# 2017 年度後期 <br> 確率モデル論（情報科学研究科） <br> 応用解析学（工学研究科） <br> 確率モデル論（国際高等研究教育院） 

## －授業科目の目的•概要及び達成目標等

確率モデルはランダム性を伴う現象の数理解析に欠かせない。講義では，時間発展するラ ンダム現象のモデルとしてマルコフ連鎖を扱う。確率論の基礎（確率変数•確率分布など） から始めて，マルコフ連鎖に関わる諸概念（推移確率•再帰性•定常分布など）を学ぶ，関連して，ランダムウォーク・出生死亡過程・ポアソン過程なども取り上げて，それらの幅広 い応用を概観する．なお，学部初年級の確率統計の知識を前提とする。

Probability models are essential in mathematical analysis of random phenomena．In these lec－ tures，we focus on Markov chains as basic models of random time evolution．Starting with fundamental concepts in probability theory（random variables，probability distributions，etc．）， we study fundamentals on Markov chains（transition probability，recurrence，stationary distri－ butions，etc．）．Moreover，we overview random walks，birth－and－death processes and Poisson processes，and their wide applications．Background knowledge on elementary probability is required．

## Basic References

拙著：確率モデル要論，牧野書店， 2012 ．
The lectures will follow this book．In fact，it is written on the basis of the lectures at GSIS in the past years．The table of contents is shown below．Some selected topics will be lectured．

1．Probability space and random variables
2．Analytic theory of probability distributions
3．Independence and dependence of random variables
4．Limit theorems
5．Random walks
6．Markov chains
7．Counting processes
8．Use of orthogonal polynomials

## －Examination

1．You will be required，at the end of January，to submit answers to the problems given during the lectures．The details later（probably at the end of December）．
2．No exceptions are made．

## －Résumé

1．A simple résumé of each lecture is distributed in the classroom．It is also available on the website （www．math．is．tohoku．ac．jp／obata）．

2．There are some relevant materials thereon．

## －Further Reading

1．D．L．Minh：Applied Probability Models，Duxbury， 2001.
This is concise and friendly．
2．S．M．Ross：Introduction to Probability Models，11th Ed．Academic Press， 2014.
This book has been read for many years since 1972．Very bulky，nealy 800 pages．The contents overlap with those of［Minh］but are more in detail．

3．J．R．Norris：Markov Chains，Cambridge UP， 1998.
An established textbook containing various concrete examples．
4．W．Feller：An Introduction to Probability Theory and Its Applications，Vol．1，Wiley， 1957.
A masterpice．This book containing many important topics covers fully our lectures．The Vol． 2 deals with more advanced topics．The japansese translation is available．

W．フェラー（河田龍夫他訳）：確率論とその応用（紀伊国屋）。こちらは 4 分冊．
5．B．V．Gnedenko：The Theory of Probability and the Elements of Statistics，AMS Chelsea Publishing Co．，6th ed． 1989.
6．R．Durrett：Probability：Theory and Examples，Duxbury Press， 1996.
Each of these two volumes，providing basic knowledge on probability models，covers also our lectures．
7．志賀徳造：ルベーグ積分から確率論（共立）， 2000.
The first part is devoted to Lebesgue integration，and the second part contains topics in random walks．
8．R．B．シナジ（今野紀雄•林俊一訳）：マルコフ連鎖から格子確率モデルヘ，シュプリンガー東京， 1999.

A handy introduction to Markov chains including the Galton－Watson processes and birth－and－death processes．

9．国沢清典：確率論とその応用（岩波全書）， 1982 ．
We aim at the level of this book．
10．舟木直久：確率論，朝倉書店， 2004 。
11．西尾真喜子：確率論，実教出版， 1978 ．
More strong mathematical flavor．
12．P．ブレモー（向井久訳）：モデルで学ぶ確率入門（新装版），シュプリンガー東京， 2004 ．
Contains many concrete applications．It seems，however，to be difficult to follow without the basic knowledge of probability theory．
13．F．Spitzer：Principles of Random Walk，Springer，2nd Ed．， 1976.
14．K．L．Chung：Markov Chains，Springer， 1960.
These are classics for professionals．
15．イアン・ハッキング（石原英樹•重田園江訳）：偶然を飼いならす，木鐸社， 1999 。
「この博物誌的な書物を好奇心に満ちたすべての読者に捧げる」とある。確率統計が 20 世紀の科学に中でいかに成功してきたかを科学史的な視点で論ずる。かなり興味深い。

16．イアン・ハッキング（広田すみれ・森元良太訳）：確率の出現，慶應義塾大学出版会， 2013.
17．キース・デブリン（原 啓介訳）：世界を変えた手紙—パスカル，フェルマーと〈確率〉の誕生，岩波書店， 2010 ．
確率論の始まりをさまざまなエピソードとともに語る．個人的にはカルダーノに大変興味がある．

Contents

## 1 Probability Spaces and Random Variables

### 1.1 Probability spaces

$\Omega$ : sample space consisting of elementary events (or sample points).
$\mathcal{F}$ : the set of events
$P$ : probability

### 1.2 Kolmogorov's axioms

Definition 1.2.1 (probability space) $(\Omega, \mathcal{F}, P)$

### 1.3 Basic Properties

Theorem 1.3.1 Let $A_{1}, A_{2}, \ldots$ be a sequence of events.
(1) If $A_{1} \subset A_{2} \subset A_{3} \subset \cdots$, then $P\left(\bigcup_{n=1}^{\infty} A_{n}\right)=\lim _{n \rightarrow \infty} P\left(A_{n}\right)$.
(2) If $A_{1} \supset A_{2} \supset A_{3} \supset \cdots$, then $P\left(\bigcap_{n=1}^{\infty} A_{n}\right)=\lim _{n \rightarrow \infty} P\left(A_{n}\right)$.

### 1.4 Random variables and their probability distributions

### 1.4.1 Discrete random variables

A random variable $X$ is called discrete if the number of values that $X$ takes is finite or countably infinite. To be more precise, for a discrete random variable $X$ there exist a (finite or infinite) sequence of real numbers $a_{1}, a_{2}, \ldots$ and corresponding nonnegative numbers $p_{1}, p_{2}, \ldots$ such that

$$
P\left(X=a_{i}\right)=p_{i}, \quad p_{i} \geq 0, \quad \sum p_{i}=1 .
$$

In this case

$$
\mu_{X}(d x)=\sum_{i} p_{i} \delta_{a_{i}}(d x)=\sum_{i} p_{i} \delta\left(x-a_{i}\right) d x
$$

is called the (probability) distribution of $X$. Obviously,

$$
P(a \leq X \leq b)=\sum_{i: a \leq a_{i} \leq b} p_{i}
$$



Example 1.4.1 (coin toss) We set

$$
X= \begin{cases}1, & \text { heads } \\ 0, & \text { tails }\end{cases}
$$

Then

$$
P(X=1)=p, \quad P(X=0)=q=1-p
$$

For a fair coin we set $p=1 / 2$.
Example 1.4.2 (waiting time) Flip a fair coin repeatedly until we get the heads. Let $T$ be the number of coin tosses to get the first heads. (If the heads occurs at the first trial, we have $T=1$; If the tails occurs at the first trial and the heads at the second trial, we have $T=2$, and so on.)

$$
P(T=k)=(1-p)^{k-1} p, \quad k=1,2, \ldots
$$

### 1.4.2 Continuous random variables

A random variable $X$ is called continuous if $P(X=a)=0$ for all $a \in \mathbf{R}$. We understand intuitively that $X$ varies continuously.

