definition 6.2.2 Let $p > 0$, $q \ge 0$ be constant numbers. A probabilty distribution on $\mathbf{R}$ whose Jacobi parameters are given by

$$\omega_1 = p, \quad \omega_2 = \omega_3 = \cdots = q, \qquad \alpha_n \equiv 0,$$

is called the *Kesten distribution* with parameters p, q. In other words, the Kesten distribution with parameters p, q is determined by

$$\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z-}\,\frac{p}{z-}\,\frac{q}{z-}\,\frac{q}{z-}\cdots .$$

Remark 6.2.3 By the Carleman condition we see that the Kesten distribution is uniquely determined by the Jacobi parameters.

Theorem 6.2.4 *The vacuum spectral distribution μ_κ of the homogeneous tree of degree κ is the Kesten distribution with parameter $\kappa, \kappa-1$.*

6.3 Explicit form of the Kesten distribution

We start with the Stieltjes transform:

$$G(z) \equiv \frac{1}{z-}\,\frac{p}{z-}\,\frac{q}{z-}\,\frac{q}{z-}\cdots .$$

Straitforward computation yields

$$G(z) = -\frac{1}{2}\,\frac{(p-2q)z + p\sqrt{z^2-4q}}{p^2-(p-q)z^2} .$$

Applying the Stieltjes inversion formula:

$$\rho(x) = -\frac{1}{\pi}\lim_{y\to+0} \operatorname{Im} G(x+iy) = \begin{cases} 0, & |x| > 2\sqrt{q}, \\ \dfrac{p}{2\pi}\,\dfrac{\sqrt{4q-x^2}}{p^2-(p-q)x^2}\,, & |x| < 2\sqrt{q}. \end{cases}$$

We now remark the following

Lemma 6.3.1

$$\frac{p}{2\pi}\int_{-2\sqrt{q}}^{+2\sqrt{q}} \frac{\sqrt{4q-x^2}}{p^2-(p-q)x^2}\,dx = \begin{cases} 1, & 0 < p \le 2q, \\ \dfrac{q}{p-q}, & 0 < 2q \le p, \end{cases}$$

PROOF. Straightforward computation. ∎

Therefore, when $0 < p \le 2q$, $\rho(x)dx$ is a probability measure so that $\mu_\kappa(dx) = \rho(x)dx$. Therefore,

$$G(z) = \int_{-2\sqrt{q}}^{+2\sqrt{q}} \frac{\rho(x)}{z-x}\, dx.$$

However, when $0 < 2q \le p$, $\rho(x)dx$ is not a probability measure and μ contains discrete or singular continuous parts. In fact, $G(z)$ has two poles at $\pm p/\sqrt{p-q}$ (which are outside of $[-2\sqrt{q}, 2\sqrt{q}]$ when $p > q$.) The residues are easily computed

$$\operatorname*{Res}_{z=\pm\frac{p}{\sqrt{p-q}}} G(z) = \frac{p-2q}{2(p-q)}\,.$$

Consequently, we come to the explicit form of the Kesten distributions.

Theorem 6.3.2 *The Kesten distribution with parameter $p > 0$, $q \ge 0$ is given by*

$$\mu(dx) = \begin{cases} \rho(x)dx, & 0 < p \le 2q, \\ \rho(x)dx + \dfrac{p-2q}{2(p-q)}\left(\delta_{-\frac{p}{\sqrt{p-q}}} + \delta_{\frac{p}{\sqrt{p-q}}}\right), & 0 < 2q \le p, \\ \dfrac{1}{2}\left(\delta_{-\sqrt{p}} + \delta_{\sqrt{p}}\right), & q = 0, \end{cases}$$

where

$$\rho(x) = \begin{cases} 0, & |x| > 2\sqrt{q}, \\ \dfrac{p}{2\pi}\,\dfrac{\sqrt{4q-x^2}}{p^2-(p-q)x^2}\,, & |x| < 2\sqrt{q}. \end{cases}$$

Theorem 6.3.3 *The vacuum spectral distribution of T_κ is given by $\mu_\kappa(dx) = \rho_\kappa(x)dx$ with*

$$\rho_\kappa(x) = \frac{\kappa}{2\pi}\,\frac{\sqrt{4(\kappa-1)-x^2}}{\kappa^2-x^2}\,.$$

6.4 Asymptotics of T_κ as $\kappa \to \infty$

We are interested in the asymptotic behavior of μ_κ as $\kappa \to \infty$. Note first that

$$\mathrm{mean}(\mu_\kappa) = \int_{-\infty}^{+\infty} x\mu_\kappa(dx) = (A)_{oo} = 0,$$

$$\mathrm{var}(\mu_\kappa) = \int_{-\infty}^{+\infty} (x - \mathrm{mean}(\mu_\kappa))^2\mu_\kappa(dx) = (A^2)_{oo} = \deg(o) = \kappa.$$

Therefore,

$$\frac{A}{\sqrt{\kappa}} = \frac{A^+}{\sqrt{\kappa}} + \frac{A^-}{\sqrt{\kappa}}$$

is a reasonable scaling for $\kappa \to \infty$.

It follows from the intersection array of T_κ that

$$\frac{A^+}{\sqrt{\kappa}}\,\Phi_0 = \Phi_1, \qquad \frac{A^+}{\sqrt{\kappa}}\,\Phi_n = \sqrt{\frac{\kappa-1}{\kappa}}\,\Phi_{n+1} \quad (n \ge 1) \tag{6.1}$$

$$\frac{A^-}{\sqrt{\kappa}}\,\Phi_0 = 0, \qquad \frac{A^-}{\sqrt{\kappa}}\,\Phi_1 = \Phi_0, \qquad \frac{A^-}{\sqrt{\kappa}}\,\Phi_n = \sqrt{\frac{\kappa-1}{\kappa}}\,\Phi_{n-1} \quad (n \ge 2) \tag{6.2}$$

The actions of $\dfrac{A_\kappa^\pm}{\sqrt{\kappa}}$ in the limit as $\kappa \to \infty$ are now easily expected. We are now in a position to introduce the following

Definition 6.4.1 An interacting Fock space associated with the Jacobi sequence $\omega_n \equiv 1$ is called the *free Fock space*. Namely, the free Fock space $(\Gamma_{\text{free}}, \{\Psi_n\}, B^+, B^-)$ is defined as

$$B^+\Phi_n = \Phi_{n+1} \quad (n \ge 0), \qquad B^-\Phi_0 = 0, \quad B^-\Phi_n = \Phi_{n-1} \quad (n \ge 1). \tag{6.3}$$

Theorem 6.4.2 (Quantum Central Limit Theorem) *For any $\epsilon_1, \dots, \epsilon_m \in \{\pm\}$ and $m = 1, 2, \dots$ we have*

$$\lim_{\kappa\to\infty} \left\langle \Phi_0, \frac{A_\kappa^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A_\kappa^{\epsilon_1}}{\sqrt{\kappa}}\,\Phi_0 \right\rangle = \langle \Psi_0, B^{\epsilon_m} \cdots B^{\epsilon_1}\Psi_0\rangle$$

In short, we say that

$$\lim_{\kappa\to\infty} \frac{A_\kappa^\pm}{\sqrt{\kappa}} = B^\pm$$

in the sense of stochastic convergence.

Proof. More generally, we may prove that

$$\lim_{\kappa\to\infty} \left\langle \Phi_i, \frac{A_\kappa^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A_\kappa^{\epsilon_1}}{\sqrt{\kappa}}\,\Phi_j \right\rangle = \langle \Psi_i, B^{\epsilon_m} \cdots B^{\epsilon_1}\Psi_j\rangle \tag{6.4}$$

for any $i, j \ge 0$. The proof is by induction on m. For $m = 1$ we need to prove that

$$\lim_{\kappa\to\infty} \left\langle \Phi_i, \frac{A_\kappa^{\epsilon_1}}{\sqrt{\kappa}}\,\Phi_j \right\rangle = \langle \Psi_i, B^{\epsilon_1}\Psi_j\rangle \tag{6.5}$$

for any $i, j \ge 1$ and $\epsilon_1 = \pm$. Suppose that $\epsilon_1 = +$. By (6.1),

$$\lim_{\kappa\to\infty} \left\langle \Phi_i, \frac{A_\kappa^+}{\sqrt{\kappa}}\,\Phi_0 \right\rangle = \lim_{\kappa\to\infty} \langle \Phi_i, \Phi_1\rangle = \langle \Psi_i, \Psi_1\rangle = \langle \Psi_i, B^+\Psi_0\rangle,$$

$$\lim_{\kappa\to\infty} \left\langle \Phi_i, \frac{A_\kappa^+}{\sqrt{\kappa}}\,\Phi_j \right\rangle = \lim_{\kappa\to\infty} \sqrt{\frac{\kappa-1}{\kappa}}\,\langle \Phi_i, \Phi_{j+1}\rangle = \langle \Psi_i, \Psi_{j+1}\rangle = \langle \Psi_i, B^+\Psi_j\rangle,$$

where $j \ge 1$. Thus, (6.5) is shown for $\epsilon_1 = +$. The case of $\epsilon_1 = -$ is similar.

We now come to the induction step, but the idea is similar. The detailed proof is left to the reader. ∎

As a direct consequence, we have

Theorem 6.4.3 *It holds that*

$$\lim_{\kappa\to\infty} \left\langle e_o, \left(\frac{A_\kappa}{\sqrt{\kappa}}\right)^m e_o \right\rangle = \langle \Phi_0, (B^+ + B^-)^m \Phi_0\rangle, \qquad m = 1, 2, \dots.$$

6.5 Chebyshev Polynomials of Second Kind

Definition 6.5.1 The *Chebyshev polynomialof second kind* $U_n(x)$ is defined by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad n = 0, 1, 2, \dots .$$

In fact, we obtain

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

Moreover, by simple calculation we see that

$$\int_{-1}^{+1} U_m(x)U_n(x)\sqrt{1-x^2}\,dx = \frac{\pi}{2}\,\delta_{mn}\,.$$

Definition 6.5.2 The probability distribution

$$\frac{1}{2\pi}\sqrt{4-x^2}\,1_{[-2,2]}(x)dx$$

is called the *Wigner semicircle law.* This is normalized to have mean 0 and variance 1.

The Wigner semicircle law is the Lesten distribution with parameter $p = q = 1$.

Theorem 6.5.3 *Set $\tilde{U}_n(x) = U_n\left(\frac{x}{2}\right)$. Then $\{\tilde{U}_n(x)\}$ is the orthogonal polynomial with respect to the Wigner semicircle law. Moreover, its Jacobi coefficients are*

$$(\{\omega_n \equiv 1\}, \{\alpha_n \equiv 0\}).$$

Proof. Direct computation. ∎

Therefore,

Theorem 6.5.4 *Let $(\Gamma_{\mathit{free}}, \{\Psi_n\}, B^+, B^-)$ be the free Fock space. Then,*

$$\langle \Phi_0, (B^+ + B^-)^m \Phi_0 \rangle = \frac{1}{2\pi}\int_{-2}^{+2} x^m\sqrt{4-x^2}\,dx, \qquad m = 1, 2, \dots .$$

Combining with Theorem 6.4.3, we obtain the following

Theorem 6.5.5 (Asymptotic spectral distribution for T_κ) *It holds that*

$$\lim_{\kappa\to\infty}\left\langle e_o, \left(\frac{A_\kappa}{\sqrt{\kappa}}\right)^m e_o \right\rangle = \frac{1}{2\pi}\int_{-2}^{+2} x^m\sqrt{4-x^2}\,dx, \qquad m = 1, 2, \dots .$$

Exercises 6

1. Fix a vertex $o \in V$ of the homogeneous tree T_κ. Let $V_n = \{x \in V\,;\, \partial(x,o) = n\}$. Show that

$$|V_0| = 1, \quad |V_1| = \kappa, \quad |V_2| = \kappa(\kappa-1), \quad \dots, \quad |V_n| = \kappa(\kappa-1)^{n-1}.$$

Then verify directly the equality:

$$\frac{|V_{n+1}|}{|V_n|} = \frac{b_n}{c_{n+1}}$$

where b_n and c_n are constant numbers appearing in the intersection array of T_κ.

2. Compute the following continued fraction:

$$\frac{1}{z} - \frac{p}{z} - \frac{q}{z} - \frac{q}{z} - \cdots$$

3. Verify the facts on the Chebyshev polynomial of second kind defined above.

4*. Verify the integral formula mentioned in Lemma 6.3.1.

5*. For the density function: $\rho_\kappa(x) = \dfrac{\kappa}{2\pi} \dfrac{\sqrt{4(\kappa-1) - x^2}}{\kappa^2 - x^2}$ compute the scaling limit:

$$\lim_{\kappa\to\infty} \sqrt{\kappa}\, \rho_\kappa(\sqrt{\kappa}x)$$

6*. Let μ be a probability distribution and $(\{\omega_n\}, \{\alpha_n\})$ the Jacobi coefficients. Show the following:

(1) The Jacobi parameters of the translated $\mu(dx - s)$ are given by $(\{\omega_n\}, \{\alpha_n + s\})$, $s \in \mathbf{R}$.

(2) The Jacobi parameters of the scaled $\mu(\lambda^{-1}dx)$ are given by $(\{\lambda^2\omega_n\}, \{\lambda\alpha_n\})$, $\lambda \in \mathbf{R}$, $\lambda \neq 0$.

7 Catalan Paths and Applications

7.1 Moments of the Wigner Semicircle Law

The Wigner semicircle law appears in the last chapter. It is absolutely continuous with respect to the Lebesgue measure and has the density function:

$$\rho(x) = \begin{cases} \dfrac{1}{2\pi}\sqrt{4-x^2}\,, & |x| \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

This is normalized to have mean 0 and variance 1.

Theorem 7.1.1 *For $m = 1, 2, \dots$ the $2m$-th moment of the Wigner semicircle law is given by*

$$\frac{1}{2\pi}\int_{-2}^{+2} x^{2m}\sqrt{4-x^2}\,dx = \frac{(2m)!}{(m+1)!m!} = \frac{1}{m+1}\binom{2m}{m}.$$

The moments of odd orders vanish.

PROOF. By direct calculation. ∎

7.2 Vacuum Distribution of Free Fock Space

Let $(\Gamma_{\mathrm{free}}, \{\Phi_n\}, B^+, B^-)$ be a free Fock space. In the last chapter we already showed (slightly less rigorously) that

$$\langle \Phi_0, (B^+ + B^-)^m \Phi_0 \rangle = \frac{1}{2\pi}\int_{-2}^{+2} x^m \sqrt{4-x^2}\,dx, \qquad m = 1, 2, \dots. \tag{7.1}$$

Therefore, it follows from Theorem 7.1.1 that for $m = 1, 2, \dots,$

$$\langle \Phi_0, (B^+ + B^-)^{2m-1} \Phi_0 \rangle = 0, \tag{7.2}$$

$$\langle \Phi_0, (B^+ + B^-)^{2m} \Phi_0 \rangle = \frac{(2m)!}{m!(m+1)!}\,. \tag{7.3}$$

Below we will show the above identities from a combinatorial viewpoint. Then, (7.1) follows from Theorem 7.1.1.

Let us start with

$$\langle \Phi_0, (B^+ + B^-)^k \Phi_0 \rangle = \sum_{\epsilon_1, \dots, \epsilon_k \in \{\pm\}} \langle \Phi_0, B^{\epsilon_k} \cdots B^{\epsilon_1} \Phi_0 \rangle,$$

where

$$\langle \Phi_0, B^{\epsilon_k} \cdots B^{\epsilon_1} \Phi_0 \rangle = \begin{cases} 1, & \text{if } B^{\epsilon_k} \cdots B^{\epsilon_1} \Phi_0 = \Phi_0, \\ 0, & \text{otherwise.} \end{cases}$$

Then (7.2) follows immediately from the actions of $B^{\pm}$ in (6.3). For $k = 2m$,

$$B^{\epsilon_{2m}} \cdots B^{\epsilon_1}\Phi_0 = \Phi_0$$

occurs if and only if

$$\begin{aligned} \epsilon_1 &\ge 0, \\ \epsilon_1 + \epsilon_2 &\ge 0, \\ &\cdots \\ \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{2m-1} &\ge 0, \\ \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{2m-1} + \epsilon_{2m} &= 0. \end{aligned}$$

In general, $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_m) \in \{+,-\}^m$ is called a *Catalan path* of length m if

$$\sum_{i=1}^{k} \epsilon_k \ge 0, \qquad k = 1, 2, \dots, m-1,$$
$$\sum_{i=1}^{m} \epsilon_k = 0.$$

Let $\mathcal{C}_m$ denote the set of Catalan paths of length m. Obviously, $\mathcal{C}_m = \emptyset$ for an odd m.

Lemma 7.2.1 *For $m = 1, 2, \dots$ we have*

$$|\mathcal{C}_{2m}| = \frac{(2m)!}{m!(m+1)!}\,.$$

PROOF. We set

$$\mathcal{D}_m = \left\{\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{2m}) \in \{+,-\}^{2m}\,;\, \epsilon_1 + \cdots + \epsilon_{2m} = 0\right\}.$$

Obviously, $\mathcal{C}_m \subset \mathcal{D}_m$. Each $\epsilon \in \mathcal{D}_m$ corresponds to a path connecting the vertices

$$(0,0),\ (1,\epsilon_1),\ (2,\epsilon_1+\epsilon_2),\ \dots,\ (2m, \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{2m}) = (2m, 0)$$

in order. Since we have

$$|\mathcal{D}_m| = \binom{2m}{m} = \frac{(2m)!}{m!m!}\,,$$

for $|\mathcal{C}_m|$ it is sufficient to count the number of paths in $\mathcal{D}_m \setminus \mathcal{C}_m$. By definition a path $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{2m})$ in $\mathcal{D}_m \setminus \mathcal{C}_m$ has one or more vertices with negative ordinates. Let k be the abscissa of the first such vertex. Then $1 \le k \le 2m-1$. If $k = 1$ we have $\epsilon_1 = -1$. Otherwise,

$$\begin{aligned} &\epsilon_1 \ge 0, \quad \epsilon_1 + \epsilon_2 \ge 0, \quad \dots, \quad \epsilon_1 + \cdots + \epsilon_{k-1} = 0, \\ &\epsilon_1 + \cdots + \epsilon_{k-1} + \epsilon_k = -1. \end{aligned}$$

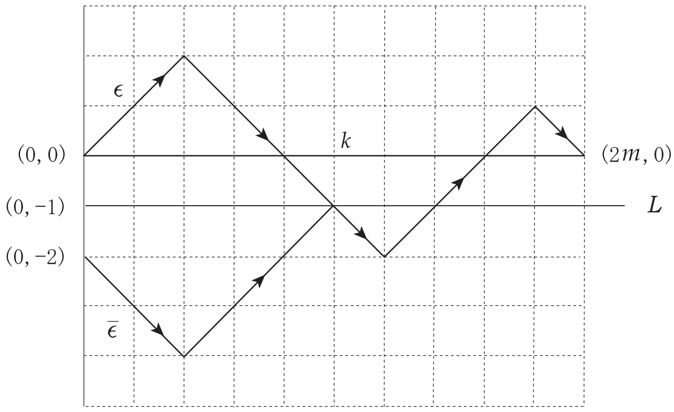


Figure 7.1: Counting the Catalan number

Let L be the horizontal line passing through $(0, -1)$. Then ϵ has one or more vertices which lie on L and $(k, -1)$ is the first one. Define $\bar{\epsilon}$ to be the path obtained from ϵ by reflecting the first part of ϵ up to $(k, -1)$ with respect to L (see Fig. 7.1). Then $\bar{\epsilon}$ becomes a path from $(0, -2)$ to $(2m, 0)$ passing through $(k, -1)$ as the first meeting point with L. It is easily verified that $\epsilon \leftrightarrow \bar{\epsilon}$ is a one-to-one correspondence between $\mathcal{D}_m \setminus \mathcal{C}_m$ and the set of paths connecting $(0, -2)$ and $(2m, 0)$. Obviously, the number of such paths is

$$\binom{2m}{m+1} = \frac{(2m)!}{(m+1)!(m-1)!} = |\mathcal{D}_m \setminus \mathcal{C}_m|.$$

Hence

$$|\mathcal{C}_m| = \frac{(2m)!}{m!m!} - \frac{(2m)!}{(m+1)!(m-1)!} = \frac{(2m)!}{m!(m+1)!},$$

which completes the proof. ∎

Definition 7.2.2 For $m = 1, 2, \dots,$

$$C_m = |\mathcal{C}_{2m}| = \frac{(2m)!}{m!(m+1)!}$$

is called the mth *Catalan number*. By definition $C_0 = 1$.

With this notation we come to

$$\langle \Phi_0, (B^+ + B^-)^{2m} \Phi_0 \rangle = |\mathcal{C}_{2m}| = C_m. \tag{7.4}$$

On the other hand, Theorem 7.1.1 is rephrased as

$$\frac{1}{2\pi} \int_{-2}^{+2} x^{2m} \sqrt{4 - x^2}\, dx = \frac{(2m)!}{(m+1)!m!} = C_m\,. \tag{7.5}$$

Consequently, we have

$$\langle \Phi_0, (B^+ + B^-)^m \Phi_0 \rangle = \frac{1}{2\pi} \int_{-2}^{+2} x^m \sqrt{4 - x^2}\, dx, \quad m = 1, 2, \dots.$$

7.3 Accardi–Bożejko Formula

Let $(\{\omega_n\}, \{\alpha_n\})$ be Jacobi coefficients and $(\Gamma, \{\Phi_n\}, B^+, B^-, B^\circ)$ the associated interacting Fock space. We are interested in the moment sequence of the real random variable $B^+ + B^- + B^\circ$:

$$M_m = \langle \Phi_0, (B^+ + B^- + B^\circ)^m \Phi_0 \rangle, \qquad m = 1, 2, \dots . \tag{7.6}$$

Expanding the right hand side, we obtain

$$M_m = \sum_{\epsilon} \langle \Phi_0, B^{\epsilon_m} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0 \rangle, \tag{7.7}$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ runs over $\{+, -, \circ\}^m$.

In order to observe the action of $B^{\epsilon_m} \cdots B^{\epsilon_2} B^{\epsilon_1}$ to the vacuum vector Φ_0 it is convenient to associate a sequence of points (i.e., a path) in $\mathbf{Z}^2$ starting at $(0,0)$ as follows. Given $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \{+, -, \circ\}^m$ we associate a sequence of points in $\mathbf{Z}^2$ defined by

$$(0,0),\ (1, \epsilon_1),\ (2, \epsilon_1 + \epsilon_2), \dots, (m, \epsilon_1 + \epsilon_2 + \cdots + \epsilon_m),$$

where numbers $+1, -1, 0$ are assigned to ϵ_i according as $\epsilon_i = +, -, \circ$. It is more instructive to draw edges connecting these points in order (see Fig. 7.2).

A sequence $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \{+, -, \circ\}^m$ is called a (generalized) *Catalan path* if

$$\sum_{i=1}^{k} \epsilon_i \ge 0, \quad k = 1, 2, \dots, m-1,$$

$$\sum_{i=1}^{m} \epsilon_i = 0.$$

Let $\tilde{\mathcal{C}}_m$ denote the set of such Catalan paths.

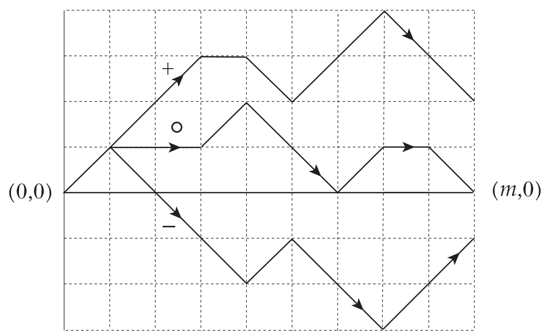


Figure 7.2: Paths in $\{+, -, \circ\}^m$ and $\tilde{\mathcal{C}}_m$

In view of the action of B^ϵ we see easily that

$$\langle \Phi_0, B^{\epsilon_m} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0 \rangle = 0, \quad (\epsilon_1, \dots, \epsilon_m) \in \{+, -, \circ\}^m \setminus \tilde{\mathcal{C}}_m \,.$$

Hence (7.7) becomes

$$M_m = \sum_{\epsilon \in \tilde{\mathcal{C}}_m} \langle \Phi_0, B^{\epsilon_m} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0 \rangle. \tag{7.8}$$

To each $\epsilon \in \tilde{\mathcal{C}}_m$ we associate a partition of natural numbers $\{1, 2, \dots, m\}$. We need notation.

Definition 7.3.1 Let $m \ge 1$ be an integer. A *partition* of $\{1, 2, \dots, m\}$ is a collection ϑ of non-empty subsets $v \subset \{1, 2, \dots, m\}$ such that

$$\{1, 2, \dots . m\} = \bigcup_{v \in \vartheta} v, \qquad v \cap v' = \emptyset, \quad v \ne v'.$$

A partition ϑ is called (i) a *pair partition* if $|v| = 2$ for all $v \in \vartheta$; (ii) a *pair partition with singletons* if $|v| = 2$ or $|v| = 1$ for all $v \in \vartheta$. An element $v \in \vartheta$ is called a *singleton* if $|v| = 1$.

Definition 7.3.2 Let ϑ be a pair partition with singleton of $\{1, 2, \dots, m\}$. For $v \in \vartheta$ we set

$$[v] = \begin{cases} \{i\}, & \text{if } v = \{i\}, \\ [i, j], & \text{if } v = \{i, j\} \text{ with } i < j. \end{cases}$$

We say that ϑ is *non-crossing* if for any pair of $u, v \in \vartheta$, one of the following relations occurs:

$$[u] \subset [v], \qquad [u] \supset [v], \qquad [u] \cap [v] = \emptyset.$$

Let $\mathcal{P}_{\mathrm{NCP}}(m)$ and $\mathcal{P}_{\mathrm{NCPS}}(m)$ denote the set of non-crossing pair partitions of $\{1, 2, \dots, m\}$ and that of non-crossing pair partitions with singletons, respectively.

We next associate with each $\epsilon \in \tilde{\mathcal{C}}_m$ a partition $\vartheta(\epsilon)$ of $\{1, 2, \dots, m\}$. In general, $\epsilon \in \{+, -, \circ\}^m$ being regarded as a map $\epsilon : \{1, 2, \dots, m\} \to \{+, -, \circ\}$, we obtain a partition:

$$\{1, 2, \dots, m\} = \epsilon^{-1}(\circ) \cup \epsilon^{-1}(+) \cup \epsilon^{-1}(-).$$

Let $\epsilon \in \tilde{\mathcal{C}}_m$. Since $|\epsilon^{-1}(+)| = |\epsilon^{-1}(-)|$ we may set

$$\epsilon^{-1}(\circ) \ = \{s_1 < \dots < s_j\}, \quad \epsilon^{-1}(\{+, -\}) = \{t_1 < \dots < t_{2k}\},$$

where $j + 2k = m$. We shall divide $\{t_1 < \dots < t_{2k}\}$ into a union of pairs. First we take $1 \le \alpha \le 2k$ such that

$$\epsilon(t_1) = \dots = \epsilon(t_\alpha) = +, \quad \epsilon(t_{\alpha+1}) = -.$$

Note that such an α always exists whenever $\epsilon^{-1}(\{+, -\}) \ne \varnothing$. Then we make a pair $\{t_\alpha < t_{\alpha+1}\}$. Setting

$$\{t'_1 < \dots < t'_{2k-2}\} = \{t_1 < \dots < t_{2k}\} \setminus \{t_\alpha < t_{\alpha+1}\},$$

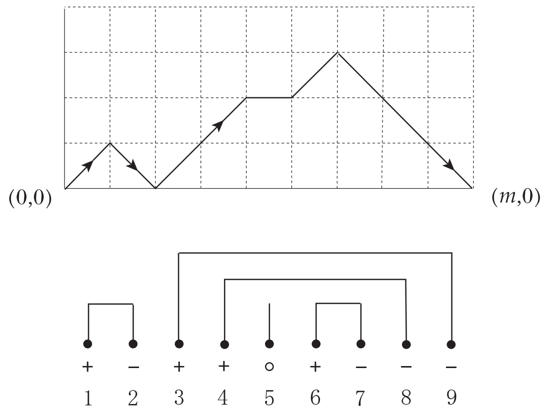


Figure 7.3: Path in $\tilde{\mathcal{C}}_m$ and partition in $\mathcal{P}_{\mathrm{NCPS}}(m)$

and applying a similar argument, we make the second pair. Repeating this procedure, we obtain a pair partition

$$\{t_1 < \cdots < t_{2k}\} = \{l_1 < r_1\} \cup \cdots \cup \{l_k < r_k\},$$

where $\epsilon(l_1) = \cdots = \epsilon(l_k) = +$ and $\epsilon(r_1) = \cdots = \epsilon(r_k) = -$. Finally we define a partition $\vartheta(\epsilon)$ by

$$\vartheta(\epsilon) = \{\{s_1\}, \dots, \{s_j\}, \{l_1 < r_1\}, \dots, \{l_k < r_k\}\}, \tag{7.9}$$

which is a pair partition with singleton (see Fig. 7.3).

Lemma 7.3.3 *Let $\epsilon \in \tilde{\mathcal{C}}_m$ and $\vartheta(\epsilon)$ the pair partition with singleton of $\{1, 2, \dots, m\}$ defined as in* (7.9)*. Then $\vartheta(\epsilon)$ is non-crossing. Moreover, the map $\epsilon \mapsto \vartheta(\epsilon)$ is a bijection from $\tilde{\mathcal{C}}_m$ onto $\mathcal{P}_{\mathrm{NCPS}}(m)$.*

PROOF. It is obvious from construction that $\vartheta(\epsilon)$ is non-crossing and that $\epsilon \mapsto \vartheta(\epsilon)$ is injective. Suppose we are given $\vartheta \in \mathcal{P}_{\mathrm{NCPS}}(m)$. Set

$$\vartheta = \{\{s_1\}, \dots, \{s_j\}, \{l_1, r_1\}, \dots, \{l_k, r_k\}\}$$

and assume that

$$s_1 < \cdots < s_j\,, \quad l_1 < \cdots < l_k\,, \quad l_1 < r_1\,, \quad \dots, \quad l_k < r_k\,. \tag{7.10}$$

Define $\epsilon \in \{+, -, \circ\}^m$ by

$$\epsilon(s_t) = \circ, \quad \epsilon(l_u) = +, \quad \epsilon(r_u) = -. \tag{7.11}$$

It is apparent that $\epsilon(1) + \cdots + \epsilon(m) = 0$. We shall prove that $\epsilon \in \tilde{\mathcal{C}}_m$, i.e.,

$$\epsilon(1) + \cdots + \epsilon(i) \geq 0, \qquad i = 1, 2, \dots, m. \tag{7.12}$$

Given i, we choose u such that

$$l_1 < \cdots < l_u \le i < l_{u+1} < \cdots < l_k\,.$$

Then, by (7.10) we have

$$\{r_1, \dots, r_k\} \cap [1, i] \subset \{r_1, \dots, r_u\}.$$

Hence in the left hand side of (7.12), $(+1)$ appears u times and (-1) at most u times, which shows that (7.12) holds. Finally, we need to prove that for ϵ defined in (7.11), $\vartheta(\epsilon) = \vartheta$. Set

$$\{l_1, \dots, l_k, r_1, \dots, r_k\} = \{w_1 < \cdots < w_{2k}\}.$$

The first step of constructing the partition $\vartheta(\epsilon)$ is to find $1 \le \alpha \le 2k$ such that

$$\epsilon(w_1) = \cdots = \epsilon(w_\alpha) = +, \quad \epsilon(w_{\alpha+1}) = -.$$

Obviously,

$$w_1 = l_1\,, \quad \dots, \quad w_\alpha = l_\alpha\,,$$

and by non-crossing condition we have

$$w_{\alpha+1} = r_\alpha\,.$$

Thus, $\vartheta(\epsilon)$ contains a pair $\{l_\alpha, r_\alpha\}$. Repeating this argument, we conclude that $\vartheta(\epsilon) = \vartheta$. ∎ ∎

Definition 7.3.4 Let $\vartheta \in \mathcal{P}_{\mathrm{NCPS}}(m)$. The *depth* of $v \in \vartheta$ is defined by

$$d_\vartheta(v) = |\{u \in \vartheta\,;\, [v] \subset [u]\}|.$$

Note that $d_\vartheta(v) \ge 1$ by definition.

For example, for ϑ in Fig. 7.3 it holds that

$$d_\vartheta(\{1,2\}) = 1, \quad d_\vartheta(\{4,8\}) = 2, \quad d_\vartheta(\{5\}) = 3.$$

The next result is easy to see.

Lemma 7.3.5 *Let $\vartheta \in \mathcal{P}_{\mathrm{NCPS}}(m)$ be corresponding to $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \tilde{\mathcal{C}}_m$. Then*

$$d_\vartheta(v) = \begin{cases} \displaystyle\sum_{i=1}^{s-1} \epsilon_i + 1, & \text{if } v = \{s\}, \\ \displaystyle\sum_{i=1}^{l-1} \epsilon_i + 1 = \sum_{i=1}^{r-1} \epsilon_i\,, & \text{if } v = \{l < r\}. \end{cases}$$

With these notation we continue calculation of (7.8) and obtain a combinatorial expression of (7.6).