If there exists a function $f(x)$ such that

$$
P(a \leq X \leq b)=\int_{a}^{b} f(x) d x, \quad a<b
$$

we say that $X$ admits a probability density function. Note that

$$
\int_{-\infty}^{+\infty} f(x) d x=1, \quad f(x) \geq 0
$$

In this case,

$$
\mu_{X}(d x)=f(x) d x
$$

is called the (probability) distribution of $X$.


It is useful to consider the distribution function:

$$
F_{X}(x)=P(X \leq x)=\int_{-\infty}^{x} f_{X}(t) d t, \quad x \in \mathbf{R}
$$

Then we have

$$
f_{X}(x)=\frac{d}{d x} F_{X}(x)
$$

Remark 1.4.3 (1) A continuous random variable does not necessarily admit a probability density function. But many continuous random variables in practical applications admit probability density functions.
(2) There is a random variable which is neither discrete nor continuous. But most random variables in practical applications are either discrete or continuous.

Example 1.4.4 (random cut) Divide the interval $[0, L](L>0)$ into two segments.
(1) Let $X$ be the coordinate of the cutting point (the length of the segment containing 0 ).

$$
F_{X}(x)=\left\{\begin{array}{ll}
0, & x<0, \\
x / L, & 0 \leq x \leq L, \\
1, & x>L,
\end{array} \quad f_{X}(x)=\right.
$$

(2) Let $M$ be the length of the longer segment.

$$
F_{M}(x)=\left\{\begin{array}{ll}
0, & x<L / 2 \\
(2 x-L) / L, & L / 2 \leq x \leq L, \\
1, & x>L
\end{array} \quad f_{M}(x)=\right.
$$

Example 1.4.5 Let $A$ be a randomly chosen point from the disc with radius $R>0$. Let $X$ be the distance between the center $O$ and $A$. We have

$$
P(a \leq X \leq b)=\frac{\pi\left(b^{2}-a^{2}\right)}{\pi R^{2}}=\frac{1}{R^{2}} \int_{a}^{b} 2 x d x, \quad 0<a<b<R
$$

so the probability density function is given by

$$
f(x)= \begin{cases}\frac{2 x}{R^{2}}, & 0 \leq x \leq R \\ 0, & \text { otherwise }\end{cases}
$$

### 1.5 Mean values and variances

Definition 1.5.1 The mean or expectation value of a random variable $X$ is defined by

$$
m=\mathbf{E}[X]=\int_{-\infty}^{+\infty} x \mu_{X}(d x)
$$

- If $X$ is discrete, we have

$$
\mathbf{E}[X]=\sum_{i} a_{i} p_{i}
$$

- If $X$ admits a probability density function $f(x)$, we have

$$
\mathbf{E}[X]=\int_{-\infty}^{+\infty} x f(x) d x
$$

Remark 1.5.2 For a function $\varphi(x)$ we have

$$
\mathbf{E}[\varphi(X)]=\int_{-\infty}^{+\infty} \varphi(x) \mu(d x)
$$

For example,

$$
\begin{aligned}
& \mathbf{E}\left[X^{m}\right]=\int_{-\infty}^{+\infty} x^{m} \mu(d x) \quad(m \text { th moment }) \\
& \mathbf{E}\left[e^{i t X}\right]=\int_{-\infty}^{+\infty} e^{i t x} \mu(d x) \quad \text { (characteristic function). }
\end{aligned}
$$

Definition 1.5.3 The variance of a random variable $X$ is defined by

$$
\sigma^{2}=\mathbf{V}[X]=\mathbf{E}\left[(X-\mathbf{E}[X])^{2}\right]=\mathbf{E}\left[X^{2}\right]-\mathbf{E}[X]^{2}
$$

or equivalently,

$$
\sigma^{2}=\mathbf{V}[X]=\int_{-\infty}^{+\infty}(x-\mathbf{E}[X])^{2} \mu(d x)=\int_{-\infty}^{+\infty} x^{2} \mu(d x)-\left(\int_{-\infty}^{+\infty} x \mu(d x)\right)^{2}
$$

Example 1.5.4 The mean value and variance of the waiting time $T$ introduced in Example 1.4.2.
Example 1.5.5 The mean value and variance of the random variables $X$ and $M$ introduced in Example 1.4.4.

### 1.6 Stochastic processes

We will study the probability models for time evolution of random phenomena. Measuring a certain quantity of the random phenomenon at each time step $n=0,1,2, \ldots$, we obtain a sequence of real values:

$$
x_{0}, x_{1}, x_{2}, \ldots, x_{n}, \ldots
$$

Because of randomness, we consider $x_{n}$ as a realized value of a random variable $X_{n}$. Here a random variable is a variable taking several different values with certain probabilities. Thus, the time evolution of a random phenomenon is modeled by a sequence of random variables

$$
\left\{X_{n} ; n=0,1,2, \ldots\right\}=\left\{X_{0}, X_{1}, X_{2}, \ldots, X_{n}, \ldots\right\}
$$

which is called a discrete-time stochastic process. If the measurement is performed along with continuous time, we need a continuous-time stochastic process:

$$
\left\{X_{t} ; t \geq 0\right\}
$$

It is our purpose to construct stochastic processes modeling typical random phenomena and to demonstrate their properties within the framework of modern probability theory.



Figure 1.1: Solar spots; Nominal exchange rate (red) and real effective exchange rate (blue)

## 2 Probability Distributions

### 2.1 One-dimensional distributions

$F_{X}(x)=P(X \leq x)$ : distribution function

### 2.2 Discrete distributions

### 2.2.1 Bernoulli distribution

For $0 \leq p \leq 1$ the distribution

$$
(1-p) \delta_{0}+p \delta_{1}
$$

is called Bernoulli distribution with success probability $p$. This is the distribution of coin toss. The mean value and variance are given by

$$
m=p, \quad \sigma^{2}=p(1-p)
$$

### 2.2.2 Binomial distribution $B(n, p)$

For $0 \leq p \leq 1$ and $n \geq 1$ the distribution

$$
\sum_{k=0}^{n}\binom{n}{k} p^{k}(1-p)^{n-k} \delta_{k}
$$

is called the binomial distribution $B(n, p)$. The quantity $\binom{n}{k} p^{k}(1-p)^{n-k}$ is the probability that $n$ coin tosses with probabilities $p$ for heads and $q=1-p$ for tails result in $k$ heads and $n-k$ tails. The mean value and variance of $B(p, n)$ are given by

$$
m=n p, \quad \sigma^{2}=n p(1-p)
$$



Figure 2.1: $B(20,0.4)$ and geometric distribution with parameter $p=0.4$

### 2.2.3 Geometric distribution

For $0 \leq p \leq 1$ the distribution

$$
\sum_{k=1}^{\infty} p(1-p)^{k-1} \delta_{k}
$$

is called the geometric distribution with success probability $p$. This is the distribution of waiting time for the first heads (Example 1.4.2). The mean value and variance are given by

$$
m=\frac{1}{p}, \quad \sigma^{2}=\frac{1}{p^{2}} .
$$

Remark 2.2.1 In some literatures, the geometric distribution with parameter $p$ is defined by

$$
\sum_{k=0}^{\infty} p(1-p)^{k} \delta_{k}
$$

### 2.2.4 Poisson distribution

For $\lambda>0$ the distribution

$$
\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^{k}}{k!} \delta_{k}
$$

is called the Poisson distribution with parameter $\lambda$. The mean and variance are given by

$$
m=\lambda, \quad \sigma^{2}=\lambda
$$



Figure 2.2: Poisson distribution $\lambda=1 / 2,1,3$

Problem 1 The probability generating function of the Poisson distribution is defined by

$$
G(z)=\sum_{k=0}^{\infty} p_{k} z^{k}, \quad p_{k}=e^{-\lambda} \frac{\lambda^{k}}{k!} .
$$

(1) Find a concise expression of $G(z)$.
(2) By using $G^{\prime}(1)$ and $G^{\prime \prime}(1)$ find the mean value and variance of the Poisson distribution with parameter $\lambda$.
(3) Show that the probability of taking even values is greater than that of odd values, i.e.,

$$
\sum_{k: \text { odd }} p_{k}<\sum_{k: \text { :even }} p_{k} .
$$

(4) [additional] Show another example of a probability generating function.