Theorem 7.3.6 *Let* $(\Gamma, \{\Phi_n\}, B^+, B^-, B^\circ)$ *be the interacting Fock space associated with* $(\{\omega_n\}, \{\alpha_n\})$. *Then,*

$$\langle \Phi_0, (B^+ + B^- + B^\circ)^m \Phi_0 \rangle = \sum_{\vartheta \in \mathcal{P}_{\mathrm{NCPS}}(m)} \prod_{\substack{v \in \vartheta \\ |v|=1}} \alpha(d_\vartheta(v)) \prod_{\substack{v \in \vartheta \\ |v|=2}} \omega(d_\vartheta(v)), \tag{7.13}$$

for $m = 1, 2, \dots$. *In particular,*

$$\begin{cases} \langle \Phi_0, (B^+ + B^-)^{2m-1} \Phi_0 \rangle = 0, \\ \langle \Phi_0, (B^+ + B^-)^{2m} \Phi_0 \rangle = \displaystyle\sum_{\vartheta \in \mathcal{P}_{\mathrm{NCP}}(2m)} \prod_{v \in \vartheta} \omega(d_\vartheta(v)). \end{cases} \tag{7.14}$$

Proof. From (7.8) we already know that

$$\langle \Phi_0, (B^+ + B^- + B^\circ)^m \Phi_0 \rangle = \sum_{\epsilon \in \tilde{\mathcal{C}}_m} \langle \Phi_0, B^{\epsilon_m} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0 \rangle.$$

We shall calculate $B^{\epsilon_m} \cdots B^{\epsilon_2} B^{\epsilon_1} \Phi_0$ for $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \tilde{\mathcal{C}}_m$. Denote by $\vartheta = \vartheta(\epsilon) \in \mathcal{P}_{\mathrm{NCPS}}(m)$ the corresponding partition and set

$$\vartheta(\epsilon) = \{\{s_1\}, \dots, \{s_j\}, \{l_1, r_1\}, \dots, \{l_k, r_k\}\}.$$

First consider a singleton $s = s_i$. Since $B^{\epsilon_{s-1}} \cdots B^{\epsilon_1} \Phi_0 \in \mathbf{C}\Phi_{\epsilon_1 + \cdots + \epsilon_{s-1}}$ and $B^{\epsilon_s} = B^\circ$, we obtain by virtue of Lemma 7.3.5

$$\begin{aligned} B^{\epsilon_s} B^{\epsilon_{s-1}} \cdots B^{\epsilon_1} \Phi_0 &= \alpha(\epsilon_1 + \cdots + \epsilon_{s-1} + 1) B^{\epsilon_{s-1}} \cdots B^{\epsilon_1} \Phi_0 \\ &= \alpha(d_\vartheta(\{s\})) B^{\epsilon_{s-1}} \cdots B^{\epsilon_1} \Phi_0 \,. \end{aligned}$$

Applying the above argument to all the singletons, we come to

$$B^{\epsilon_m} \cdots B^{\epsilon_1} \Phi_0 = \left\{ \prod_{i=1}^{j} \alpha(d_\vartheta(\{s_i\})) \right\} [[B^{\epsilon_m} \cdots B^{\epsilon_1}]] \Phi_0 \,, \tag{7.15}$$

where $[[B^{\epsilon_m} \cdots B^{\epsilon_1}]]$ stands for omission of B°. Then $[[B^{\epsilon_m} \cdots B^{\epsilon_1}]]$ is a product of k creation operators B^+ and k annihilation operators B^- which form a non-crossing pair partition. Hence there exists $\{l, r\} = \{l_i, r_i\}$ such that B^{ϵ_r} and B^{ϵ_l} are consecutive. In that case

$$[[B^{\epsilon_m} \cdots B^{\epsilon_r} B^{\epsilon_l} \cdots B^{\epsilon_1}]] \Phi_0 = [[B^{\epsilon_m} \cdots B^- B^+ \cdots B^{\epsilon_1}]] \Phi_0 \,.$$

Since the action of B° does not change the level of the number vectors, in the above expression $[[\cdots B^{\epsilon_1}]] \Phi_0 \in \mathbf{C}\Phi_{\epsilon_1 + \cdots + \epsilon_{l-1}}$ so that the action of $B^- B^+$ on it becomes a scalar $\omega(\epsilon_1 + \cdots + \epsilon_{l-1} + 1) = \omega(d_\vartheta(\{l, r\}))$, where Lemma 7.3.5 is taken into account. Thus, we have

$$[[B^{\epsilon_m} \cdots B^{\epsilon_r} B^{\epsilon_l} \cdots B^{\epsilon_1}]] \Phi_0 = \omega(d_\vartheta(\{l, r\})) [[B^{\epsilon_m} \cdots \check{B}^{\epsilon_r} \check{B}^{\epsilon_l} \cdots B^{\epsilon_1}]] \Phi_0 \,,$$

where $\check{B}^{\epsilon_r}\check{B}^{\epsilon_l}$ means that $B^{\epsilon_r}B^{\epsilon_l}$ is omitted. Repeating this argument, we come to

$$[[B^{\epsilon_m}\cdots B^{\epsilon_1}]]\Phi_0 = \left\{\prod_{i=1}^{k}\omega(d_\vartheta(\{l_i,r_i\}))\right\}\Phi_0\,. \tag{7.16}$$

Now the formula (7.13) follows immediately from (7.15) and (7.16). The formula (7.14) follows from (7.13). ∎ ∎

Theorem 7.3.7 (Accardi–Bożejko formula) *For $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ let $\{M_m\}$ be its moment sequence and $(\{\omega_n\},\{\alpha_n\})$ its Jacobi coefficient. Then it holds that*

$$M_m = \sum_{\vartheta\in\mathcal{P}_{\mathrm{NCPS}}(m)}\prod_{\substack{v\in\vartheta\\|v|=1}}\alpha(d_\vartheta(v))\prod_{\substack{v\in\vartheta\\|v|=2}}\omega(d_\vartheta(v)),\qquad m=1,2,\dots. \tag{7.17}$$

Moreover, if μ is symmetric,

$$\begin{cases} M_{2m-1}=0,\\ M_{2m}=\displaystyle\sum_{\vartheta\in\mathcal{P}_{\mathrm{NCP}}(2m)}\prod_{v\in\vartheta}\omega(d_\vartheta(v)),\qquad m=1,2,\dots.\end{cases} \tag{7.18}$$

Proof. Let $(\Gamma,\{\Phi_n\},B^+,B^-,B^\circ)$ be the interacting Fock space associated with $(\{\omega_n\},\{\alpha_n\})$. We know that

$$M_m = \langle\Phi_0,(B^++B^-+B^\circ)^m\Phi_0\rangle,\qquad m=1,2,\dots.$$

Then we need only to apply Theorem 7.3.6. ∎

In Remark 5.1.5 we mentioned that there is a bijection $F:\mathfrak{M}\to\mathfrak{J}$. In fact, $F^{-1}:\mathfrak{J}\to\mathfrak{M}$ is expressed explicitly by the Accardi–Bożejko formula.

7.4 Quantum Decomposition of a Real Random Variable

Let $(\mathcal{A},\varphi)$ be an algebraic probability space and $a\in\mathcal{A}$ a real random variable. Then there exists a probability distribution $\mu\in\mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ such that

$$\varphi(a^m)=\int_{-\infty}^{+\infty}x^m\mu(dx),\qquad m=1,2,\dots.$$

This μ is not uniquely determined by a but its Jacobi coefficients. Let $(\{\omega_n\},\{\alpha_n\})$ be the Jacobi coefficients of μ and consider the associated interacting Fock space $(\Gamma,\{\Phi_n\},B^+,B^-,B^\circ)$. Then we know that

$$\langle\Phi_0,(B^++B^-+B^\circ)^m\Phi_0\rangle=\int_{-\infty}^{+\infty}x^m\mu(dx),\qquad m=1,2,\dots.$$

Consequently,

$$\varphi(a^m)=\langle\Phi_0,(B^++B^-+B^\circ)^m\Phi_0\rangle,\qquad m=1,2,\dots.$$

From the above identity we say that a and $B^++B^-+B^\circ$ are stochastically equivalent. For brevity we write

$$a=B^++B^-+B^\circ$$

and call the *quantum decomposition* of a.

Remark 7.4.1 Recall that the map $\mathfrak{P}_{\mathrm{fm}}(\mathbf{R}) \to \mathfrak{M}$ is not injective (determinate moment problem). Therefore, $\mathfrak{P}_{\mathrm{fm}}(\mathbf{R}) \to \mathfrak{J}$ is not either. A simple sufficient condition for μ to be the solution of a determinate moment problem is that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{\omega_n}} = +\infty. \tag{7.19}$$

This is knwon as Carleman's condition. If $\omega_n = 0$ happens, we understand (7.19) is fulfilled automatically. In that case, μ is the solution of a determinate moment problem. Indeed, the Jacobi coefficient is of finite type so that μ is a finite sum of δ-measures.

It may be worthwhile to mention a few words about how to deal with a classical random variable. Let X be a classical $\mathbf{R}$-valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. Let μ be the distribution of X and assume that $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$, that is, $\mathbb{E}(|X|^m) < \infty$ for all $m = 1, 2, \dots$. Then, taking the Jacobi coefficient $(\{\omega_n\}, \{\alpha_n\})$ of μ, we obtain

$$\mathbb{E}(X^m) = \int_{-\infty}^{+\infty} x^m \mu(dx) = \langle \Phi_0, (B^+ + B^- + B^\circ)^m \Phi_0 \rangle, \quad m = 1, 2, \dots.$$

We thereby write

$$X = B^+ + B^- + B^\circ$$

and call it the *quantum decomposition* of a classical random variable X. The quantum decomposition brings a classical variable X into a non-commutative paradigm where X is studied by means of its quantum components.

Exercises 7

1. For $m = 1, 2, \dots$ calculate the following integral:

$$\frac{1}{2\pi} \int_{-2}^{+2} x^{2m} \sqrt{4 - x^2}\, dx$$

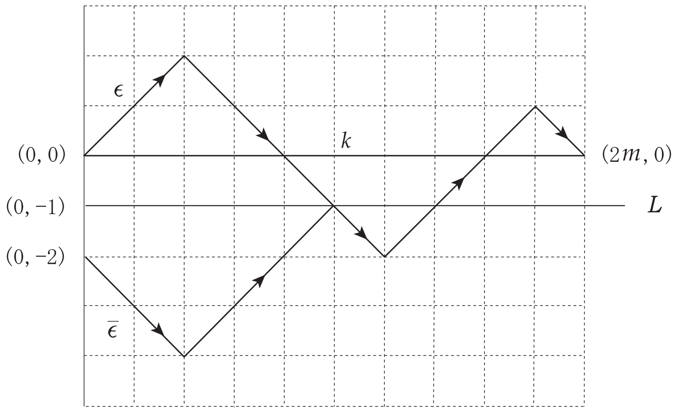
There are many ways of computation. For example, the Beta-function may be applied.

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt = 2 \int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta \, d\theta = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

2. Show that the Catalan number is given by

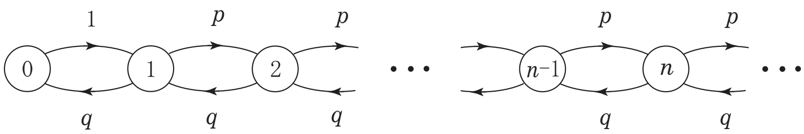
$$C_m = \frac{(2m)!}{m!(m+1)!}, \qquad m = 1, 2, \dots.$$

Hint: $C_m = \binom{2m}{m} - \binom{2m}{m+1}$



3. Find a quantum decomposition of a Bernoulli random variable X defined by $P(X = 1) = p$ and $P(X = 0) = 1 - p$. Hint: Find the Jacobi parameters.

4*. Let $\{X_n\}$ be a random walk on $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ determined by the transition probabilities as below:



where $p + q = 1$. Applying the idea of the Accardi–Bożejko formula find a probability distribution $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ such that

$$P(X_m = 0 | X_0 = 0) = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots.$$

8 Graph Products and Independence

8.1 Motivation

A growing graph models a revolution of networks in the real world.

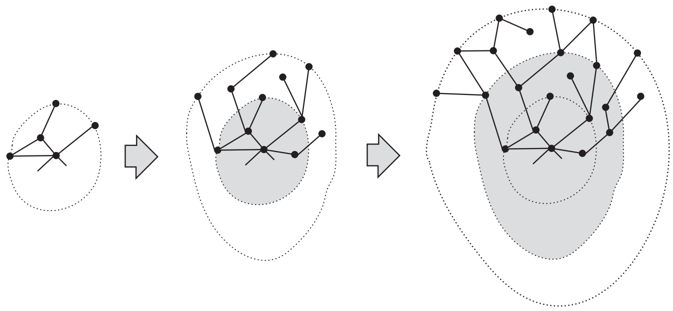


Figure 8.1: Growing graph

It would be interesting if the growing graph $G^{(\nu)}$ is considered as an analogue of an independent increment process in classical probability theory. It is our hope that the evolution is formulated as

$$G^{(\nu)} = G^{(\nu-1)} \sharp H^{(\nu)}, \tag{8.1}$$

where $\sharp H^{(\nu)}$ is an operation to form a new graph $G^{(\nu)}$ and $H^{(\nu)}$ is given at each evolution step. We hope that $H^{(\nu)}$ shares a common sprit with independent random variables.

In this chapter we discuss graph products. Given two graphs G_1 and G_2, we form a new graph $G_1 \sharp G_2$ as a "product." This graph product gives rise to a product of the adjacency matrices

$$A = A_1 \sharp A_2 \,. \tag{8.2}$$

When the evoluton of graphs is formulated in terms of a graph product, (8.1) yields

$$A^{(\nu)} = A^{(\nu-1)} \sharp B^{(\nu)} = \cdots = (\cdots((A^{(0)} \sharp B^{(0)}) \sharp B^{(1)}) \cdots) \sharp B^{(\nu)}.$$

We may expect that the spectral properties of $A^{(\nu)}$ follow from the study of some interrelation among $B^{(\nu)}$ with respect to the operation $\sharp$. From this aspect various types of independence in quantum probability would be useful.

8.2 Direct (Cartesian) Products

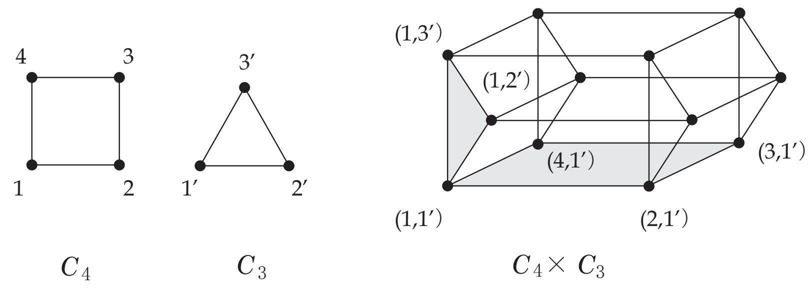
Definition 8.2.1 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. For $(x, y), (x', y') \in V_1 \times V_2$ we write $(x, y) \sim (x', y')$ if one of the following conditions is satisfied:

(i) $x = x'$ and $y \sim y'$;

(ii) $x \sim x'$ and $y = y'$.

Then $V_1 \times V_2$ becomes a graph in such a way that $(x, y), (x', y') \in V_1 \times V_2$ are adjacent if $(x, y) \sim (x', y')$. This graph is called the *direct product* of G_1 and G_2, and is denoted by $G_1 \times G_2$.

Example 8.2.2 $C_4 \times C_3$



Lemma 8.2.3 (1) $G_1 \times G_2 \cong G_2 \times G_1$.
(2) $(G_1 \times G_2) \times G_3 \cong G_1 \times (G_2 \times G_3)$.

Proof. Straightforward. ∎

Example 8.2.4 $\mathbb{Z}^N \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$ (N times)

Example 8.2.5 Let n, d be natural numbers. Set

$$V = \{x = (\xi_1, \xi_2, \dots, \xi_d)\,;\, \xi_i \in F\}, \qquad F = \{1, 2, \dots, n\}.$$

For $x = (\xi_i), y = (\eta_i) \in V$ define

$$\partial(x, y) = |\{1 \le i \le d\,;\, \xi_i \ne \eta_i\}|,$$

and draw an edge between x, y if $\partial(x, y) = 1$. Thus we obtain a graph $G = (V, E)$, called a *Hamming graph* and denoted by $H(d, n)$. The Hamming graph $H(d, n)$ is isomorphic to the direct product of d copies of the complete graph K_n, i.e.,

$$H(d, n) = K_n \times \cdots \times K_n \quad (d \text{ times}).$$

The adjacency matrix A_i acts on $C_0(V_i)$ by usual matrix multiplication, hence the adjacency matrix A of the direct product $G_1 \times G_2$ acts on $C_0(V_1 \times V_2) \cong C_0(V_1) \otimes C_0(V_2)$, where the canonical isomorphism is defined by the correspondence of basis $\delta_{(x,y)} \mapsto \delta_x \otimes \delta_y$.

Theorem 8.2.6 *As an operator acting on $C_0(V_1) \otimes C_0(V_2)$, the adjacency matrix A of the direct product $G_1 \times G_2$ is of the form:*

$$A = A_1 \otimes E_2 + E_1 \otimes A_2\,, \tag{8.3}$$

where E_i is the identity matrix on $C_0(V_i)$.

PROOF. We see that

$$(A_1 \otimes E_2)_{(x,y),(x',y')} = (A_1)_{xx'}(E_2)_{y,y'} = \begin{cases} 1, & \text{if } x \sim x' \text{ and } y = y', \\ 0, & \text{otherwise.} \end{cases}$$

Similarly,

$$(E_1 \otimes A_2)_{(x,y),(x',y')} = (E_1)_{xx'}(A_2)_{y,y'} = \begin{cases} 1, & \text{if } x = x' \text{ and } y \sim y', \\ 0, & \text{otherwise.} \end{cases}$$

Since the two conditions (i) $x \sim x'$ and $y = y'$; (ii) $x = x'$ and $y \sim y'$ do not occur simultaneously, we have

$$(A_1 \otimes E_2 + E_1 \otimes A_2)_{(x,y),(x',y')} = \begin{cases} 1, & \text{if } (x,y) \sim (x',y'), \\ 0, & \text{otherwise.} \end{cases}$$

This means that $A_1 \otimes E_2 + E_1 \otimes A_2$ coincides with the adjacency matrix of $G_1 \times G_2$. ∎

8.3 Star Products

Definition 8.3.1 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with distinguished vertices $o_1 \in V_1$ and $o_2 \in V_2$. Define a subset of $V_1 \times V_2$ by

$$V_1 \star V_2 = \{(x, o_2) \,;\, x \in V_1\} \cup \{(o_1, y) \,;\, y \in V_2\}$$

The induced subgraph of $G_1 \times G_2$ spanned by $V_1 \star V_2$ is called the *star product* of G_1 and G_2 (with contact vertices o_1 and o_2), and is denoted by $G_1 \star G_2 = G_1 \,{}_{o_1}\!\star_{o_2} G_2$.

In general, $H = (W, F)$ is called a *subgraph* of a graph $G = (V, E)$ if $W \subset V$ and $F \subset E$. A subgraph $H = (W, F)$ is called an *induced subgraph* of $G = (V, E)$ spanned by W if $F = \{\{x, y\} \in E \,;\, x, y \in W\}$.

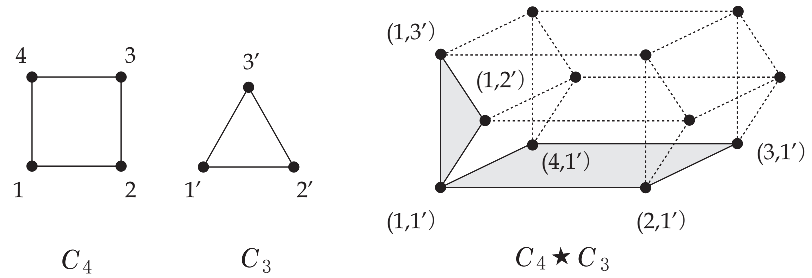
Lemma 8.3.2 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with distinguished vertices $o_1 \in V_1$, $o_2 \in V_2$. Let $G = G_1 \star G_2$ be the star product. Then two vertices $(x, y), (x', y') \in V_1 \star V_2$ are adjacent if and only if one of the following conditions is satisfied:*

(i) *$x = x' = o_1$ and $y \sim y'$;*

(ii) *$x \sim x'$ and $y = y' = o_2$.*

PROOF. Straightforward. ∎

Example 8.3.3 $C_4 \star C_3$



Lemma 8.3.4 (1) $G_1 \star G_2 \cong G_1 \star G_2$.

(2) $(G_1 \star G_2) \star G_3 \cong G_1 \star (G_2 \star G_3)$.

PROOF. Exercises. ∎

As usual, we regard the adjacency matrix A_i as an operator acting on $C_0(V_i)$. Since $G_1 \star G_2$ is an induced subgraph of $G_1 \times G_2$ spanned by $V_1 \star V_2$, its adjacency matrix A acts on $C_0(V_1 \star V_2)$, which is a subspace of $C(V_1 \times V_2) = C(V_1) \otimes C(V_2)$.

Theorem 8.3.5 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with fixed origins $o_1 \in V_1$ and $o_2 \in V_2$. Let A be the adjacency matrix of the star product $G_1 \star G_2$. Then, as an operator acting on $C(V_1 \star V_2)$ we have*

$$A = (A_1 \otimes P_2 + P_1 \otimes A_2) \restriction_{C(V_1 \star V_2)}$$

PROOF. It follows from the above argument that $A = A_{G_1 \times G_2} \restriction_{C(V_1 \star V_2)}$. By Theorem 8.2.6 we see that

$$A = A_{G_1 \times G_2} \restriction_{C(V_1 \star V_2)} = (A_1 \otimes E_2 + E_1 \otimes A_2) \restriction_{C(V_1 \star V_2)}$$

It is easily verified by definition that

$$(A_1 \otimes E_2 + E_1 \otimes A_2) \restriction_{C(V_1 \star V_2)} = (A_1 \otimes P_2 + P_1 \otimes A_2) \restriction_{C(V_1 \star V_2)},$$

which completes the proof. ∎

8.4 Comb Products

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We fix a vertix $o_2 \in V_2$. For $(x, y), (x', y') \in V_1 \times V_2$ we write $(x, y) \sim (x', y')$ if one of the following conditions is satisfied:

(i) $x = x'$ and $y \sim y'$;

(ii) $x \sim x'$ and $y = y' = o_2$.

Then $V_1 \times V_2$ becomes a graph in such a way that $(x, y), (x', y') \in V_1 \times V_2$ are adjacent if $(x, y) \sim (x', y')$. This graph is denoted by $G_1 \rhd_{o_2} G_2$ and is called the *comb product*.

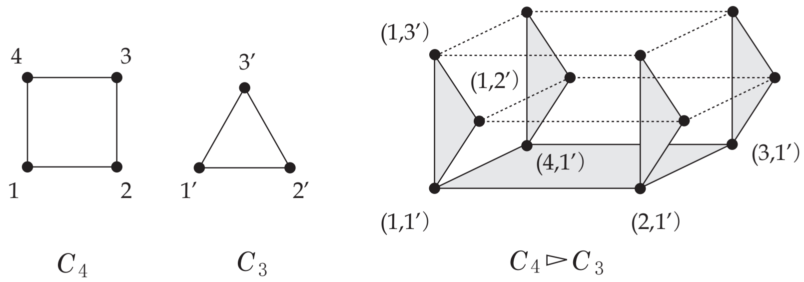
Theorem 8.4.1 *As an operator on $C_0(V_1) \otimes C_0(V_2)$ the adjacency matrix of $G_1 \rhd_{o_2} G_2$ is given by*

$$A = A_1 \otimes P_2 + E_1 \otimes A_2$$

where $P_2 : C_0(V_2) \to C_0(V_2)$ is the projection onto the space spanned by δ_{o_2} and E_1 is the identity matrix acting on $C_0(V_1)$.

PROOF. Exercise. ∎

Example 8.4.2 $C_4 \rhd C_3$



The comb product is not commutative, but associative.

Lemma 8.4.3 $(G_1 \rhd G_2) \rhd G_3 \cong G_1 \rhd (G_2 \rhd G_3)$.

8.5 Notions of Independence

Consider two classical random variables X, Y defined on a probability space $(\Omega, \mathcal{F}, P)$. If they are independent, by the product formula we obtain

$$\mathbb{E}(XYXXYXY) = \mathbb{E}(X^4Y^3) = \mathbb{E}(X^4)\mathbb{E}(Y^3). \tag{8.4}$$

In general, such a statistical quantity as above is called a *mixed moment* or a *correlation coefficient*. We understand that the independence gives a rule of calculating mixed moments. In quantum probability theory many different rules can be introduced because of non-commutativity of random variables, where, for example, the first equality in (8.4) may be no longer guaranteed. In this section, we shall mention four different notions of independence, which have been up to now considered most fundamental.

Definition 8.5.1 (Commutative independence) *Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. A family $\{\mathcal{A}_\lambda\}$ of $*$-subalgebras of $\mathcal{A}$ is called commutative independent or tensor independent (with respect to φ) if*

$$\varphi(a_1 \cdots a_m), \qquad a_i \in \mathcal{A}_{\lambda_i},$$

is factorized as follows:

(i) *when* $\lambda_1 \notin \{\lambda_2, \ldots, \lambda_m\}$,

$$\varphi(a_1 \cdots a_m) = \varphi(a_1)\varphi(a_2 \cdots a_m);$$

(ii) *otherwise, letting* r *be the smallest number such that* $\lambda_1 = \lambda_r$,

$$\varphi(a_1 \cdots a_m) = \varphi(a_2 \cdots a_{r-1}(a_1 a_r)a_{r+1} \cdots a_m).$$

Note that neither $\mathcal{A}_\lambda$ *nor* $\mathcal{A}$ *is assumed to be commutative.*

Definition 8.5.2 (Free independence) *Let* $(\mathcal{A}, \varphi)$ *be an algebraic probability space. A family* $\{\mathcal{A}_\lambda\}$ *of* $*$*-subalgebras of* $\mathcal{A}$ *is called free independent (with respect to* φ*) if*

$$\varphi(a_1 \cdots a_m) = 0$$

holds for any $a_i \in \mathcal{A}_{\lambda_i}$ *with* $\varphi(a_i) = 0$, $i = 1, 2, \ldots, m$, *and* $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_m$ *(any two consecutive indices are different).*

Definition 8.5.3 (Boolean independence) *Let* $(\mathcal{A}, \varphi)$ *be an algebraic probability space and* $\mathcal{A}_\lambda \subset \mathcal{A}$ *a subset which is closed under the algebraic operations and involution (i.e., a* $*$*-subalgebra which does not necessarily contain the identity* $1_\mathcal{A}$ *of* $\mathcal{A}$*). We say that* $\{\mathcal{A}_\lambda\}$ *is Boolean independent (with respect to* φ*) if*

$$\varphi(a_1 \cdots a_m) = \varphi(a_1)\varphi(a_2 \cdots a_m)$$

for any $a_i \in \mathcal{A}_{\lambda_i}$ *with* $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_m$.

We need notation. Let $(\Lambda, <)$ be a totally ordered set and consider a finite sequence

$$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_p \neq \cdots \neq \lambda_m \tag{8.5}$$

of elements in Λ, $m \geq 2$. Then λ_p is called a *peak* in (8.5) if (i) $1 < p < m$, $\lambda_{p-1} < \lambda_p$ and $\lambda_p > \lambda_{p+1}$; or (ii) $p = 1$ and $\lambda_1 > \lambda_2$; or (iii) $p = m$ and $\lambda_{m-1} < \lambda_m$.

Definition 8.5.4 (Monotone independence) Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. Let $(\Lambda, <)$ be a totally ordered set and for each $\lambda \in \Lambda$, $\mathcal{A}_\lambda \subset \mathcal{A}$ a subset which is closed under the algebraic operations and involution. We say that $\{\mathcal{A}_\lambda\}$ is *monotone independent* (with respect to φ) if

$$\varphi(a_1 \cdots a_m) = \varphi(a_p)\varphi(a_1 \cdots \check{a}_p \cdots a_m) \qquad (\check{a}_p\text{: omission})$$

for any $a_i \in \mathcal{A}_{\lambda_i}$ with λ_p being a peak in $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_m$.

Remark 8.5.5 The Boolean independence yields a rather trivial situation when $\mathcal{A}_\lambda$ contains the identity. This remains even if the condition in Definition 8.5.3 is weakened in such a way that a_i is taken from $\mathcal{A}_{\lambda_i} \setminus \mathbf{C}$. Assume that $\{\mathcal{A}_1, \mathcal{A}_2\}$ is Boolean independent and that $\mathcal{A}_1$

contains the identity, i.e., is a $*$-subalgebra. Take $a_i \in \mathcal{A}_i \setminus \mathbf{C}$ and consider $\varphi(a_2^*(a_1+1)a_2)$. Applying the independence and then linearity, we come to

$$\begin{aligned}\varphi(a_2^*(a_1+1)a_2) &= \varphi(a_2^*)\varphi(a_1+1)\varphi(a_2) \\ &= \varphi(a_2^*)\varphi(a_1)\varphi(a_2) + \varphi(a_2^*)\varphi(a_2). \end{aligned} \tag{8.6}$$

On the other hand, applying first linearity and then independence, we obtain

$$\begin{aligned}\varphi(a_2^*(a_1+1)a_2) &= \varphi(a_2^*a_1a_2) + \varphi(a_2^*a_2) \\ &= \varphi(a_2^*)\varphi(a_1)\varphi(a_2) + \varphi(a_2^*a_2). \end{aligned} \tag{8.7}$$

We then see from (8.6) and (8.7) that

$$\varphi(a_2^*a_2) = \varphi(a_2^*)\varphi(a_2) = |\varphi(a_2)|^2.$$

Similarly, from $\varphi(a_2(a_1+1)a_2^*)$ we obtain

$$\varphi(a_2a_2^*) = |\varphi(a_2)|^2.$$

Consequently, $a_2 = \varphi(a_2)1$ (the Schwarz *equality* holds). In other words, $\mathcal{A}_2$ is reduced essentially to the $*$-subalgebra $\mathbf{C}1$. A similar situation occurs in the case of monotone independence.

The above definitions indicate only the first step of calculating a mixed moment $\varphi(a_1 \cdots a_m)$. Table 8.1 shows how mixed moments of $a \in \mathcal{A}_1$ and $b \in \mathcal{A}_2$ are factorized when $\{\mathcal{A}_1, \mathcal{A}_2\}$ is commutative, free, Boolean, or monotone independent (for the monotone independence the natural order $1 < 2$ is adopted). In actual computation the following formulae are useful.

Lemma 8.5.6 *Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. Let $a_i \in \mathcal{A}$ and set $\bar{a}_i = a_i - \varphi(a_i)$, $i = 1, 2, \dots, m$. Then*

$$a_1 \cdots a_m = a_1 \cdots \bar{a}_i \cdots a_m + \varphi(a_i) a_1 \cdots \check{a}_i \cdots a_m\,, \tag{8.8}$$

$$\varphi(a_1 \cdots a_m) = \varphi(\bar{a}_1 \cdots \bar{a}_m) + \sum_{i=1}^m \varphi(a_i)\varphi(\underbrace{a_1 \cdots a_{i-1}}_{i-1}\, \check{a}_i\, \underbrace{\bar{a}_{i+1} \cdots \bar{a}_m}_{m-i}). \tag{8.9}$$

PROOF. Exercise. ∎ ∎

Definition 8.5.7 *Let $(\mathcal{A}, \varphi)$ be an algebraic probability space and $\{a_n\}$ be a sequence of random variables. Let $\mathcal{A}_n^0$ be the linear span of elements of the form*

$$a_n^{\epsilon_1} \cdots a_n^{\epsilon_m}, \qquad \epsilon_i \in \{1, *\}, \quad m = 1, 2, \dots,$$

and set $\mathcal{A}_n = \mathcal{A}_n^0 + \mathbf{C}1$, which is the $*$-subalgebra generated by a_n. We say that $\{a_n\}$ is commutative or free independent if so is $\{\mathcal{A}_n\}$. We say that $\{a_n\}$ is Boolean or monotone independent if so is $\{\mathcal{A}_n^0\}$.

Table 8.1: Illustrating the factorization rules

	commutative	free	Boolean	monotone
$\varphi(aba)$	$\varphi(a^2)\varphi(b)$	$\varphi(a^2)\varphi(b)$	$\varphi(a)^2\varphi(b)$	$\varphi(a^2)\varphi(b)$
$\varphi(bab)$	$\varphi(a)\varphi(b^2)$	$\varphi(a)\varphi(b^2)$	$\varphi(a)\varphi(b)^2$	$\varphi(a)\varphi(b)^2$
$\varphi(abab)$	$\varphi(a^2)\varphi(b^2)$	$\varphi(a)^2\varphi(b^2)$ $+\varphi(a^2)\varphi(b)^2$ $-\varphi(a)^2\varphi(b)^2$	$\varphi(a)^2\varphi(b)^2$	$\varphi(a^2)\varphi(b)^2$

Remark 8.5.8 *$\mathcal{A}_n^0$ is closed under the algebraic operations and involution. But it can not be decided by definition whether or not the identity $1_{\mathcal{A}}$ is contained in $\mathcal{A}_n^0$.*

Theorem 8.5.9 *Let $G = G_1 \times G_2$ be the direct product of two graphs and*

$$A = A_1 \otimes E + E \otimes A_2 \tag{8.10}$$

be the adjacency matrix expressed as an operator on $C_0(V_1) \otimes C_0(V_2)$, see Theorem 8.2.6. Then the right hand side of (8.10) is a sum of commutative independent random variables with respect to the vacuum state $e_{o_1} \otimes e_{o_2}$, where $o_1 \in V_1$ and $o_2 \in V_2$.

Proof. The full proof is omitted. For simplicity we set

$$\varphi(a) = \langle e_{o_1} \otimes e_{o_2}, a(e_{o_1} \otimes e_{o_2})\rangle.$$

We will only observe that

$$\varphi((A_1 \otimes E_2)^\alpha (E_1 \otimes A_2)^\beta) = \varphi((A_1 \otimes E_2)^\alpha)\varphi((E_1 \otimes A_2)^\beta). \tag{8.11}$$

First the left hand side becomes

$$\begin{aligned}\varphi((A_1 \otimes E_2)^\alpha (E_1 \otimes A_2)^\beta) &= \langle e_{o_1} \otimes e_{o_2}, (A_1 \otimes E_2)^\alpha (E_1 \otimes A_2)^\beta (e_{o_1} \otimes e_{o_2})\rangle \\ &= \langle e_{o_1}, A_1^\alpha E_1^\beta e_{o_1}\rangle\langle e_{o_2}, E_2^\alpha A_2^\beta e_{o_2}\rangle \\ &= \langle e_{o_1}, A_1^\alpha e_{o_1}\rangle\langle e_{o_2}, A_2^\beta e_{o_2}\rangle.\end{aligned}$$

On the other hand, for the right hand side we have

$$\begin{aligned}\varphi((A_1 \otimes E_2)^\alpha) &= \langle e_{o_1} \otimes e_{o_2}, (A_1 \otimes E_2)^\alpha (e_{o_1} \otimes e_{o_2})\rangle \\ &= \langle e_{o_1} A_1^\alpha e_{o_1}\rangle\langle e_{o_2}, E_2^\alpha e_{o_2}\rangle \\ &= \langle e_{o_1} A_1^\alpha e_{o_1}\rangle.\end{aligned}$$

Similarly,

$$\varphi((E_1 \otimes A_2)^\beta) = \langle e_{o_2} A_2^\beta e_{o_2}\rangle.$$

Thus, (8.11) is verified. ∎

Theorem 8.5.10 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with fixed origins $o_1 \in V_1$ and $o_2 \in V_2$. Let A be the adjacency matrix of the star product $G_1 \star G_2$. Then, as an operator acting on $C(V_1 \star V_2)$*

$$A = (A_1 \otimes P_2 + P_1 \otimes A_2) \upharpoonright_{C(V_1 \star V_2)}$$

is a sum of Boolean independent random variables with respect to the vacuum state at (o_1, o_2), see also Theorem 8.3.5.