### 2.3 Continuous distributions and density functions

### 2.3.1 Uniform distribution

For a finite interval $[a, b]$,

$$
f(x)= \begin{cases}\frac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text { otherwise }\end{cases}
$$

becomes a density function, which determines the uniform distribution on $[a, b]$. The mean value and the variance are given by

$$
m=\int_{a}^{b} x \frac{d x}{b-a}=\frac{a+b}{2}, \quad \sigma^{2}=\int_{a}^{b} x^{2} \frac{d x}{b-a}-m^{2}=\frac{(b-a)^{2}}{12}
$$



Figure 2.3: Uniform distribution on $[a, b]$

### 2.3.2 Exponential distribution

The exponential distribution with parameter $\lambda>0$ is defined by the density function

$$
f(x)= \begin{cases}\lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text { otherwise }\end{cases}
$$

This is a model for waiting time (continuous time). The mean value and variance are given by

$$
m=\frac{1}{\lambda}, \quad \sigma^{2}=\frac{1}{\lambda^{2}}
$$



Figure 2.4: Exponential distribution with parameter $\lambda$

### 2.3.3 Normal distribution

For $m \in \mathbb{R}$ and sigma $>0$ we may check that

$$
f(x)=\frac{1}{\sqrt{2 \pi \sigma^{2}}} \exp \left\{-\frac{(x-m)^{2}}{2 \sigma^{2}}\right\}
$$

becomes a density function. The distribution defined by the above density function is called the normal distribution or Gaussian distribution and denoted by $N\left(m, \sigma^{2}\right)$. In particular, $N(0,1)$ is called the standard normal distribution or the standard Gaussian distribution.


The mean value and variance of $N\left(m, \sigma^{2}\right)$ are $m$ and $\sigma^{2}$, respectively. This is verified by explicit calculation of the integrals:

$$
\begin{aligned}
& \frac{1}{\sqrt{2 \pi \sigma^{2}}} \int_{-\infty}^{+\infty} x \exp \left\{-\frac{(x-m)^{2}}{2 \sigma^{2}}\right\} d x=m \\
& \frac{1}{\sqrt{2 \pi \sigma^{2}}} \int_{-\infty}^{+\infty}(x-m)^{2} \exp \left\{-\frac{(x-m)^{2}}{2 \sigma^{2}}\right\} d x=\sigma^{2}
\end{aligned}
$$

Use the famous integral formula:

$$
\int_{0}^{+\infty} e^{-t x^{2}} d x=\frac{\sqrt{\pi}}{2 \sqrt{t}}, \quad t>0
$$

which is a standard exercise of double integrals in the first year course of calculus.
Problem 2 Choose randomly a point $A$ from the disc with radius one and let $X$ be the radius of the inscribed circle with center $A$.
(1) For $x \geq 0$ find the probability $P(X \leq x)$.
(2) Find the probability density function $f_{X}(x)$ of $X$. (Note that $x$ varies over all real numbers.)
(3) Calculate the mean and variance of $X$.
(4) Calculate the mean and variance of the area of inscribed circle $S=\pi X^{2}$.
(5) [additional] Discuss similar questions for a ball.


## 3 Independence and Dependence

### 3.1 Independent events and conditional probability

Definition 3.1.1 (Pairwise independence) A (finite or infinite) sequence of events $A_{1}, A_{2}, \ldots$ is called pairwise independent if any pair of events $A_{i_{1}}, A_{i_{2}}\left(i_{1} \neq i_{2}\right)$ verifies

$$
P\left(A_{i_{1}} \cap A_{i_{2}}\right)=P\left(A_{i_{1}}\right) P\left(A_{i_{2}}\right) .
$$

Definition 3.1.2 (Independence) A (finite or infinite) sequence of events $A_{1}, A_{2}, \ldots$ is called independent if any choice of finitely many events $A_{i_{1}}, \ldots, A_{i_{n}}\left(i_{1}<i_{2}<\cdots<i_{n}\right)$ satisfies

$$
P\left(A_{i_{1}} \cap A_{i_{2}} \cap \cdots \cap A_{i_{n}}\right)=P\left(A_{i_{1}}\right) P\left(A_{i_{2}}\right) \cdots P\left(A_{i_{n}}\right) .
$$

Example 3.1.3 Consider the trial to randomly draw a card from a deck of 52 cards. Let $A$ be the event that the result is an ace and $B$ the event that the result is spades. Then $A, B$ are independent.

Remark 3.1.4 It is allowed to consider whether the sequence of events $\{A, A\}$ is independent or not. If they are independent, by definition we have $P(A \cap A)=P(A) P(A)$, from which $P(A)=0$ or $P(A)=1$ follows. Notice that $P(A)=0$ does not imply $A=\emptyset$ (empty event). Similarly, $P(A)=1$ does not imply $A=\Omega$ (whole event).

Definition 3.1.5 (Conditional probability) For two events $A, B$ the conditional probability of $A$ relative to $B$ (or on the hypothesis $B$, or for given $B$ ) is defined by

$$
P(A \mid B)=\frac{P(A \cap B)}{P(B)} \quad \text { whenever } P(B)>0 .
$$

Theorem 3.1.6 Let $A, B$ be events with $P(A)>0$ and $P(B)>0$. Then,

$$
A, B \text { are independent } \Leftrightarrow P(A \mid B)=P(A) \quad \Leftrightarrow \quad P(B \mid A)=P(B)
$$

### 3.2 Independent random variables

Definition 3.2.1 A (finite or infinite) sequence of random variables $X_{1}, X_{2}, \ldots$ is independent (resp. pairwise independent) if so is the sequence of events $\left\{X_{1} \leq a_{1}\right\},\left\{X_{1} \leq a_{2}\right\}, \ldots$ for any $a_{1}, a_{2}, \cdots \in \mathbf{R}$.

In other words, a (finite or infinite) sequence of random variables $X_{1}, X_{2}, \ldots$ is independent if for any finite $X_{i_{1}}, \ldots, X_{i_{n}}\left(i_{1}<i_{2}<\cdots<i_{n}\right)$ and constant numbers $a_{1}, \ldots, a_{n}$

$$
\begin{equation*}
P\left(X_{i_{1}} \leq a_{1}, X_{i_{2}} \leq a_{2}, \ldots, X_{i_{n}} \leq a_{n}\right)=P\left(X_{i_{1}} \leq a_{1}\right) P\left(X_{i_{2}} \leq a_{2}\right) \cdots P\left(X_{i_{n}} \leq a_{n}\right) \tag{3.1}
\end{equation*}
$$

holds. Similar assertion holds for the pairwise independence. If random variables $X_{1}, X_{2}, \ldots$ are discrete, (3.1) may be replaced with

$$
P\left(X_{i_{1}}=a_{1}, X_{i_{2}}=a_{2}, \ldots, X_{i_{n}}=a_{n}\right)=P\left(X_{i_{1}}=a_{1}\right) P\left(X_{i_{2}}=a_{2}\right) \cdots P\left(X_{i_{n}}=a_{n}\right) .
$$

Example 3.2.2 Choose at random a point from the rectangle $\Omega=\{(x, y) ; a \leq x \leq b, c \leq y \leq d\}$. Let $X$ denote the $x$-coordinates of the chosen point and $Y$ the $y$-coordinates. Then $X, Y$ are independent.