Proof. Detailed argument is left to the reader. We only show that

$$\langle (A_1 \otimes P_2)^\alpha (P_1 \otimes A_2)^\beta (A_1 \otimes P_2)^\gamma \rangle = \langle (A_1 \otimes P_2)^\alpha \rangle \langle (P_1 \otimes A_2)^\beta \rangle \langle (A_1 \otimes P_2)^\gamma \rangle.$$

In fact, we first observe that

$$\langle (A_1 \otimes P_2)^\alpha (P_1 \otimes A_2)^\beta (A_1 \otimes P_2)^\gamma \rangle = \langle e_{o_1}, A_1^\alpha P_1 A_1^\gamma e_{o_1} \rangle \langle e_{o_2}, P_2 A_2^\beta P_2 e_{o_2} \rangle. \tag{8.12}$$

Here $P_1 A_1^\gamma \delta_{o_1} = \langle \delta_{o_1}, A_1^\gamma \delta_{o_1} \rangle \delta_{o_1}$ so that

$$\langle e_{o_1}, A_1^\alpha P_1 A_1^\gamma e_{o_1} \rangle = \langle e_{o_1}, A_1^\alpha e_{o_1} \rangle \langle e_{o_1}, A_1^\gamma e_{o_1} \rangle. \tag{8.13}$$

On the other hand,

$$\langle e_{o_2}, P_2 A_2^\beta P_2 e_{o_2} \rangle = \langle e_{o_2}, A_2^\beta e_{o_2} \rangle. \tag{8.14}$$

Incerting (8.13) and (8.14) into (8.12), we obtain the desired relation. ∎

Theorem 8.5.11 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with fixed origins $o_2 \in V_2$. Let A be the adjacency matrix of the comb product $G_1 \rhd G_2$. Then, as an operator acting on $C(V_1 \times V_2)$*

$$A = A_1 \otimes P_2 + E_1 \otimes A_2$$

is a sum of monotone independent random variables with respect to the vacuum state at (o_1, o_2), see also Theorem 8.4.1.

The proof is omitted.

In fact, Theorem 8.5.10 is a consequence from a more general result.

Theorem 8.5.12 *For $n = 1, 2, \dots, N$ let $\mathcal{H}_n$ be a Hilbert space with a distinguished unit vector $\Omega_n \in \mathcal{H}_n$ and consider an algebraic probability space:*

$$(\mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N), \Omega_1 \otimes \cdots \otimes \Omega_N).$$

Let $P_n \in \mathcal{B}(\mathcal{H}_n)$ be the projection onto the one-dimensional subspace spanned by Ω_n and $\mathcal{A}_n$ the set of operators of the form

$$P_1 \otimes \cdots \otimes P_{n-1} \otimes S_n \otimes P_{n+1} \otimes \cdots \otimes P_N\,, \qquad S_n \in \mathcal{B}(\mathcal{H}_n). \tag{8.15}$$

Then $\{\mathcal{A}_n\}$ is Boolean independent.

PROOF. Note that $\mathcal{A}_n$ is closed under the algebraic operations and involution. ($\mathcal{A}_n$ might contain the identity or might not.) For simplicity we set

$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N, \quad \Omega = \Omega_1 \otimes \cdots \otimes \Omega_N.$$

Let $m \geq 2$ and take $n_1 \neq n_2 \neq \cdots \neq n_m$ from $\{1, 2, \ldots, N\}$. For $a_i \in \mathcal{A}_{n_i}$ we need to show that

$$\langle \Omega, a_1 \cdots a_m \Omega \rangle = \langle \Omega, a_1 \Omega \rangle \langle \Omega, a_2 \cdots a_m \Omega \rangle. \tag{8.16}$$

We set

$$a_i = P_1 \otimes \cdots \otimes \overset{n_i\text{th}}{S_i} \otimes \cdots \otimes P_N, \qquad S_i \in \mathcal{B}(\mathcal{H}_{n_i}).$$

Noting that $n_1 \neq n_2$, we observe that

$$\begin{aligned} a_2^* a_1^* \Omega &= (P_1 \otimes \cdots \otimes \overset{n_2\text{th}}{S_2^*} \otimes \cdots \otimes P_N)(P_1 \otimes \cdots \otimes \overset{n_1\text{th}}{S_1^*} \otimes \cdots \otimes P_N)\Omega \\ &= \Omega_1 \otimes \cdots \otimes S_2^* \Omega_{n_2} \otimes \cdots \otimes P_{n_1} S_1^* \Omega_{n_1} \otimes \cdots \otimes \Omega_N. \end{aligned}$$

Since $P_{n_1} S_1^* \Omega_{n_1} = \langle \Omega_{n_1}, S_1^* \Omega_{n_1} \rangle \Omega_{n_1} = \langle \Omega, a_1^* \Omega \rangle \Omega_{n_1}$, we have

$$\begin{aligned} a_2^* a_1^* \Omega &= \langle \Omega, a_1^* \Omega \rangle \, \Omega_1 \otimes \cdots \otimes S_2^* \Omega_{n_2} \otimes \cdots \otimes \Omega_{n_1} \otimes \cdots \otimes \Omega_N \\ &= \langle \Omega, a_1^* \Omega \rangle \, a_2^* \Omega. \end{aligned}$$

Hence

$$\begin{aligned} \langle \Omega, a_1 \cdots a_m \Omega \rangle &= \langle a_2^* a_1^* \Omega, a_3 \cdots a_m \Omega \rangle \\ &= \overline{\langle \Omega, a_1^* \Omega \rangle} \, \langle a_2^* \Omega, a_3 \cdots a_m \Omega \rangle \\ &= \langle \Omega, a_1 \Omega \rangle \, \langle \Omega, a_2 a_3 \cdots a_m \Omega \rangle, \end{aligned}$$

which proves (8.16). ∎ ∎

Similarly, Theorem 8.5.11 is generalized as follows:

Theorem 8.5.13 *For $n = 1, 2, \ldots, N$ let $\mathcal{H}_n$ be a Hilbert space. Consider an algebraic probability space:*

$$(\mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N), \psi \otimes \Omega_2 \cdots \otimes \Omega_N),$$

where ψ is a state on $\mathcal{B}(\mathcal{H}_1)$ and Ω_n a vector state on $\mathcal{B}(\mathcal{H}_n)$ corresponding to a unit vector (denoted by the same symbol) in $\mathcal{H}_n$ for $n = 2, 3, \ldots, N$. Let $\mathcal{A}_n$ be the set of operators of the form

$$1_1 \otimes \cdots \otimes 1_{n-1} \otimes S_n \otimes P_{n+1} \otimes \cdots \otimes P_N, \qquad S_n \in \mathcal{B}(\mathcal{H}_n),$$

where 1_n is the identity of $\mathcal{B}(\mathcal{H}_n)$ and $P_n \in \mathcal{B}(\mathcal{H}_n)$ the projection onto the one-dimensional subspace spanned by Ω_n. Then $\{\mathcal{A}_n\}$ is monotone independent. (Here $\{1, 2, \ldots, N\}$ is equipped with the usual total order.)

PROOF. Similar to Theorem 8.5.12. ∎

Exercises 8

1. Draw a picture of the Hamming graph $H(2, 3)$. (This is known as *rook's graph.*)

2. Let G_1 and G_2 be two graphs and $G = G_1 \times G_2$ the cartesian product. Prove that

$$\deg_G((x, y)) = \deg_{G_1}(x) + \deg_{G_2}(y)$$

3. Find the degree and diameter of $H(d, N)$.

4^*. Let G_1, G_2 be two graphs and A_1, A_2 their adjacency matrices, respectively. Let $G = G_1 \times G_2$ the cartesian product and A its adjacency matrix. If $A_1 f = \lambda f$ and $A_2 g = \mu g$, show that $A(f \otimes g) = (\lambda + \mu)(f \otimes g)$. Describe $\mathrm{Spec}\,(G)$ in terms of $\mathrm{Spec}\,(G_1)$ and $\mathrm{Spec}\,(G_2)$.

5. Let G_1 and G_2 be connected graphs. Show that $G = G_1 \times G_2$ is connected and prove

$$\partial_G((x, y), (x', y')) = \partial_{G_1}(x, x') + \partial_{G_2}(y, y').$$

6. Let G_1 and G_2 be connected graphs. Show that $G = G_1 \times G_2$ is connected and prove

$$\partial_G = \partial_{G_1 \times G_2} \restriction_{V_1 \star V_2} .$$

7^*. Verify Table 8.1.

9 Quantum Central Limit Theorems

9.1 Singleton Condition

We first recall roughly the central limit theorem in classical probability theory. Let $X_1, X_2, \dots$ be independent, identically distributed random variables with mean 0 and variance 1. Then the sum

$$\sum_{n=1}^{N} X_n$$

obeys approximately the Gaussian distribution $N(0, N)$ for a large N. More precisely,

$$\lim_{N\to\infty} P\left(a \le \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \le b\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2} dx, \qquad a < b.$$

We should like to discuss a similar question in quantum probability.

Let $(\mathcal{A}, \varphi)$ be an algebraic probability space and $a_1, a_2, \dots \in \mathcal{A}$ a sequence of random variables. We are interested in the asymptotic behaviour of the partial sum defined by

$$S_N = \sum_{n=1}^{N} a_n \tag{9.1}$$

as $N \to \infty$. In the following we restrict ourselves to the case of real random variables, i.e., $a_n = a_n^*$. The moments of S_N are given by

$$\varphi(S_N^m) = \sum_{n_1,\dots,n_m=1}^{N} \varphi(a_{n_1} \cdots a_{n_m}), \qquad m = 1, 2, \dots. \tag{9.2}$$

We will study $\varphi(S_N^m)$ for a large N under the condition that $a_1, a_2, \dots$ are "independent."

Definition 9.1.1 For a finite sequence of natural numbers:

$$n_1, \dots, n_s, \dots, n_m \tag{9.3}$$

we say that n_s is a *singleton* in (9.3) if n_s appears just once, i.e., if $n_s \ne n_i$ for all $i \ne s$.

Definition 9.1.2 Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. Let $a_1, a_2, \dots \in \mathcal{A}$ be a sequence of random variables satisfying $a_n^* = a_n$ and $\varphi(a_n) = 0$ for all n. We say that the sequence $\{a_n\}$ satisfies the *singleton condition* if

$$\varphi(a_{n_1} \cdots a_{n_s} \cdots a_{n_m}) = 0$$

holds for any choice of finitely many natural numbers $n_1, \dots, n_s, \dots, n_m$ with a singleton n_s.

Remark 9.1.3 In some literatures the singleton condition is defined for a sequence of subalgebras. Let $\mathcal{B}_1, \mathcal{B}_2, \dots \subset \mathcal{A}$ be $*$-subalgebras without assuming $1_{\mathcal{A}}$, namely $\mathcal{B}_n$ is only assumed to be closed under the algebraic operations and the involution. We say that $\{\mathcal{B}_n\}$ satisfies the *singleton condition* if

$$\varphi(b_1 \cdots b_s \cdots b_m) = 0$$

holds for any choice of finitely many natural numbers $n_1, \dots, n_s, \dots, n_m$ with a singleton n_s and for any $b_i \in \mathcal{B}_{n_i}$ with $\varphi(b_s) = 0$. We mention relation between two definitions. Let $a_1, a_2, \dots \in \mathcal{A}$ be a sequence of random variables satisfying $a_n^* = a_n$ and $\varphi(a_n) = 0$ for all n. Define $\mathcal{A}_n^0$ to be the linear space spanned by $a_n, a_n^2, a_n^3, \dots$. If $\{\mathcal{A}_n^0\}$ satisfies the singleton condition, so does $\{a_n\}$. However, the converse is not valid.

Theorem 9.1.4 *Let $(\mathcal{A}, \varphi)$ an algebraic probability space and $a_1, a_2, \dots \in \mathcal{A}$ a sequence of random variables satisfying $a_n^* = a_n$ and $\varphi(a_n) = 0$ for all n. If $\{a_n\}$ is commutative, free, Boolean or monotone independent, it satisfies the singleton condition.*

PROOF. The proof is rather simple for the case of commutative, Boolean and monotone independence. Here we prove only for the free independence.

Let $\mathcal{A}_n$ be the $*$-subalgebra generated by a_n, that is, the polynomials in a_n. By definition $\{\mathcal{A}_n\}$ is free independent. For any choice of natural numbers $n_1, \dots, n_s, \dots, n_m$ with a singleton n_s we need to show

$$\varphi(a_{n_1} \cdots a_{n_s} \cdots a_{n_m}) = 0.$$

Here, in $n_1, n_2, \dots, n_s, \dots, n_m$ a pair of successive numbers may coincide. So we rewite

$$a_{n_1} \cdots a_{n_s} \cdots a_{n_m} = a_{i_1}^{p_1} \cdots a_{i_t} \cdots a_{i_k}^{p_k}$$

where $i_1 \neq i_2 \neq \cdots \neq i_k$, $i_t = n_s$ is a singleton therein, and $p_j \geq 1$. Thus, it is sufficient to show that

$$\varphi(b_1 \cdots b_s \cdots b_m) = 0 \tag{9.4}$$

holds for any choice of $n_1 \neq n_2 \neq \cdots \neq n_m$ with a singleton n_s and $b_i \in \mathcal{A}_{n_i}$ with $\varphi(b_s) = 0$.

We employ the mathematical induction on m. For $m = 1$ the assertion is obvious. Assume that the assertion is true up to $m - 1$, $m \geq 2$. Taking $i \neq s$, we observe that

$$\begin{aligned} &\varphi(b_1 \cdots b_i \cdots b_s \cdots b_m) \\ &\quad = \varphi(b_1 \cdots (b_i - \varphi(b_i)) \cdots b_s \cdots b_m) + \varphi(b_i)\varphi(b_1 \cdots \check{b}_i \cdots b_s \cdots b_m). \end{aligned} \tag{9.5}$$

Here $\varphi(b_1 \cdots \check{b}_i \cdots b_s \cdots b_m) = 0$ by the induction hypothesis. For simplicity we write $\bar{b}_i = b_i - \varphi(b_i)$. Then (9.5) becomes

$$\varphi(b_1 \cdots b_i \cdots b_s \cdots b_m) = \varphi(b_1 \cdots \bar{b}_i \cdots b_s \cdots b_m).$$

Repeating this procedure we come to

$$\varphi(b_1 \cdots b_i \cdots b_s \cdots b_m) = \varphi(\bar{b}_1 \cdots \bar{b}_i \cdots b_s \cdots \bar{b}_m).$$

The last expression is 0 by free independence. ∎

9.2 Singleton CLT

We now go back to (9.2), namely, we study the asymptotic behaviour of the mth moment of S_N

$$\varphi(S_N^m) = \sum_{n_1,\dots,n_m=1}^{N} \varphi(a_{n_1}\cdots a_{n_m}), \qquad m = 1, 2, \dots. \tag{9.6}$$

We now assume the following conditions:

(i) a_n is real, i.e., $a_n = a_n^*$;

(ii) a_n is normalized in such a way that $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$;

(iii) $\{a_n\}$ has uniformly bounded mixed moments, i.e., for each $m = 1, 2, \dots$,

$$K_m = \sup\{|\varphi(a_{n_1}\cdots a_{n_m})|\,;\, n_1,\dots,n_m = 1, 2, \dots\} < \infty;$$

(iv) $\{a_n\}$ satisfies the singleton condition.

Our strategy is simple. We eliminate the terms $\varphi(a_{n_1}\cdots a_{n_m})$ in the right hand side of (9.6) which do not contribute to the limit.