Problem 3 (1) An urn contains four balls with numbers 112, 121, 211, 222. We draw a ball at random and let $X_{1}$ be the first digit, $X_{2}$ the second digit, and $X_{3}$ the last digit. For $i=1,2,3$ we define an event $A_{i}$ by $A_{i}=\left\{X_{i}=1\right\}$. Show that $\left\{A_{1}, A_{2}, A_{3}\right\}$ is pairwise independent but is not independent.
(2) Two dice are tossed. Let $A$ be the event that the first die gives a $4, B$ be the event that the sum is 6, and $C$ be the event that the sum is 7. Calculate $P(B \mid A)$ and $P(C \mid A)$, and study the independence among $\{A, B, C\}$.

### 3.3 Bernoulli trials

This is a model of coin-toss and is the most fundamental stochastic process. A sequence of random variables (or a discrete-time stochastic process) $\left\{X_{1}, X_{2}, \ldots, X_{n}, \ldots\right\}$ is called the Bernoulli trials with success probability $p(0 \leq p \leq 1)$ if they are independent and have the same distribution as

$$
P\left(X_{n}=1\right)=p, \quad P\left(X_{n}=0\right)=q=1-p
$$

By definition we have

$$
P\left(X_{1}=\xi_{1}, X_{2}=\xi_{2}, \ldots, X_{n}=\xi_{n}\right)=\prod_{k=1}^{n} P\left(X_{k}=\xi_{k}\right) \quad \text { for all } \xi_{1}, \xi_{2}, \ldots, \xi_{n} \in\{0,1\} .
$$

In general, statistical quantity in the left-hand side is called the finite dimensional distribution of the stochastic process $\left\{X_{n}\right\}$. The total set of finite dimensional distributions characterizes a stochastic process.

### 3.4 Covariance and correlation coefficients

Recall that the mean of a random variable $X$ is defined by

$$
m_{X}=\mathbf{E}(X)=\int_{-\infty}^{+\infty} x \mu_{X}(d x)
$$

Theorem 3.4.1 (Linearity) For two random variables $X, Y$ and two constant numbers $a, b$ it holds that

$$
\mathbf{E}(a X+b Y)=a \mathbf{E}(X)+b \mathbf{E}(Y)
$$

Theorem 3.4.2 (Multiplicativity) If random variables $X_{1}, X_{2}, \ldots, X_{n}$ are independent, we have

$$
\begin{equation*}
\mathbf{E}\left[X_{1} X_{2} \cdots X_{n}\right]=\mathbf{E}\left[X_{1}\right] \cdots \mathbf{E}\left[X_{n}\right] . \tag{3.2}
\end{equation*}
$$

Proof We first prove the assertion for $X_{k}=1_{A_{k}}$ (indicator random variable). By definition $X_{1}, \ldots, X_{n}$ are independent if and only if so are $A_{1}, \ldots, A_{n}$. Therefore,

$$
\begin{aligned}
\mathbf{E}\left[X_{1} \cdots X_{n}\right] & =\mathbf{E}\left[1_{A_{1} \cap \cdots \cap A_{n}}\right]=P\left(A_{1} \cap \cdots \cap A_{n}\right) \\
& =P\left(A_{1}\right) \cdots P\left(A_{n}\right)=\mathbf{E}\left[X_{1}\right] \cdots \mathbf{E}\left[X_{n}\right] .
\end{aligned}
$$

Thus (3.2) is verified. Then, by linearity the assertion is valid for $X_{k}$ taking finitely many values (finite linear combination of indicator random variables). Finally, for general $X_{k}$, coming back to the definition of Lebesgue integration, we can prove the assertion by approximation argument.

Remark 3.4.3 $\mathbf{E}[X Y]=\mathbf{E}[X] \mathbf{E}[Y]$ is not a sufficient condition for the random variables $X$ and $Y$ being independent. It is merely a necessary condition!

The variance of $X$ is defined by

$$
\sigma_{X}^{2}=\mathbf{V}(X)=\mathbf{E}\left[\left(X-m_{X}\right)^{2}\right]=\mathbf{E}\left[X^{2}\right]-\mathbf{E}[X]^{2}
$$

By means of the distribution $\mu(d x)$ of $X$ we may write

$$
\mathbf{V}(X)=\int_{-\infty}^{+\infty}\left(x-m_{X}\right)^{2} \mu(d x)=\int_{-\infty}^{+\infty} x^{2} \mu(d x)-\left(\int_{-\infty}^{+\infty} x \mu(d x)\right)^{2}
$$

Definition 3.4.4 The covariance of two random variables $X, Y$ is defined by

$$
\operatorname{Cov}(X, Y)=\sigma_{X Y}=\mathbf{E}[(X-\mathbf{E}(X))(Y-\mathbf{E}(Y))]=\mathbf{E}[X Y]-\mathbf{E}[X] \mathbf{E}[Y]
$$

In particular, $\sigma_{X X}=\sigma_{X}^{2}$ becomes the variance of $X$. The correlation coefficient of two random variables $X, Y$ is defined by

$$
\rho_{X Y}=\frac{\sigma_{X Y}}{\sigma_{X} \sigma_{Y}}
$$

whenever $\sigma_{X}>0$ and $\sigma_{Y}>0$.
Definition 3.4.5 $X, Y$ are called uncorrelated if $\sigma_{X Y}=0$. They are called positively (resp. negatively) correlated if $\sigma_{X Y}>0\left(\right.$ resp. $\left.\sigma_{X Y}<0\right)$.

Theorem 3.4.6 If two random variables $X, Y$ are independent, they are uncorrelated.
Remark 3.4.7 The converse of Theorem 3.4.6 is not true in general. Let $X$ be a random variable satisfying

$$
P(X=-1)=P(X=1)=\frac{1}{4}, \quad P(X=0)=\frac{1}{2}
$$

and set $Y=X^{2}$. Then, $X, Y$ are not independent, but $\sigma_{X Y}=0$. On the other hand, for random variables $X, Y$ taking only two values, the converse of Theorem 3.4.6 is valid (see Problem 5).

Theorem 3.4.8 (Additivity of variance) Let $X_{1}, X_{2}, \ldots, X_{n}$ be random variables, any pair of which is uncorrelated. Then

$$
\mathbf{V}\left[\sum_{k=1}^{n} X_{k}\right]=\sum_{k=1}^{n} \mathbf{V}\left[X_{k}\right] .
$$

Theorem 3.4.9 $-1 \leq \rho_{X Y} \leq 1$ for two random variables $X, Y$ with $\sigma_{X}>0, \sigma_{Y}>0$.
Proof Note that $\mathbf{E}\left[\left\{t\left(X-m_{X}\right)+\left(Y-m_{Y}\right)\right\}^{2}\right] \geq 0$ for all $t \in \mathbf{R}$.

Problem 4 Throw two dice and let $L$ be the larger spot and $S$ the smaller. (If double spots, set $L=S$.)
(1) Show the joint probability of $(L, S)$ by a table.
(2) Calculate the correlation coefficient $\rho_{L S}$ and explain the meaning of the signature of $\rho_{L S}$.

Problem 5 Let $X$ and $Y$ be random variables such that

$$
P(X=a)=p_{1}, \quad P(X=b)=q_{1}=1-p_{1}, \quad P(Y=c)=p_{2}, \quad P(Y=d)=q_{2}=1-p_{2}
$$

where $a, b, c, d$ are constant numbers and $0<p_{1}<1,0<p_{2}<1$. Show that $X, Y$ are independent if and only if $\sigma_{X Y}=0$. Explain the significance of this case. [Hint: In general, uncorrelated random variables are not necessarily independent.]