We prepare some notation. Let $\mathfrak{M}(m,N)$ denote the set of maps from $\{1,2,\dots,m\}$ into $\{1,2,\dots,N\}$. Then, (9.6) becomes

$$\varphi(S_N^m) = \sum_{n\in\mathfrak{M}(m,N)} \varphi(a_{n_1}\cdots a_{n_m}). \tag{9.7}$$

By singleton condition if $n_1,\dots,n_m$ contains a singleton, the corresponding term vanishes. Setting

$$\mathfrak{M}'(m,N) = \{n \in \mathfrak{M}(m,N)\,;\, |n^{-1}(i)| \neq 1 \text{ for all } i \in \{1,2,\dots,N\}\},$$

we have

$$\varphi(S_N^m) = \sum_{n\in\mathfrak{M}'(m,N)} \varphi(a_{n_1}\cdots a_{n_m}). \tag{9.8}$$

For $n \in \mathfrak{M}'(m,N)$ we have

$$|\mathrm{Im}\, n| \le \frac{m}{2} \quad \text{for even } m; \qquad |\mathrm{Im}\, n| \le \frac{m-1}{2} \quad \text{for odd } m.$$

For $p = 1, 2, \dots$ we set

$$\mathfrak{M}'_p(m,N) = \{n \in \mathfrak{M}'(m,N)\,;\, |\mathrm{Im}\, n| = p\}.$$

Then we have

$$\varphi(S_N^{2m}) = \sum_{p=1}^{m} \sum_{n\in\mathfrak{M}'_p(2m,N)} \varphi(a_{n_1}\cdots a_{n_{2m}}), \tag{9.9}$$

$$\varphi(S_N^{2m-1}) = \sum_{p=1}^{m-1} \sum_{n\in\mathfrak{M}'_p(2m-1,N)} \varphi(a_{n_1}\cdots a_{n_{2m-1}}), \tag{9.10}$$

We now examine (9.9). First observe that

$$|\mathfrak{M}'_m(2m,N)| = \binom{N}{m}\frac{(2m)!}{2^m} = O(N^m), \quad \sum_{p=1}^{m-1}|\mathfrak{M}'_m(2m,N)| = O(N^{m-1}).$$

Hence under the condition (iii) uniformly bounded mixed moments we see that

$$\lim_{N\to\infty} N^{-m}\sum_{p=1}^{m-1}\sum_{n\in\mathfrak{M}'_p(2m,N)}\varphi(a_{n_1}\cdots a_{n_{2m}}) = 0.$$

Therefore, we see from (9.9) that

$$\lim_{N\to\infty} N^{-m}\varphi(S_N^{2m}) = \lim_{N\to\infty} N^{-m}\sum_{n\in\mathfrak{M}'_m(2m,N)}\varphi(a_{n_1}\cdots a_{n_{2m}}).$$

In other words,

$$\lim_{N\to\infty}\varphi\left(\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N}a_n\right)^{2m}\right) = \lim_{N\to\infty} N^{-m}\sum_{n\in\mathfrak{M}'_m(2m,N)}\varphi(a_{n_1}\cdots a_{n_{2m}}). \tag{9.11}$$

We next consider (9.10). Since

$$\sum_{p=1}^{m-1}|\mathfrak{M}'_p(2m-1,N)| = O(N^{m-1}),$$

we have

$$\lim_{N\to\infty} N^{-(2m-1)/2}\sum_{p=1}^{m-1}\sum_{n\in\mathfrak{M}'_p(2m-1,N)}\varphi(a_{n_1}\cdots a_{n_{2m-1}}) = 0.$$

In other words,

$$\lim_{N\to\infty}\varphi\left(\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N}a_n\right)^{2m-1}\right) = 0. \tag{9.12}$$

Summing up, we obtain the following

Theorem 9.2.1 *Let $\{a_n\}$ be a sequence of random variables in an algebraic probability space $(\mathcal{A},\varphi)$ satisfying the four conditions (i)–(iv) above. Then for $m = 1, 2, \dots$ we have*

$$\lim_{N\to\infty}\varphi\left(\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N}a_n\right)^{2m-1}\right) = 0, \tag{9.13}$$

$$\lim_{N\to\infty}\varphi\left(\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N}a_n\right)^{2m}\right) = \lim_{N\to\infty} N^{-m}\sum_{n\in\mathfrak{M}'_m(2m,N)}\varphi(a_{n_1}\cdots a_{n_{2m}}), \tag{9.14}$$

where $\mathfrak{M}'_m(2m,N)$ is the collection of maps n from $\{1,2,\dots,2m\}$ into $\{1,2,\dots,N\}$ such that $|n^{-1}(i)| = 0$ or 2 for $i \in \{1,2,\dots,N\}$.

Remark 9.2.2 If $\{a_n\}$ is a sequence of bounded operators on a Hilbert space such that $\sup_n \|a_n\| < \infty$, then $\{a_n\}$ has uniformly bounded mixed moments in a vector state. This criterion is also valid for a C^*-probability space $(\mathcal{A}, \varphi)$.

Remark 9.2.3 One might consider another scaling such as

$$\lim_{N\to\infty} N^{-\alpha m}\varphi(S_N^m), \qquad \alpha > 0.$$

However, as is seen during the above discussion, $\alpha = 1/2$ is the unique choice for the reasonable limit under condition (iii).

9.3 Quantum Central Limit Theorems

Throughout this section we keep the assumptions

(i) a_n is real, i.e., $a_n^* = a_n$;

(ii) a_n is normalized, i.e., $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$;

(iii) $\{a_n\}$ has uniformly bounded mixed moments.

Replacing the condition (iv) of singleton condition with one of the four independence, see Theorem 9.1.4, we proceed computation of (9.14).

Let $\mathcal{P}_{\mathrm{P}}(2m)$ denote the set of all pair partitions of $\{1, 2, \dots, 2m\}$. With each $n \in \mathfrak{M}'_m(2m, N)$ we associate a pair partition $\vartheta \in \mathcal{P}_{\mathrm{P}}(2m)$ by

$$\vartheta = \{n^{-1}(i)\,;\, i \in \{1, 2, \dots, N\}, n^{-1}(i) \neq \emptyset\}.$$

The blocks in ϑ may be arranged in such a way that

$$\{l_1, r_1\}, \quad \{l_2, r_2\}, \quad \dots, \quad \{l_m, r_m\},$$

with

$$l_1 < r_1, \quad l_2 < r_2, \quad \dots, \quad l_m < r_m, \quad l_1 < l_2 < \dots < l_m\,.$$

Moreover, $l_1, \dots, l_m, r_1, \dots, r_m$ are uniquely determined. Define a map $\sigma : \{1, 2, \dots, m\} \to \{1, 2, \dots, N\}$ by

$$\sigma(k) = n_{l_k}\,.$$

Then σ is an injection. Let $\mathfrak{M}_i(m, N)$ denote the set of injective maps from $\{1, 2, \dots, m\}$ into $\{1, 2, \dots, N\}$. Thus we obtain a map $n \mapsto (\vartheta, \sigma) \in \mathcal{P}_{\mathrm{P}}(2m) \times \mathfrak{M}_i(m, N)$, $n \in \mathfrak{M}'_m(2m, N)$. It is easily seen that this map is bijective. With these notation (9.14) becomes

$$\lim_{N\to\infty} \varphi\biggl(\biggl(\frac{1}{\sqrt{N}}\sum_{n=1}^N a_n\biggr)^{2m}\biggr) = \lim_{N\to\infty} N^{-m} \sum_{\vartheta\in\mathcal{P}_{\mathrm{P}}(2m)} \sum_{\sigma\in\mathfrak{M}_i(m,N)} \varphi(a_{n_1}\cdots a_{n_{2m}}), \tag{9.15}$$

where n is determined by (ϑ, σ) as above. The alternative expression (9.15) is also useful.

Theorem 9.3.1 (Commutative CLT) *Let $\{a_n\}$ satisfy the above three conditions (i)–(iii) and assume that it is commutative independent. Then*

$$\lim_{N\to\infty} \varphi\left(\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} a_n\right)^m\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^m e^{-x^2/2}dx, \quad m=1,2,\dots,$$

where the probability measure appearing in the right hand side is the standard Gaussian distribution.

Proof. By elementary calculus, we know that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^{2m-1} e^{-x^2/2}dx = 0, \qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^{2m} e^{-x^2/2}dx = \frac{(2m)!}{2^m m!}\,.$$

Hence it is sufficient to show that

$$\lim_{N\to\infty} N^{-m} \sum_{n\in\mathfrak{M}'_m(2m,N)} \varphi(a_{n_1}\cdots a_{n_{2m}}) = \frac{(2m)!}{2^m m!}\,.$$

Since $\{a_n\}$ is commutative independent,

$$\varphi(a_{n_1}\cdots a_{n_{2m}}) = \varphi(a_{i_1}^2)\cdots\varphi(a_{i_m}^2) = 1, \qquad n\in\mathfrak{M}'_m(2m,N).$$

Hence

$$\begin{aligned} N^{-m} \sum_{n\in\mathfrak{M}'_m(2m,N)} \varphi(a_{n_1}\cdots a_{n_{2m}}) &= N^{-m}|\mathfrak{M}'_m(2m,N)| \\ &= N^{-m}\binom{N}{m}\frac{(2m)!}{2^m} \to \frac{(2m)!}{2^m m!}\,, \end{aligned}$$

as desired. ∎

Theorem 9.3.2 (Free CLT) *Notations and assumptions being as in (CC), if $\{a_n\}$ is free independent, we have*

$$\lim_{N\to\infty} \varphi\left(\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} a_n\right)^m\right) = \frac{1}{2\pi}\int_{-2}^{+2} x^m \sqrt{4-x^2}\,dx, \quad m=1,2,\dots,$$

where the probability measure appearing in the right hand side is the Wigner semicircle law.

Proof. The proof is similar to that of Theorem 9.3.1. We already know that

$$\frac{1}{2\pi}\int_{-2}^{+2} x^{2m-1}\sqrt{4-x^2}\,dx = 0, \qquad \frac{1}{2\pi}\int_{-2}^{+2} x^{2m}\sqrt{4-x^2}\,dx = \frac{(2m)!}{(m+1)!m!}\,.$$

Hence it is sufficient to show that

$$\lim_{N\to\infty} N^{-m} \sum_{n\in\mathfrak{M}'_m(2m,N)} \varphi(a_{n_1}\cdots a_{n_{2m}}) = \frac{(2m)!}{(m+1)!m!}\,. \tag{9.16}$$

We observe easily that

$$\varphi(a_{n_1}\cdots a_{n_{2m}}) = \begin{cases} 1, & \vartheta \in \mathcal{P}_{\mathrm{NCP}}(2m), \\ 0, & \text{otherwise}, \end{cases}$$

where ϑ is a pair partition associated with n. Hence,

$$\sum_{n\in\mathfrak{M}'_m(2m,N)} \varphi(a_{n_1}\cdots a_{n_{2m}}) = |\mathfrak{M}_i(m,N)\times\mathcal{P}_{\mathrm{NCP}}(2m)| = \binom{N}{m} m!\,\frac{(2m)!}{(m+1)!m!}\,,$$

from which (9.16) follows. ∎

Theorem 9.3.3 (Boolean CLT) *Notations and assumptions being as in (CC), if $\{a_n\}$ is Boolean independent, we have*

$$\lim_{N\to\infty}\varphi\left(\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} a_n\right)^m\right) = \frac{1}{2}\int_{-\infty}^{+\infty} x^m(\delta_{-1}+\delta_{+1})(dx), \quad m=1,2,\dots,$$

where the probability measure appearing in the right hand side is the Bernoulli distribution.

Proof. The proof is similar to those of Theorems 9.3.1 and 9.3.2. We readily know that

$$\frac{1}{2}\int_{-\infty}^{+\infty} x^{2m-1}(\delta_{-1}+\delta_{+1})(dx) = 0, \qquad \frac{1}{2}\int_{-\infty}^{+\infty} x^{2m}(\delta_{-1}+\delta_{+1})(dx) = 1,$$

so it is sufficient to show that

$$\lim_{N\to\infty} N^{-m}\sum_{n\in\mathfrak{M}'_m(2m,N)} \varphi(a_{n_1}\cdots a_{n_{2m}}) = 1. \tag{9.17}$$

By Boolean independence we see that

$$\varphi(a_{n_1}\cdots a_{n_{2m}}) = \begin{cases} 1, & n_1=n_2,\ \dots,\ n_{2m-1}=n_{2m}\,, \\ 0, & \text{otherwise}. \end{cases}$$

The number of such n's is $\binom{N}{m}m!$. Hence

$$\lim_{N\to\infty} N^{-m}\sum_{n\in\mathfrak{M}'_m(2m,N)} \varphi(a_{n_1}\cdots a_{n_{2m}}) = \lim_{N\to\infty} N^{-m}\binom{N}{m}m! = 1,$$

as desired. ∎

Theorem 9.3.4 (Monotone CLT) *Notations and assumptions being as in (CC), if $\{a_n\}$ is monotone independent, we have for $m=1,2,\dots,$*

$$\lim_{N\to\infty}\varphi\left(\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} a_n\right)^m\right) = \frac{1}{\pi}\int_{-\sqrt{2}}^{+\sqrt{2}} \frac{x^m}{\sqrt{2-x^2}}\,dx, \tag{9.18}$$

where the probability measure appearing in the right hand side is the normalized arcsine law.

PROOF. By elementary calculus we obtain

$$\frac{1}{\pi}\int_{-\sqrt{2}}^{+\sqrt{2}} \frac{x^{2m-1}}{\sqrt{2-x^2}}\,dx = 0, \qquad \frac{1}{\pi}\int_{-\sqrt{2}}^{+\sqrt{2}} \frac{x^{2m}}{\sqrt{2-x^2}}\,dx = \frac{(2m)!}{2^m m! m!}\,.$$

It is then sufficient to show that

$$\lim_{N\to\infty} N^{-m} \sum_{n\in\mathfrak{M}'_m(2m,N)} \varphi(a_{n_1}\cdots a_{n_{2m}}) = \frac{(2m)!}{2^m m! m!}\,. \tag{9.19}$$

Let $n \in \mathfrak{M}'_m(2m,N)$. Then $n_1, n_2, \dots, n_{2m}$ is an arrangement of $1 \le i_1 < i_2 < \cdots < i_m \le N$ with each number appearing twice. By monotone independence $\varphi(a_{n_1}\cdots a_{n_{2m}}) = 0$ if i_m appears as a peak, i.e., if i_m does not appear successively. If i_m appears successively, we take out $\varphi(a_{i_m}^2)$. For the rest we repeat a similar consideration, we see that

$$\varphi(a_{n_1}\cdots a_{n_{2m}}) = 1$$

if i_m appears side by side, i_{m-1} appears side by side in the sequence obtained by eliminating i_m, and so forth;

$$\varphi(a_{n_1}\cdots a_{n_{2m}}) = 0$$

otherwise. The number of such arrangements of a fixed $i_1 < i_2 < \cdots < i_m$ is

$$(2m-1)(2m-3)\dots 3\cdot 1 = \frac{(2m)!}{2^m m!}\,.$$

Therefore,

$$\lim_{N\to\infty} N^{-m} \sum_{n\in\mathfrak{M}'_m(2m,N)} \varphi(a_{n_1}\cdots a_{n_{2m}}) = \lim_{N\to\infty} N^{-m}\binom{N}{m}\frac{(2m)!}{2^m m!} = \frac{(2m)!}{2^m m! m!}$$

which completes the proof. ∎

Exercises 9

1. Let $(\mathcal{A}, \varphi)$ an algebraic probability space and $a_1, a_2, \dots \in \mathcal{A}$ a sequence of random variables satisfying $a_n^* = a_n$ and $\varphi(a_n) = 0$ for all n. Assume that $\{a_n\}$ is commutative independent, i.e., $\{\mathcal{A}_n\}$ is commutative independent, where $\mathcal{A}_n$ is the $*$-subalgebra spanned by a_n (polynomials in a_n).

(1) Prove that $\varphi(a_1 a_2 a_1 a_3 a_2 a_1) = 0$.

(2) Prove that $\{a_n\}$ satisfies the singleton condition.

2. Let p, m, N be natural numbers with $m < N$. Let $\mathfrak{M}'_p(m, N)$ denote the set of all maps from $\{1, 2, \dots, m\}$ into $\{1, 2, \dots, N\}$ such that

(i) $|n^{-1}(i)| \neq 1$ for all $i \in \{1, 2, \dots, N\}$; (ii) $|\mathrm{Im}\, n| = p$.

Show that

(1) $|\mathfrak{M}'_m(2m, N)| = \binom{N}{m} \dfrac{(2m)!}{2^m} = O(N^m)$

(2) $|\mathfrak{M}'_p(m, N)| = O(N^p)$.

3. Compute the cardinalities $|\mathcal{P}_{\mathrm{P}}(2m)|$ and $|\mathfrak{M}_i(m, N)|$. Then examine directly

$$|\mathfrak{M}'_m(2m, N)| = |\mathcal{P}_{\mathrm{P}}(2m) \times \mathfrak{M}_i(m, N)|.$$

4. Let $(\mathcal{A}, \varphi)$ an algebraic probability space and $a_1, a_2, \dots \in \mathcal{A}$ a sequence of random variables satisfying $a_n^* = a_n$, $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$ for all n. Assume that $\{a_n\}$ is free independent. Calculate the following

$$\varphi(a_1 a_2 a_1) \qquad \varphi(a_1 a_2 a_1 a_2) \qquad \varphi(a_1 a_1 a_2 a_2) \qquad \varphi(a_1 a_2 a_2 a_1)$$

5. Keeping the same assumptions as above, prove that

$$\varphi(a_{n_1} \cdots a_{n_{2m}}) = \begin{cases} 1, & \vartheta \in \mathcal{P}_{\mathrm{NCP}}(2m), \\ 0, & \text{otherwise}, \end{cases}$$

where ϑ is a pair partition associated with n.

6. Show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^{2m} e^{-x^2/2} dx = \frac{(2m)!}{2^m m!}, \qquad \frac{1}{\pi} \int_{-\sqrt{2}}^{+\sqrt{2}} \frac{x^{2m}}{\sqrt{2 - x^2}}\, dx = \frac{(2m)!}{2^m m! m!}.$$

10 Deformed Vacuum States and Q-Matrices

10.1 Q-Matrices

Definition 10.1.1 Let $G = (V, E)$ be a connected graph. Given $q \in \mathbf{C}$, the matrix $Q = Q_q$ indexed by $V \times V$ defined by

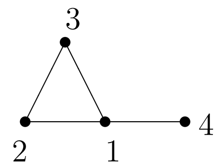
$$(Q)_{xy} = q^{\partial(x,y)}, \qquad x, y \in V$$

is called the *Q-matrix* of G. For $q = 0$ we understand that $0^0 = 1$ and $Q_0 = E$ (the identity matrix).

When a Q-matrix is considered, the graph is pressumed to be connected. The Q-matrix is related to the adjacency matrix: $\dfrac{d}{dq} Q \Big|_{q=0} = A$.

Example 10.1.2 (1)

$$\begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix}$$

(2) 

$$\begin{bmatrix} 1 & q & q & q \\ q & 1 & q & q^2 \\ q & q & 1 & q^2 \\ q & q^2 & q^2 & 1 \end{bmatrix}$$

The Q-matrix gives rise to a one-parameter deformation of the vacuum state. Let us define

$$\langle a \rangle_q = \langle Qe_o, ae_o \rangle = \sum_{x \in V} q^{\partial(x,o)} \langle e_x, ae_o \rangle, \qquad a \in \mathcal{A}(G). \tag{10.1}$$

Obviously, $\mathcal{A}(G) \ni a \mapsto \langle a \rangle_q$ is a normalized linear function on $\mathcal{A}(G)$.

Definition 10.1.3 A normalized linear function defined in (10.1) is called a *deformed vacuum functional* on $\mathcal{A}(G)$.

A deformed vacuum functional is not necessarily a state. We are interested in when $\langle \cdot \rangle_q$ is positive. We recall the following general notion.

Definition 10.1.4 Let T be a matrix indexed by $V \times V$. We say that T is *positive definite* if

$$\langle f, Tf \rangle \geq 0 \qquad \text{for all } f \in C_0(V).$$

A positive definite matrix T is called *strictly positive definite* if

$$\langle f, Tf \rangle > 0 \qquad \text{for all } f \in C_0(V),\ f \neq 0.$$

Theorem 10.1.5 *The normalized linear function $\langle \cdot \rangle_q$ defined by* (10.1) *is positive, hence a state on $\mathcal{A}(G)$ if the following two conditions are fulfilled:*

(i) *Q is a positive definite kernel on V;*

(ii) $QA = AQ$. *(Note that Q is not necessarily locally finite but A is. Therefore the matrix elements of both sides are well-defined.)*

PROOF. Let $a \in \mathcal{A}(G)$. Since a is a polynomial in A, we have $Qa = aQ$. Then, by the definition (10.1) we have

$$\langle a^* a \rangle_q = \langle Qe_o, a^* ae_o \rangle = \langle aQe_o, ae_o \rangle = \langle Qae_o, ae_o \rangle \geq 0,$$

which proves the assertion. ∎ ∎

Lemma 10.1.6 *Let $\mathcal{G} = (V, E)$ be a graph with $|V| \geq 2$. If $Q = (q^{\partial(x,y)})$ is a positive definite kernel on V, then $-1 \leq q \leq 1$.*

PROOF. By assumption there is a pair of $a, b \in V$ such that $\partial(a, b) = 1$. Since $Q = (q^{\partial(x,y)})$ is a positive definite kernel on V, taking $f = \alpha e_a + \beta e_b$ in $C_0(V)$, we obtain

$$\left\langle \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\rangle \geq 0, \qquad \alpha, \beta \in \mathbf{C}, \tag{10.2}$$

where $\langle \cdot, \cdot \rangle$ is the usual Hermitian inner product of $\mathbf{C}^2$. Therefore, the 2×2 matrix appearing in (10.2) is positive definite. Hence $q \in \mathbf{R}$ and $1 - q^2 \geq 0$. ∎

It is an important question, which is quite open, to determine the range of $q \in [-1, 1]$ for which Q becomes positive definite. For a graph G we set

$$q[G] = \{-1 \leq q \leq 1\,;\, Q_q \text{ is strictly positive definite}\},$$
$$\tilde{q}[G] = \{-1 \leq q \leq 1\,;\, Q_q \text{ is positive definite}\}.$$

Lemma 10.1.7 (1) $q[G] \subset \tilde{q}[G]$.
(2) *$\overline{q[G]} \subset \tilde{q}[G]$ and $\tilde{q}[G]$ is a closed subset of $[-1, 1]$.*

PROOF. Immediate. ∎

Example 10.1.8 (1) The eigenvalues of the Q-matrix of $P_2 = K_2$ are $1 \pm q$. Hence

$$q[P_2] = (-1, 1), \qquad \tilde{q}[P_2] = [-1, 1].$$

(2) The eigenvalues of the Q-matrix of C_3 are $1 + 2q$ and $1 - q$ (multiplicity 2).

$$q[C_3] = \left(-\frac{1}{2}, 1\right), \qquad \tilde{q}[C_3] = \left[-\frac{1}{2}, 1\right].$$

(3) For a complete bipartite graph $K_{m,n}$ with $2 \le m \le n$,

$$q[K_{m,n}] = \left(-\frac{1}{\sqrt{(m-1)(n-1)}}, \frac{1}{\sqrt{(m-1)(n-1)}} \right),$$

$$\tilde{q}[K_{m,n}] = \left[-\frac{1}{\sqrt{(m-1)(n-1)}}, \frac{1}{\sqrt{(m-1)(n-1)}} \right] \cup \{-1, 1\}.$$

More discussion on $q[G]$ and $\tilde{q}[G]$ will be found in the next sections.

In order to derive a sufficient condition for the equality $QA = AQ$ we consider a geometric property of a graph. A graph $G = (V, E)$ is called *quasi-distance-regular* if

$$\left| \left\{ z \in V\,;\ \begin{array}{c} \partial(z,x) = n \\ \partial(z,y) = 1 \end{array} \right\} \right| = \left| \left\{ z \in V\,;\ \begin{array}{c} \partial(z,x) = 1 \\ \partial(z,y) = n \end{array} \right\} \right| \tag{10.3}$$

holds for any choice of $x, y \in V$ and $n = 0, 1, 2, \dots$. Here the number defined by (10.3) may depend on the choice of $x, y \in V$. By definition, a distance-regular graph is quasi-distance-regular. On the other hand, if (10.3) depends only on $\partial(x, y)$, the graph G becomes distance-regular.

Proposition 10.1.9 *If a graph is quasi-distance-regular, then $QA = AQ$ for all $q \in \mathbf{R}$. Conversely, if $QA = AQ$ holds for q running over a non-empty open interval, then the graph is quasi-distance-regular.*

PROOF. Let $x, y \in V$. Then

$$\begin{aligned} (QA)_{xy} &= \sum_{z \in V} q^{\partial(x,z)} A_{zy} = \sum_{z \sim y} q^{\partial(x,z)} \\ &= \sum_{n=0}^{\infty} q^n |\{z \in V\,;\ \partial(z,x) = n,\ \partial(z,y) = 1\}|, \end{aligned} \tag{10.4}$$

which is in fact a finite sum. Similarly, we have

$$(AQ)_{xy} = \sum_{n=0}^{\infty} q^n |\{z \in V\,;\ \partial(z,x) = 1,\ \partial(z,y) = n\}|. \tag{10.5}$$

Hence, if the graph is quasi-distance-regular, the coefficients of q^n in (10.4) and (10.5) coincide and we obtain $(QA)_{xy} = (AQ)_{xy}$ for all $x, y \in V$. The converse assertion is readily clear. ∎

10.2 Cartesian Product

Lemma 10.2.1 *Let $G = G_1 \times G_2$. Then,*

$$\partial_G((x,y),(x',y')) = \partial_{G_1}(x,x') + \partial_{G_2}(y,y'). \tag{10.6}$$

PROOF. Set $s = \partial_G((x,y),(x',y'))$. Then we may find a sequence of vertices of $G_1 \times G_2$ such that

$$(x,y) = (x_0,y_0) \sim (x_1,y_1) \sim (x_2,y_2) \sim \cdots \sim (x_{s-1},y_{s-1}) \sim (x_s,y_s) = (x',y').$$

Then, every pair of consecutive vertices in the sequence

$$x = x_0, \quad x_1, \quad x_2, \quad \dots, \quad x_{s-1}, \quad x_s = x'$$

are identical or adjacent. Hence, reducing consecutively identical vertices into one vertex, we obtain a walk connecting x and x', of which the length is, say, α. Similarly, from

$$y = y_0, \quad y_1, \quad y_2, \quad \dots, \quad y_{s-1}, \quad y_s = y'$$

we obtain a walk connecting y and y', of which the length is, say, β. By the definition of a direct product graph, $x_i = x_{i+1}$ happens if and only if $y_i \sim y_{i+1}$. Hence

$$\alpha + \beta = s.$$

Since $\partial_{G_1}(x,x') \le \alpha$ and $\partial_{G_2}(y,y') \le \beta$, we have

$$\partial_{G_1}(x,x') + \partial_{G_2}(y,y') \le \alpha + \beta = s.$$

That $\partial_{G_1}(x,x') + \partial_{G_2}(y,y') \ge s$ is shown by constructing a walk. ∎

Lemma 10.2.2 *Let Q_1, Q_2 and Q be the Q-matrices of graphs G_1, G_2 and $G = G_1 \times G_2$, with a common parameter q. Then*

$$Q = Q_1 \otimes Q_2 .$$

PROOF. First by definition

$$(Q)_{(x,y),(x',y')} = q^{\partial_G((x,y),(x',y'))}.$$

Applying Lemma 10.2.1, we obtain

$$= q^{\partial_{G_1}(x,x')} q^{\partial_{G_2}(y,y')} = (Q_1)_{xx'}(Q_2)_{yy'} = (Q_1 \otimes Q_2)_{(x,y),(x',y')}.$$

Therefore, $Q = Q_1 \otimes Q_2$. ∎

Theorem 10.2.3 *Let $G = G_1 \times G_2$.*

(1) $q[G] = q[G_1] \cap q[G_2]$.

(2) $\tilde{q}[G] = \tilde{q}[G_1] \cap \tilde{q}[G_2]$.

PROOF. We see from Lemma 10.2.2 that the eigenvalues of Q are of the form $\alpha\beta$, where α and β are eigenvalues of Q_1 and Q_2, respectively.

(1) Let $q \in q[G_1] \cap q[G_2]$, namely, $Q_i = Q_i(q)$ is a strictly positive definite kernel for G_i. Since the eigenvalues of Q_i are all positive, every eigenvalues of Q are also positive. Therefore, $q[G_1] \cap q[G_2] \subset q[G]$.

We show that Q contains Q_1 as a principal submatrix. Take a vetex $o_2 \in V_2$ and set

$$W = \{(x, o_2)\,;\, x \in V_1\}.$$

Let H_1 be the induced subgraph of $G_1 \times G_2$ spanned by W. Then, H_1 is isomorphic to G_1 and $\partial_H = \partial_{G_1}$ coincides with the restriction of ∂_G to H. Hence Q_1 is regarded as a principal submatrix of Q. The situation is similar for Q_2. Now let $q \in q[G]$. Then Q is strictly positive definite so are all the principal submatrices. In particular, so are Q_1 and Q_2. Consequently, $q[G] \subset q[G_1] \cap q[G_2]$.

(2) The proof is similar. Let $q \in \tilde{q}[G_1] \cap \tilde{q}[G_2]$, namely, $Q_i = Q_i(q)$ is a positive definite kernel for G_i. Since the eigenvalues of Q_i are all non-negative, every eigenvalues of Q are also non-negative. Therefore, $\tilde{q}[G_1] \cap \tilde{q}[G_2] \subset \tilde{q}[G]$.

The second half is also similar to the argument in (1). ∎

10.3 Star Product and Comb Product

We now consider the graph distance of the star product.

Lemma 10.3.1 *Let $G = G_1 \star G_2$. Then,*

$$\partial_G = \partial_{G_1 \times G_2} \restriction_{V_1 \star V_2} .$$

PROOF. Take a pair of vertices of $G_1 \star G_2$. For $(x, o_2), (x', o_2)$ we have

$$\begin{aligned}
\partial_G((x, o_2), (x', o_2)) &= \partial_{G_1}(x, x') \\
&= \partial_{G_1}(x, x') + \partial_{G_2}(o_2, o_2) \\
&= \partial_{G_1 \times G_2}((x, o_2), (x', o_2)).
\end{aligned}$$

For $(x, o_2), (o_1, y)$ we have

$$\begin{aligned}
\partial_G((x, o_2), (o_1, y)) &= \partial_G((x, o_2), (o_1, o_2)) + \partial_G((o_1, o_2), (o_1, y)) \\
&= \partial_{G_1}(x, o_1) + \partial_{G_2}(o_2, y) \\
&= \partial_{G_1 \times G_2}((x, o_2), (o_1, y)).
\end{aligned}$$

∎

As an immediate consequence from Lemma 10.3.1 we obtain

Lemma 10.3.2 *The Q-matrix of the star product $G = G_1 \star G_2$ is a principal submatrix of the Q-matrix of $G_1 \times G_2$ as follows:*

$$Q_{G_1 \star G_2} = Q_{G_1 \times G_2} \restriction_{C(V_1 \star V_2)}$$

Theorem 10.3.3 *Let $G = G_1 \star G_2$.*

(1) $q[G] = q[G_1] \cap q[G_2]$.

(2) $\tilde{q}[G] = \tilde{q}[G_1] \cap \tilde{q}[G_2]$.

PROOF. (1) Supose $q \in q[G_1] \cap q[G_2]$. We see from Theorem 10.2.3 that $Q_{G_1 \times G_2}(q)$ is strictly positive definite. Since $Q_{G_1 \star G_2}$ is a principal submatrix by Lemma 10.3.2, it is also strictly positive definite. Namely, $q[G_1] \cap q[G_2] \subset q[G]$.

Conversely, let $q \in q[G]$. Then $Q_{G_1 \star G_2}(q)$ is strictly positive definite. Since G_i is isometrically imbedded in $G_1 \star G_2$, its Q-matrix is a principal submatrix of $Q_{G_1 \star G_2}(q)$. Therefore, $Q_{G_i}(q)$ is also a strictly positive definite. Thus, $q[G] \subset q[G_1] \cap q[G_2]$.

(2) is proved similarly. ∎

Remark 10.3.4 Theorem 10.3.3 was first obtained as a corollary to Bożejko's theorem on Markov sum. The above argument provides an alternative proof.

Theorem 10.3.5 *Let $G = G_1 \rhd G_2$.*

(1) $q[G] = q[G_1] \cap q[G_2]$.

(2) $\tilde{q}[G] = \tilde{q}[G_1] \cap \tilde{q}[G_2]$.

PROOF. Since

$$G_1 \rhd G_2 \cong (\cdots((G_1 \star \overbrace{G_2) \star G_2 \star \cdots) \star G_2}^{|V_1| \text{ times}},$$

the assertion follows from Theorem 10.3.3. ∎

10.4 Haagerup States

Let T_κ denote the homogeneous tree of degree κ. We start with the following fundamental fact.

Theorem 10.4.1 *The deformed vacuum functional $\langle \cdot \rangle_q$ on the homogeneous tree T_κ is a state for all $-1 \le q \le 1$.*

PROOF. We check the conditions (i) and (ii) in Theorem 10.1.5. First, (i) is clear because T_κ is distance-regular. For (ii) it is sufficient to show that the Q-matrix of a finite tree is positive definite for all $-1 \le q \le 1$. But a tree is formed by repeated application of star product with $P_2 = K_2$. Since $\tilde{q}[P_2] = [-1, 1]$ we see from Theorem 10.3.3 that $\tilde{q}[T_\kappa] = [-1, 1]$. ∎

Definition 10.4.2 The deformed vacuum state $\langle \cdot \rangle_q$ on the adjacency algebra $\mathcal{A}(T_\kappa)$ is called the *Haagerup state*.

Remark 10.4.3 In fact, Theorem 10.4.1 is originally due to Haagerup. His proof uses some specific structure of free group. Later Bożejko, introducing a concept of Markov sum of positive definite kernels, drastically simplified the proof. Our proof is based on our argument.

We are interested in the asymptotics of the spectral distribution $\mu_{\kappa,q}$ determined by

$$\langle A^m \rangle_q = \int_{-\infty}^{+\infty} x^m \mu_{\kappa,q}(dx), \qquad m = 1, 2, \dots .$$

It is reasonable to call $\mu_{\kappa,q}$ a *deformed Kesten distribution*. We first note the following

Lemma 10.4.4 (1) $\mathrm{mean}\,(\mu_{\kappa,q}) = \langle A \rangle_q = \kappa q$.

(2) $\mathrm{var}\,(\mu_{\kappa,q}) = \Sigma_q^2(A) = \kappa(1-q^2)$.