### 3.5 Convolutions of probability distributions

Perhaps omitted.

## 4 Limit Theorems

### 4.1 Simulation of Coin Toss

Let $\left\{X_{n}\right\}$ be a Bernoulli trial with success probability $1 / 2$, namely, tossing a fair coin, and consider the binomial process defined by

$$
S_{n}=\sum_{k=1}^{n} X_{k}
$$

Since $S_{n}$ counts the number of heads during the first $n$ trials,

$$
\frac{S_{n}}{n}=\frac{1}{n} \sum_{k=1}^{n} X_{k}
$$

gives the relative frequency of heads during the first $n$ trials.
The following is just one example showing that the relative frequency of heads $S_{n} / n$ tends to $1 / 2$. It is our question how to describe this phenomenon mathematically. A naive formula:

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{S_{n}}{n}=\frac{1}{2} \tag{4.1}
\end{equation*}
$$

is not acceptable. Why?


Figure 4.1: Relative frequency of heads $S_{n} / n$

### 4.2 Law of Large Numbers (LLN)

Theorem 4.2.1 (Weak law of large numbers) Let $X_{1}, X_{2}, \ldots$ be identically distributed random variables with mean $m$ and variance $\sigma^{2}$. (This means that $X_{i}$ has a finite variance.) If $X_{1}, X_{2}, \ldots$ are uncorrelated, for any $\epsilon>0$ we have

$$
\lim _{n \rightarrow \infty} P\left(\left|\frac{1}{n} \sum_{k=1}^{n} X_{k}-m\right| \geq \epsilon\right)=0
$$

We say that $\frac{1}{n} \sum_{k=1}^{n} X_{k}$ converges to $m$ in probability.
Remark 4.2.2 In many literatures the weak law of large numbers is stated under the assumption that $X_{1}, X_{2}, \ldots$ are independent. It is noticeable that the same result holds under the weaker assumption of being uncorrelated.

Theorem 4.2.3 (Chebyshev inequality) Let $X$ be a random variable with mean $m$ and variance $\sigma^{2}$. Then, for any $\epsilon>0$ we have

$$
P(|X-m| \geq \epsilon) \leq \frac{\sigma^{2}}{\epsilon^{2}}
$$

Proof Set $A=\{|X-m| \geq \epsilon\}$ and let $1_{A}$ be the indicator random variable. Then we have

$$
\begin{aligned}
\sigma^{2} & =\mathbf{E}\left[(X-m)^{2}\right]=\mathbf{E}\left[(X-m)^{2} 1_{A}+(X-m)^{2} 1_{A^{c}}\right] \\
& \geq \mathbf{E}\left[(X-m)^{2} 1_{A}\right] \geq \mathbf{E}\left[\epsilon^{2} 1_{A}\right]=\epsilon^{2} P(A),
\end{aligned}
$$

where we used the obvious relation $\mathbf{E}\left[1_{A}\right]=P(A)$.
Proof [Theorem 4.2.1 (Weak Law of Large Numbers)] For simplicity we set

$$
Y=Y_{n}=\frac{1}{n} \sum_{k=1}^{n} X_{k}
$$

The mean value is given by

$$
\mathbf{E}[Y]=\frac{1}{n} \sum_{k=1}^{n} \mathbf{E}\left[X_{k}\right]=m
$$

Since $X_{1}, X_{2}, \ldots$ are pairwise uncorrelated, the variance is computed by using the additive property of variance. In fact, we have

$$
\mathbf{V}[Y]=\frac{1}{n^{2}} \mathbf{V}\left[\sum_{k=1}^{n} X_{k}\right]=\frac{1}{n^{2}} \sum_{k=1}^{n} \mathbf{V}\left[X_{k}\right]=\frac{1}{n^{2}} \times n \sigma^{2}=\frac{\sigma^{2}}{n}
$$

On the other hand, applying Chebyshev inequality, we have

$$
P(|Y-m| \geq \epsilon) \leq \frac{\mathbf{V}[Y]}{\epsilon^{2}}=\frac{\sigma^{2}}{n \epsilon^{2}}
$$

Consequently,

$$
\lim _{n \rightarrow \infty} P\left(\left|Y_{n}-m\right| \geq \epsilon\right)=0
$$

as desired.

## Example 4.2.4 (Coin toss)

Theorem 4.2.5 (Strong law of large numbers) Let $X_{1}, X_{2}, \ldots$ be identically distributed random variables with mean $m$. (This means that $X_{i}$ has a mean but is not assumed to have a finite variance.) If $X_{1}, X_{2}, \ldots$ are pairwise independent, we have

$$
P\left(\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} X_{k}=m\right)=1
$$

In other words,

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} X_{k}=m \quad \text { a.s. }
$$

Remark 4.2.6 Kolmogorov proved the strong law of large numbers under the assumption that $X_{1}, X_{2}, \ldots$ are independent. In many literatures, the strong law of large numbers is stated as Kolmogorov proved. Its proof being based on the so-called "Kolmogorov's almost sure convergence theorem," we cannot relax the assumption of independence. Theorem 4.2.5 is due to N. Etemadi (1981), where the assumption is relaxed to being pairwise independent and the proof is more elementary, see also books by Sato, by Durrett, etc.

### 4.3 Central Limit Theorem (CLT)

Theorem 4.3.1 (Central Limit Theorem) Let $Z_{1}, Z_{2}, \ldots$ be independent identically distributed (iid) random variables with mean 0 and variance 1 . Then, for any $x \in \mathbf{R}$ it holds that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} P\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} Z_{k} \leq x\right)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{x} e^{-t^{2} / 2} d t \tag{4.2}
\end{equation*}
$$

In short,

$$
\frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_{k} \quad \longrightarrow \quad N(0,1) \quad \text { weakly as } n \rightarrow \infty
$$

Proof is by characteristic functions (Fourier transform), see the textbooks.
Example 4.3.2 The de Moivre-Laplace theorem claims that

$$
\begin{equation*}
B(n, p) \approx N(n p, n p(1-p)) \tag{4.3}
\end{equation*}
$$



Figure 4.2: The normal distribution whose mean and variance are the same as $B(100,0.4)$

This is a special case of CLT. Let $\left\{X_{1}, X_{2}, \ldots\right\}$ be a Bernoulli trials with success probability $p$. Set

$$
Z_{k}=\frac{X_{k}-m}{\sigma}, \quad m=\mathbf{E}\left[X_{k}\right]=p, \quad \sigma^{2}=\mathbf{V}\left[X_{k}\right]=p(1-p)
$$

so that $\left\{Z_{k}\right\}$ are iid random variables with 0 and variance 1 . Apply the central limit theorem we have (4.2). For the left-hand side we see that

$$
\frac{1}{\sqrt{n}} \sum_{k=1}^{n} Z_{k}=\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{X_{k}-m}{\sigma}=\frac{1}{\sigma \sqrt{n}}\left(\sum_{k=1}^{n} X_{k}-n m\right)
$$

Then (4.2) becomes

$$
\lim _{n \rightarrow \infty} P\left(\sum_{k=1}^{n} X_{k} \leq n m+x \sigma \sqrt{n}\right)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{x} e^{-t^{2} / 2} d t
$$

Setting $y=n m+x \sigma \sqrt{n}$, we have

$$
P\left(\sum_{k=1}^{n} X_{k} \leq y\right) \approx \frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\frac{y-n m}{\sigma \sqrt{n}}} e^{-t^{2} / 2} d t=\frac{1}{\sqrt{2 \pi \sigma^{2}}} \int_{-\infty}^{y} e^{-\frac{(t-n m)^{2}}{2 n \sigma^{2}}} d t
$$

Thus, for a large $n$ we have

$$
\sum_{k=1}^{n} X_{k} \approx N\left(n m, n \sigma^{2}\right)=N(n p, n p(1-p))
$$

On the other hand, we know that $\sum_{k=1}^{n} X_{k}$ obeys $B(n, p)$, of which the mean value and variance are given by $n p$ and $n p(1-p)$. Consequently, for a large $n$ we have (4.3). The approximation (4.3) means that distribution functions are almost the same.

Problem 6 (Monte Carlo simulation) Let $f(x)$ be a continuous function on the interval $[0,1]$ and consider the integral

$$
\begin{equation*}
\int_{0}^{1} f(x) d x \tag{4.4}
\end{equation*}
$$

(1) Let $X$ be a random variable obeying the uniform distribution on [0,1]. Give expressions of the mean value $\mathbf{E}[f(X)]$ and variance $\mathbf{V}[f(X)]$ of the random variable $f(X)$.
(2) Let $x_{1}, x_{2}, \ldots$ is a sequence random numbers taken from $[0,1]$. Explain that the arithmetic mean

$$
\frac{1}{n} \sum_{k=1}^{n} f\left(x_{k}\right)
$$

is a good approximation of the integral (4.4) by means of law of large numbers and central limit theorem.
(3) By using a computer, verify the above fact for $f(x)=\sqrt{1-x^{2}}$.

## 5 Markov Chains

### 5.1 Conditional Probability

For two events $A, B$ the conditional probability of $A$ relative (subject) to $B$ is defined by

$$
\begin{equation*}
P(A \mid B)=\frac{P(A \cap B)}{P(B)}, \quad \text { whenever } P(B)>0 \tag{5.1}
\end{equation*}
$$

see Section 3.1. Formula (5.1) is often used in the following form:

$$
\begin{equation*}
P(A \cap B)=P(B) P(A \mid B) \tag{5.2}
\end{equation*}
$$

This is the so-called theorem on compound probabilities, giving a ground to the usage of tree diagram in computation of probability. For example, for two events $A, B$ see Fig. 5.1.


Figure 5.1: Tree diagram

Theorem 5.1.1 (Compound probabilities) For events $A_{1}, A_{2}, \ldots, A_{n}$ we have

$$
\begin{equation*}
P\left(A_{1} \cap A_{2} \cap \cdots \cap A_{n}\right)=P\left(A_{1}\right) P\left(A_{2} \mid A_{1}\right) P\left(A_{3} \mid A_{1} \cap A_{2}\right) \cdots P\left(A_{n} \mid A_{1} \cap A_{2} \cap \cdots \cap A_{n-1}\right) . \tag{5.3}
\end{equation*}
$$

Proof Straightforward by induction on $n$.

### 5.2 Markov Chains

Let $S$ be a finite or countable set. Consider a discrete time stochastic process $\left\{X_{n} ; n=0,1,2, \ldots\right\}$ taking values in $S$. This $S$ is called a state space and is not necessarily a subset of $\mathbb{R}$ in general. In the following we often meet the cases of $S=\{0,1\}, S=\{1,2, \ldots, N\}$ and $S=\{0,1,2, \ldots\}$.

Definition 5.2.1 Let $\left\{X_{n} ; n=0,1,2, \ldots\right\}$ be a discrete time stochastic process over $S$. It is called a Markov process over $S$ if

$$
P\left(X_{m}=j \mid X_{n_{1}}=i_{1}, X_{n_{2}}=i_{2}, \ldots, X_{n_{k}}=i_{k}, X_{n}=i\right)=P\left(X_{m}=j \mid X_{n}=i\right)
$$

holds for any $0 \leq n_{1}<n_{2}<\cdots<n_{k}<n<m$ and $i_{1}, i_{2}, \ldots, i_{k}, i, j \in S$.
If $\left\{X_{1}, X_{2}, \ldots\right\}$ are independent random variables with values in $S$, obviously they form a Markov chain. Hence the Markov property is weaker than independence.

Theorem 5.2.2 (multiplication rule) Let $\left\{X_{n}\right\}$ be a Markov chain on $S$. Then, for any $0 \leq n_{1}<n_{2}<\cdots<$ $n_{k}$ and $i_{1}, i_{2}, \ldots, i_{k} \in S$ we have

$$
\begin{aligned}
& P\left(X_{n_{1}}=i_{1}, X_{n_{2}}=i_{2}, \ldots, X_{n_{k}}=i_{k}\right) \\
& =P\left(X_{n_{1}}=i_{1}\right) P\left(X_{n_{2}}=i_{2} \mid X_{n_{1}}=i_{1}\right) P\left(X_{n_{3}}=i_{3} \mid X_{n_{2}}=i_{2}\right) \cdots P\left(X_{n_{k}}=i_{k} \mid X_{n_{k-1}}=i_{k-1}\right)
\end{aligned}
$$

Definition 5.2.3 For a Markov chain $\left\{X_{n}\right\}$ over $S$,

$$
P\left(X_{n+1}=j \mid X_{n}=i\right)
$$

is called the transition probability at time $n$ from a state $i$ to $j$. If this is independent of $n$, the Markov chain is called time homogeneous.

Hereafter a Markov chain is always assumed to be time homogeneous. In this case the transition probability is denoted by

$$
p_{i j}=p(i, j)=P\left(X_{n+1}=j \mid X_{n}=i\right)
$$

and the transition matrix is defined by $P=\left[p_{i j}\right]$
Definition 5.2.4 A matrix $P=\left[p_{i j}\right]$ with index set $S$ is called a stochastic matrix if

$$
p_{i j} \geq 0 \quad \text { and } \quad \sum_{j \in S} p_{i j}=1
$$

Theorem 5.2.5 The transition matrix of a Markov chain is a stochastic matrix. Conversely, given a stochastic matrix we can construct a Markov chain of which the transition matrix coincides with the given stochastic matrix.

Example 5.2.6 (2-state Markov chain) A Markov chain over the state space $\{0,1\}$ is determined by the transition probabilities:

$$
p(0,1)=p, \quad p(0,0)=1-p, \quad p(1,0)=q, \quad p(1,1)=1-q .
$$

The transition matrix is defined by

$$
\left[\begin{array}{cc}
1-p & p \\
q & 1-q
\end{array}\right]
$$

The transition diagram is as follows:


Example 5.2.7 (3-state Markov chain) An animal is healthy, sick or dead, and changes its state every day. Consider a Markov chain on $\{H, S, D\}$ described by the following transition diagram:


The transition matrix is defined by

$$
\left[\begin{array}{lll}
a & b & 0 \\
p & r & q \\
0 & 0 & 1
\end{array}\right], \quad a+b=1, \quad p+q+r=1
$$

Example 5.2.8 (Random walk on $\mathbb{Z}^{1}$ ) The random walk on $\mathbb{Z}^{1}$ is illustrated as


The transition probabilities are given by

$$
p(i, j)= \begin{cases}p, & \text { if } j=i+1 \\ q=1-p, & \text { if } j=i-1 \\ 0, & \text { otherwise }\end{cases}
$$

The transition matrix is a two-sided infinite matrix given by

$$
\left[\begin{array}{llllllll}
\ddots & \ddots & \ddots & \ddots & & & & \\
\ddots & q & 0 & p & 0 & & & \\
& 0 & q & 0 & p & 0 & & \\
& & 0 & q & 0 & p & 0 & \\
& & & 0 & q & 0 & p & \ddots \\
& & & & \ddots & \ddots & \ddots & \ddots
\end{array}\right]
$$

Example 5.2.9 (Random walk with absorbing barriers) Let $A>0$ and $B>0$. The state space of a random walk with absorbing barriers at $-A$ and $B$ is $S=\{-A,-A+1, \ldots, B-1, B\}$. Then the transition probabilities are given as follows. For $-A<i<B$,

$$
p(i, j)= \begin{cases}p, & \text { if } j=i+1 \\ q=1-p, & \text { if } j=i-1 \\ 0, & \text { otherwise }\end{cases}
$$