Proof. (1) By definition

$$\begin{aligned}\langle A \rangle_q &= \langle Q\delta_o, A\delta_o \rangle = \langle \delta_o, QA\delta_o \rangle = (QA)_{oo} \\ &= \sum_{x \in V} (Q)_{ox} (A)_{xo} = \sum_{x \sim o} (Q)_{ox} = \sum_{x \sim o} q^{\partial(o,x)} \\ &= q|\{x \in V \,;\, x \sim o\}| = q\kappa.\end{aligned}$$

(2) Since

$$\Sigma_q^2(A) = \langle A^2 \rangle_q - \langle A \rangle_q^2$$

by definition, we need to compute $\langle A^2 \rangle_q$. In a similar manner as in (1) we see that

$$\langle A^2 \rangle_q = \kappa(\kappa - 1)q^2 + \kappa,$$

from which the assertion follows. ∎

Lemma 10.4.4 suggests that a reasonable object to study is not A itself but the normalized adacency matrix defined by

$$\frac{A - \langle A \rangle_q}{\Sigma_q(A)} = \frac{A - \kappa q}{\sqrt{\kappa(1-q^2)}} .$$

We will study the moments:

$$\left\langle \left(\frac{A - \kappa q}{\sqrt{\kappa(1-q^2)}} \right)^m \right\rangle_q, \qquad m = 1, 2, \dots .$$

Having already chosen an origin o of T_κ, we have the natural stratification and the quantum decomposition of $A = A^+ + A^-$ ($A^\circ = 0$ for a tree). Accordingly, the normalized adjacency matrix is decomposed into three parts:

$$\frac{A - \kappa q}{\sqrt{\kappa(1-q^2)}} = \frac{A^+}{\sqrt{\kappa(1-q^2)}} + \frac{A^-}{\sqrt{\kappa(1-q^2)}} + \frac{-\kappa q}{\sqrt{\kappa(1-q^2)}} .$$

For simplicity we introduce $C^\epsilon = C^\epsilon(\kappa, q)$ by

$$C^+ = \frac{A^+}{\sqrt{\kappa(1-q^2)}}\,, \qquad C^- = \frac{A^-}{\sqrt{\kappa(1-q^2)}} \qquad C^\circ = \frac{-\kappa q}{\sqrt{\kappa(1-q^2)}}\,. \tag{10.7}$$

Using the actions of $A^\pm$ on $\Gamma(T_\kappa)$, see Section 6.4, we obtain easily

$$C^+\Phi_0 = \frac{1}{\sqrt{1-q^2}}\,\Phi_1, \qquad C^+\Phi_n = \sqrt{\frac{\kappa-1}{\kappa(1-q^2)}}\,\Phi_{n+1} \quad (n \ge 1)$$

$$C^-\Phi_0 = 0, \qquad C^-\Phi_1 = \frac{1}{\sqrt{1-q^2}}\,\Phi_1, \qquad C^-\Phi_n = \sqrt{\frac{\kappa-1}{\kappa(1-q^2)}}\,\Phi_{n-1} \quad (n \ge 2)$$

$$C^\circ\Phi_n = -\sqrt{\frac{q^2\kappa}{1-q^2}}\,\Phi_n \quad (n \ge 0)$$

We are interested in the asymptotics as $\kappa \to \infty$ (the growing trees) so we need to take a suitable balance with q. The reasonable scaling is as follows:

$$\kappa \to \infty, \qquad q\sqrt{\kappa} \to \gamma, \qquad q \to 0, \tag{10.8}$$

where $\gamma \in \mathbf{R}$ is a constant. Under this scaling limit the limit actions of C^ϵ are rather apparent. In particular, in view of the actions of $C^\pm$, we expect that the limit is described in terms of the free Fock space.

We need to discuss the mixed moments:

$$\langle C^{\epsilon_m} \cdots C^{\epsilon_1} \rangle_q = \langle Q\Phi_0, C^{\epsilon_m} \cdots C^{\epsilon_1}\Phi_0 \rangle,$$

where the limit actions of $C^{\epsilon_m}, \dots, C^{\epsilon_1}$ are readily observed. Consider the vector $Q\Phi_0$. By definition

$$\begin{aligned} Q\Phi_0 &= \sum_{x\in V} \langle \delta_x, Q\Phi_0 \rangle \delta_x = \sum_{x\in V} (Q)_{xo}\delta_x \\ &= \sum_{x\in V} q^{\partial(x,o)}\delta_x = \sum_{n=0}^{\infty} \sum_{x\in V_n} q^n \delta_x \\ &= \sum_{n=0}^{\infty} q^n |V_n|^{1/2} \Phi_n \end{aligned}$$

Since $|V_n| = \kappa(\kappa-1)^{n-1}$ for $n \ge 1$, under the scaling limit as in (10.8) the coefficient converges:

$$q^n |V_n|^{1/2} \to \gamma^n.$$

Definition 10.4.5 Let $(\Gamma_{\mathrm{free}}, \{\Psi_n\}, B^+, B^-)$ be a free Fock space. For $z \in \mathbf{C}$,

$$\Omega_z = \sum_{n=0}^{\infty} z^n \Psi_n\,. \tag{10.9}$$

is called a *coherent vector*.

(10.9) is a formal sum but makes sense as a linear functional on the $*$-algebra $\mathcal{A}_{\rm free}$ generated by B^+, B^- and diagonal operators. Namely, for $a \in \mathcal{A}_{\rm free}$,

$$\langle \Omega_z, a\Psi_0 \rangle = \sum_{n=0}^{\infty} \bar{z}^n \langle \Psi_n, a\Phi_0 \rangle$$

is a finite sum and

$$a \mapsto \langle \Omega_z, a\Psi_0 \rangle$$

is a linear functional on $\mathcal{A}_{\rm free}$.

Remark 10.4.6 (1) The infinite series (10.9) converges in norm for $|z| < 1$.

(2) Ω_z is an eigenvector of B^-, i.e., $B^-\Omega_z = z\Omega_z$. More precisely, $\langle \Omega_z, B^+\Psi_n \rangle = \langle z\Omega_z, \Psi_n \rangle$ for n. This motivated us to call Ω_z a coherent vector.

Theorem 10.4.7 (Quantum Central Limit Theorem) *Let $A = A_\kappa$ be the adjacency matrix of T_κ and define $C^\epsilon = C^\epsilon(\kappa, q)$ as in* (10.7). *Let $(\Gamma_{\rm free}, \{\Psi_n\}, B^+, B^-)$ be the free Fock space and set $B^\circ = -\gamma I$ (scalar operator). Then*

$$\lim \langle C^{\epsilon_m} \cdots C^{\epsilon_1} \rangle_q = \langle \Omega_\gamma, B^{\epsilon_m} \cdots B^{\epsilon_1} \Psi_0 \rangle_{\rm free},$$

where the limit is taken as $\kappa \to \infty$, $q \to 0$ with $q\sqrt{\kappa} \to \gamma \in \mathbf{R}$ (constant).

PROOF. The proof is already clear from the above argument. ∎

Theorem 10.4.8 *For the normalized adjacency matrix of T_κ we have*

$$\lim \left\langle \left(\frac{A - \kappa q}{\sqrt{\kappa(1-q^2)}} \right)^m \right\rangle_q = \langle \Omega_\gamma, (B^+ + B^- - \gamma I)^m \Psi_0 \rangle_{\rm free}, \qquad m = 1, 2, \dots .$$

10.5 Free Poisson Distributions

In this section we meet one of the most basic result on the free Fock space. Let P be the vacuum projection, i.e.,

$$P\Psi_0 = \Psi_0, \qquad P\Psi_n = 0 \quad (n \ge 1).$$

Note that $B^+B^- = I - P$.

Lemma 10.5.1 *For $z \in \mathbf{C}$ and $m = 1, 2, \dots$ we have:*

$$\langle \Omega_{\bar{z}}, (B^+ + B^-)^m \Psi_0 \rangle = \langle \Psi_0, (B^+ + B^- + zP)^m \Psi_0 \rangle, \tag{10.10}$$
$$\langle \Omega_{\bar{z}}, (B^+ + B^- - z)^m \Psi_0 \rangle = \langle \Phi_0, (B^+ + B^- - zB^+B^-)^m \Psi_0 \rangle, \tag{10.11}$$

where $\Omega_{\bar{z}}$ is the coherent vector with parameter $\bar{z}$.

PROOF. (10.11) follows from (10.10). In fact,

$$
\begin{aligned}
&\langle \Omega_{\bar{z}}, (B^+ + B^- - z)^m \Psi_0 \rangle \\
&\quad = \sum_{n=0}^{m} \binom{m}{n} (-z)^{m-n} \langle \Omega_{\bar{z}}, (B^+ + B^-)^n \Psi_0 \rangle \\
&\quad = \sum_{n=0}^{m} \binom{m}{n} (-z)^{m-n} \langle \Psi_0, (B^+ + B^- + zP)^n \Psi_0 \rangle \\
&\quad = \langle \Psi_0, (B^+ + B^- + zP - z)^m \Psi_0 \rangle.
\end{aligned}
$$

Since $B^+B^- = 1 - P$, the last expression becomes

$$
= \langle \Psi_0, (B^+ + B^- - zB^+B^-)^m \Psi_0 \rangle,
$$

which proves (10.11). The proof of (10.10) is left to the reader. ∎

In particular, for any $\gamma \in \mathbf{R}$ there exists a probability measure μ_γ such that

$$
\langle \Omega_\gamma, (B^+ + B^- - \gamma)^m \Psi_0 \rangle = \langle \Psi_0, (B^+ + B^- - \gamma B^+B^-)^m \Psi_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu_\gamma(dx)
$$

for $m = 1, 2, \dots$. In fact, the Jacobi coefficients of μ_γ is given by

$$
\omega_1 = \omega_2 = \cdots = 1, \qquad \alpha_1 = 0, \quad \alpha_2 = \alpha_3 = \cdots = -\gamma. \tag{10.12}
$$

Then, Corollary 10.4.8 yields the following

Theorem 10.5.2 (CLT) *For the normalized adjacency matrix of T_κ we have*

$$
\lim \left\langle \left(\frac{A - \kappa q}{\sqrt{\kappa(1 - q^2)}} \right)^m \right\rangle_q = \int_{-\infty}^{+\infty} x^m \mu_\gamma(dx), \qquad m = 1, 2, \dots,
$$

where μ_γ is uniquely determined by the Jacobi coefficients given by (10.12).

We are now in a good position to give the following

Definition 10.5.3 Let $(\Gamma_{\text{free}}, \{\Psi_n\}, B^+, B^-)$ be the free Fock space and $\lambda > 0$ a constant. The vacuum spectral distribution of $(B^+ + \sqrt{\lambda})(B^- + \sqrt{\lambda})$ is called the *free Poisson distribution* or *Marchenko–Pastur distribution* with parameter λ. In other words, the free Poisson distribution with parameter λ is a probability measure ν_λ uniquely specified by

$$
\langle \Psi_0, ((B^+ + \sqrt{\lambda})(B^- + \sqrt{\lambda}))^m \Psi_0 \rangle = \int_{-\infty}^{+\infty} x^m \nu_\lambda(dx), \quad m = 1, 2, \dots. \tag{10.13}
$$

Lemma 10.5.4 (1) mean $(\nu_\lambda) = \operatorname{var}(\nu_\lambda) = \lambda$.

(2) *The Jacobi coefficients of ν_λ are given by*

$$
\omega_1 = \omega_2 = \cdots = \lambda, \qquad \alpha_1 = \lambda, \quad \alpha_2 = \alpha_3 = \cdots = \lambda + 1. \tag{10.14}
$$

PROOF. (1) follows from (2) since mean $(\nu_\lambda) = \alpha 1$ and var $(\nu_\lambda) = \omega 1$.
(2) Note that

$$(B^+ + \sqrt{\lambda})(B^- + \sqrt{\lambda}) = \sqrt{\lambda}\, B^+ + \sqrt{\lambda}\, B^- + (\lambda + B^+B^-).$$

Since

$$\sqrt{\lambda}\, B^+\Phi_n = \sqrt{\lambda}\, \Phi_{n+1}\,, \qquad n \geq 0,$$

we obtain $\omega_1 = \omega_2 = \cdots = \lambda$. Similarly, from

$$(\lambda + B^+B^-)\Phi_0 = \lambda\Phi_0, \qquad (\lambda + B^+B^-)\Phi_n = (\lambda+1)\Phi_n \quad (n \geq 1)$$

we see that $\alpha_1 = \lambda$ and $\alpha_2 = \alpha_3 = \cdots = \lambda + 1$. ∎

Comparing (10.12) and (10.14), we claim the following

Theorem 10.5.5 *For $\gamma \neq 0$, μ_γ is obtained from the free Poisson distribution ν_{1/γ^2} with parameter $1/\gamma^2$ by reflection and normalization. For $\gamma = 0$, μ_γ is the Wigner semicircle law.*

Remark 10.5.6 The density function of the free Poisson distribution is given explicitly. For $\lambda > 0$ we set

$$\rho_\lambda(x) = \begin{cases} \dfrac{\sqrt{4\lambda - (x-1-\lambda)^2}}{2\pi x}, & (1-\sqrt{\lambda})^2 \leq x \leq (1+\sqrt{\lambda})^2, \\ 0, & \text{otherwise.} \end{cases}$$

The free Poisson distribution with parameter λ is given by

$$\begin{cases} (1-\lambda)\delta_0 + \rho_\lambda(x)dx, & 0 < \lambda < 1, \\ \rho_\lambda(x)dx, & \lambda \geq 1. \end{cases}$$

Exercises 10

1. Prove that

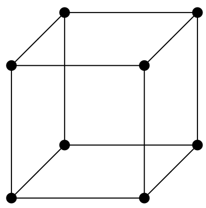
$$q[K_3] = \left(-\frac{1}{2}, 1\right), \qquad \tilde{q}[K_3] = \left[-\frac{1}{2}, 1\right].$$

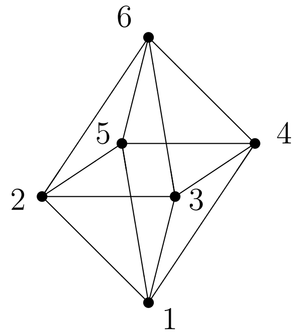
Then for a general complete graph K_n prove that

$$q[K_n] = \left(-\frac{1}{n-1}, 1\right), \qquad \tilde{q}[K_n] = \left[-\frac{1}{n-1}, 1\right]$$

2. Let G be a cube. Find $q[G]$ and $\tilde{q}[G]$.

3*. Let G be an octahedron. Find $q[G]$ and $\tilde{q}[G]$.





4. Let $T_\kappa = (V, E)$ be a homogeneous tree with a distinguished vertex $o \in V$. Let

$$V = \bigcup_{n=0}^{\infty} V_n\,, \qquad V_n = \{x \in V\,;\, \partial(x, o) = n\},$$

be the stratification.

(1) Find the cardinality $|V_n|$.

(2) Set

$$\Phi_n = \frac{1}{\sqrt{|V_n|}} \sum_{x \in V_n} e_x\,.$$

Then compute $\langle \Phi_n, Q\Phi_0 \rangle$.

5. Let $\langle \cdot \rangle_q$ be the Haagerup state on $\mathcal{A}(T_\kappa)$. Show that the mean and the variance of the adjacency matrix A are given by

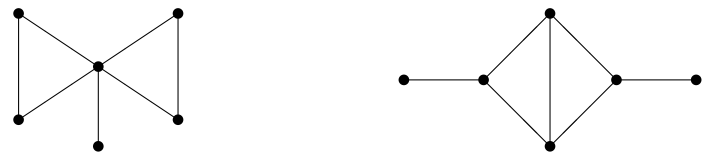
$$\langle A \rangle_q = \kappa q, \qquad \langle (A - \langle A \rangle_q)^2 \rangle_q = \kappa(1 - q^2).$$

6*. Let $(\Gamma, \{\Psi_n\}, B^+, B^-)$ be a free Fock space. Let Ω_z be a coherent vector, $z \in \mathbf{C}$ and P the vacuum projection. Show that

$$\langle \Omega_{\bar{z}}, (B^+ + B^-)^m \Psi_0 \rangle = \langle \Psi_0, (B^+ + B^- + zP)^m \Psi_0 \rangle,$$
$$\langle \Omega_{\bar{z}}, (B^+ + B^- - z)^m \Psi_0 \rangle = \langle \Psi_0, (B^+ + B^- - zB^+B^-)^m \Psi_0 \rangle.$$

Addendum: An Experimental Mathematics

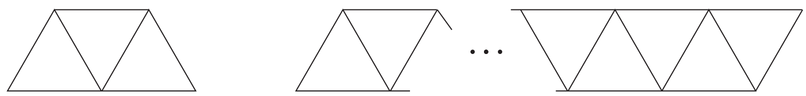
1. The following pair of graphs have the same spectra. Find the positivity regions $q[G]$ and $\tilde{q}[G]$.



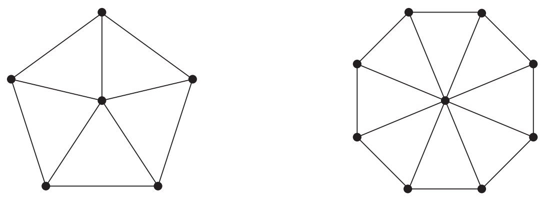
2. (1) It is desirable to find the positivity regions $q[G]$ and $\tilde{q}[G]$.

(2) If (1) is difficult, it would be interesting, as an easier question, to determine the region of q such that $\det Q > 0$ or $\det Q \geq 0$.

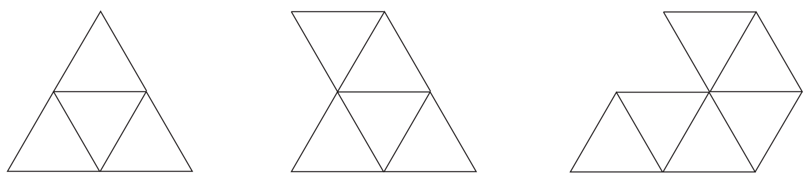
(a) sequence of triangles



(b) polygons with center, as a blocks of triangles



(c) block of triangles



(d) block of squares