For $i=-A$ or $i=B$,

$$
p(-A, j)=\left\{\begin{array}{ll}
1, & \text { if } j=-A, \\
0, & \text { otherwise },
\end{array} \quad p(B, j)= \begin{cases}1, & \text { if } j=B \\
0, & \text { otherwise }\end{cases}\right.
$$

In a matrix form we have

$$
\left[\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
q & 0 & p & 0 & 0 & \cdots & 0 \\
0 & q & 0 & p & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & q & 0 & p & 0 \\
0 & 0 & \cdots & 0 & q & 0 & p \\
0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{array}\right]
$$



Example 5.2.10 (Random walk with reflecting barriers) Let $A>0$ and $B>0$. The state space of a random walk with absorbing barriers at $-A$ and $B$ is $S=\{-A,-A+1, \ldots, B-1, B\}$. The transition probabilities are given as follows. For $-A<i<B$,

$$
p(i, j)= \begin{cases}p, & \text { if } j=i+1 \\ q=1-p, & \text { if } j=i-1 \\ 0, & \text { otherwise }\end{cases}
$$

For $i=-A$ or $i=B$,

$$
p(-A, j)=\left\{\begin{array}{ll}
1, & \text { if } j=-A+1, \\
0, & \text { otherwise },
\end{array} \quad p(B, j)= \begin{cases}1, & \text { if } j=B-1, \\
0, & \text { otherwise }\end{cases}\right.
$$

In a matrix form we have

$$
\left[\begin{array}{ccccccc}
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
q & 0 & p & 0 & 0 & \cdots & 0 \\
0 & q & 0 & p & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & q & 0 & p & 0 \\
0 & 0 & \cdots & 0 & q & 0 & p \\
0 & 0 & \cdots & 0 & 0 & 1 & 0
\end{array}\right]
$$



### 5.3 Distribution of a Markov Chain

Let $S$ be a state space as before. In general, a row vector $\pi=\left[\cdots \pi_{i} \cdots\right]$ indexed by $S$ is called a distribution on $S$ if

$$
\begin{equation*}
\pi_{i} \geq 0 \quad \text { and } \quad \sum_{i \in S} \pi_{i}=1 \tag{5.4}
\end{equation*}
$$

For a Markov chain $\left\{X_{n}\right\}$ on $S$ we set

$$
\pi(n)=\left[\cdots \pi_{i}(n) \cdots\right], \quad \pi_{i}(n)=P\left(X_{n}=i\right)
$$

which becomes a distribution on $S$. We call $\pi(n)$ the distribution of $X_{n}$. In particular, $\pi(0)$, the distribution of $X_{0}$, is called the initial distribution. We often take

$$
\pi(0)=[\cdots 0,1,0, \cdots], \quad \text { where } 1 \text { occurs at } i \text { th posotion. }
$$

In this case the Markov chain $\left\{X_{n}\right\}$ starts from the state $i$.

For a Markov chain $\left\{X_{n}\right\}$ with a transition matrix $P=\left[p_{i j}\right]$ the $n$-step transition probability is defined by

$$
p_{n}(i, j)=P\left(X_{m+n}=j \mid X_{m}=i\right), \quad i, j \in S .
$$

The right-hand side is independent of $n$ because our Markov chain is assumed to be time homogeneous.

Theorem 5.3.1 (Chapman-Kolmogorov equation) For $0 \leq r \leq n$ we have

$$
\begin{equation*}
p_{n}(i, j)=\sum_{k \in S} p_{r}(i, k) p_{n-r}(k, j) \tag{5.5}
\end{equation*}
$$

Proof First we note the obvious identity:

$$
p_{n}(i, j)=P\left(X_{m+n}=j \mid X_{m}=i\right)=\sum_{k \in S} P\left(X_{m+n}=j, X_{m+r}=k \mid X_{m}=i\right)
$$

Moreover,

$$
\begin{aligned}
P\left(X_{m+n}=j, X_{m+r}=k \mid X_{m}=i\right) & =\frac{P\left(X_{m+n}=j, X_{m+r}=k, X_{m}=i\right)}{P\left(X_{m+r}=k, X_{m}=i\right)} \times \frac{P\left(X_{m+r}=k, X_{m}=i\right)}{P\left(X_{m}=i\right)} \\
& =P\left(X_{m+n}=j \mid X_{m+r}=k, X_{m}=i\right) P\left(X_{m+r}=k \mid X_{m}=i\right) .
\end{aligned}
$$

Using the Markov property, we have

$$
P\left(X_{m+n}=j \mid X_{m+r}=k, X_{m}=i\right)=P\left(X_{m+n}=j \mid X_{m+r}=k\right)
$$

so that

$$
P\left(X_{m+n}=j, X_{m+r}=k \mid X_{m}=i\right)=P\left(X_{m+n}=j \mid X_{m+r}=k\right) P\left(X_{m+r}=k \mid X_{m}=i\right) .
$$

Finally, by the property of being time homogeneous, we come to

$$
P\left(X_{m+n}=j, X_{m+r}=k \mid X_{m}=i\right)=p_{n-r}(k, j) p_{r}(i, k) .
$$

Thus we have obtained (5.5).
Applying (5.5) repeatedly and noting that $p_{1}(i, j)=p(i, j)$, we obtain

$$
\begin{equation*}
p_{n}(i, j)=\sum_{k_{1}, \ldots, k_{n-1} \in S} p\left(i, k_{1}\right) p\left(k_{1}, k_{2}\right) \cdots p\left(k_{n-1}, j\right) . \tag{5.6}
\end{equation*}
$$

The right-hand side is nothing else but the multiplication of matrices, i.e., the $n$-step transition probability $p_{n}(i, j)$ is the $(i, j)$-entry of the $n$-power of the transition matrix $P$. Summing up, we obtain the following important result.

Theorem 5.3.2 For $m, n \geq 0$ and $i, j \in S$ we have

$$
P\left(X_{m+n}=j \mid X_{m}=i\right)=p_{n}(i, j)=\left(P^{n}\right)_{i j}
$$

Proof Immediate from Theorem 5.3.1.

Remark 5.3.3 As a result, the Chapman-Kolmogorov equation is nothing else but an entrywise expression of the obvious relation for the transition matrix:

$$
P^{n}=P^{r} P^{n-r}
$$

(As usual, $P^{0}=E$ (identity matrix).)

Theorem 5.3.4 We have

$$
\pi(n)=\pi(n-1) P, \quad n \geq 1
$$

or equivalently,

$$
\pi_{j}(n)=\sum_{i} \pi_{i}(n-1) p_{i j} .
$$

Therefore,

$$
\pi(n)=\pi(0) P^{n} .
$$

Proof We first note that

$$
\pi_{j}(n)=P\left(X_{n}=j\right)=\sum_{i \in S} P\left(X_{n}=j \mid X_{n-1}=i\right) P\left(X_{n-1}=i\right)=\sum_{i \in S} p_{i j} \pi_{i}(n-1),
$$

which proves $\pi(n)=\pi(n-1) P$. By repeated application we have

$$
\pi(n)=\pi(n-1) P=(\pi(n-2) P) P=\left(\pi(n-2) P^{2}=\cdots=\pi(0) P^{n},\right.
$$

as desired.
Example 5.3.5 (2-state Markov chain) Let $\left\{X_{n}\right\}$ be the Markov chain introduced in Example 5.2.6. The eigenvalues of the transition matrix

$$
P=\left[\begin{array}{cc}
1-p & p \\
q & 1-q
\end{array}\right] .
$$

are $1,1-p-q$. These are distinct if $p+q>0$. Omitting the case of $p+q=0$, i.e., $p=q=0$, we assume that $p+q>0$. By standard argument we obtain

$$
P^{n}=\frac{1}{p+q}\left[\begin{array}{ll}
q+p r^{n} & p-p r^{n} \\
q-q r^{n} & p+q r^{n}
\end{array}\right], \quad r=1-p-q .
$$

Let $\pi(0)=\left[\pi_{0}(0) \pi_{1}(0)\right]$ be the distriution of $X_{0}$. Then the distribution of $X_{n}$ is given by

$$
\pi(n)=\left[P\left(X_{n}=0\right), P\left(X_{n}=1\right)\right]=\left[\pi_{0}(0) \pi_{1}(0)\right] P^{n}=\pi(0) P^{n} .
$$

Problem 7 There are two parties, say, A and B, and their supporters of a constant ratio exchange at every election. Suppose that just before an election, $25 \%$ of the supporters of A change to support B and $20 \%$ of the supporters of B change to support A. At the beginning, $85 \%$ of the voters support A and $15 \%$ support B.
(1) When will the party B command a majority?
(2) Find the final ratio of supporters after many elections if the same situation continues.
(3) (optional) Discuss relevant topics at your own choice.

Problem 8 Study the $n$-step transition probability of the three-state Markov chain introduced in Example 5.2.7. Explain that every animal dies within finite time.

Problem 9 Let $\left\{X_{n}\right\}$ be a Markov chain on $\{0,1\}$ given by the transition matrix $P=\left[\begin{array}{cc}1-p & p \\ q & 1-q\end{array}\right]$ with the initial distribution $\pi_{0}=\left[\frac{q}{p+q}, \frac{p}{p+q}\right]$. Calculate the following statistical quantities:

$$
\mathbf{E}\left[X_{n}\right], \quad \mathbf{V}\left[X_{n}\right], \quad \operatorname{Cov}\left(X_{m+n}, X_{n}\right)=\mathbf{E}\left[X_{m+n} X_{n}\right]-\mathbf{E}\left[X_{m+n}\right] \mathbf{E}\left[X_{n}\right], \quad \rho\left(X_{m+n}, X_{n}\right)=\frac{\operatorname{Cov}\left(X_{m+n}, X_{n}\right)}{\sqrt{\mathbf{V}\left[X_{m+n}\right] \mathbf{V}\left[X_{n}\right]}}
$$

## 6 Stationary Distributions

### 6.1 Definition and Examples

Definition 6.1.1 Let $\left\{X_{n}\right\}$ be a Markov chain on $S$ with transition probability matrix $P$. A distribution $\pi$ on $S$ is called stationary (or invariant) if

$$
\begin{equation*}
\pi=\pi P, \tag{6.1}
\end{equation*}
$$

or equivalently if

$$
\begin{equation*}
\pi_{j}=\sum_{i \in S} \pi_{i} p_{i j}, \quad j \in S . \tag{6.2}
\end{equation*}
$$

Thus, in order to find a stationary distribution of a Markov chain with a transition probability matrix $P=\left[p_{i j}\right]$, we need to solve the linear system (6.1) (or equivalently (6.2)) together with the conditions:

$$
\sum_{i} \pi_{i}=1 \quad \text { and } \pi_{i} \geq 0 \text { for all } i \in S
$$

Example 6.1.2 (2-state Markov chain) Consider the transition matrix:

$$
P=\left[\begin{array}{cc}
1-p & p \\
q & 1-q
\end{array}\right] .
$$

Solving the equation $\pi P=\pi, \pi=\left[\pi_{0} \pi_{1}\right]$, which is equivalent to $p \pi_{0}-q \pi_{1}=0$, together with $\pi_{0}+\pi_{1}=1$, we obtain

$$
\begin{equation*}
\pi_{0}=\frac{q}{p+q}, \quad \pi_{1}=\frac{p}{p+q} . \tag{6.3}
\end{equation*}
$$

Thus we conclude:
(i) If $p+q>0$, there exists a unique stationary distribution as in (6.3).
(ii) If $p=q=0$, a stationary distribution is not uniquely determined. In fact, any distribution $\pi=\left[\pi_{0}, \pi_{1}\right]$ is stationary.

Moreover, we see from Example 5.3.5 that if $0<p+q<2$, or equivalently, if $|r|<1$, we have

$$
\lim _{n \rightarrow \infty} P^{n}=\frac{1}{p+q}\left[\begin{array}{ll}
q & p \\
q & p
\end{array}\right] .
$$

Then

$$
\lim _{n \rightarrow \infty} \pi(n)=\lim _{n \rightarrow \infty} \pi(0) P^{n}=\left[\pi_{0}(0) \pi_{1}(0)\right] \times \frac{1}{p+q}\left[\begin{array}{cc}
q & p \\
q & p
\end{array}\right]=\left[\frac{q}{p+q} \frac{p}{p+q}\right] .
$$

It is noteworthy that the stationary distribution is obtained as a limit distribution.
Example 6.1 .3 (3-state Markov chain) We discuss the Markov chain $\left\{X_{n}\right\}$ introduced in Example 5.2.7. If $q>0$ and $b>0$, a stationary distribution is unique and given by $\pi=\left[\begin{array}{lll}0 & 0 & 1\end{array}\right]$.

Example 6.1.4 (One-dimensional RW) Consider the 1-dimensional random walk with right-move probability $p>0$ and left-move probability $q=1-p>0$. Let $[\cdots \pi(k) \cdots]$ be a distribution on $\mathbf{Z}$. If it is stationary, we have

$$
\begin{equation*}
\pi(k)=p \pi(k-1)+q \pi(k+1), \quad k \in \mathbf{Z} . \tag{6.4}
\end{equation*}
$$

(Case 1) $p \neq q$. Then a general solution to (6.4) is given by

$$
\pi(k)=C_{1} 1^{k}+C_{2}\left(\frac{p}{q}\right)^{k}=C_{1}+C_{2}\left(\frac{p}{q}\right)^{k}, \quad k \in \mathbf{Z} .
$$

This never becomes a probability distribution for any choice of $C_{1}$ and $C_{2}$. Namely, there is no stationary distribution.
(Case 2) $p=q$. In this case a general solution to (6.4) is given by

$$
\pi(k)=\left(C_{1}+C_{2} k\right) 1^{k}=C_{1}+C_{2} k, \quad k \in \mathbf{Z}
$$

This never becomes a probability distribution for any choice of $C_{1}$ and $C_{2}$. Namely, there is no stationary distribution.

Example 6.1.5 (One-dimensional RW with reflection barrier) There is a unique stationary distribution when $p<q$. In fact,

$$
\pi(0)=C p, \quad \pi(k)=C\left(\frac{p}{q}\right)^{k}, \quad k \geq 1, \quad \text { where } \quad C=\frac{q-p}{2 p q}
$$

If $p \geq q$, then there is no stationary distribution.

### 6.2 Existence and Uniqueness

Theorem 6.2.1 A Markov chain over a finite state space $S$ has a stationary distribution.
A simple proof is based on the Brouwer's fixed-point theorem, for details see the textbooks. Note that the stationary distribution mentioned in the above theorem is not necessarily unique.

Definition 6.2.2 We say that a state $j$ can be reached from a state $i$ if there exists some $n \geq 0$ such that $p_{n}(i, j)>0$. By definition every state $i$ can be reached from itself. We say that two states $i$ and $j$ intercommunicate if $i$ can be reached form $j$ and $j$ can be reached from $i$, i.e., there exist $m \geq 0$ and $n \geq 0$ such that $p_{n}(i, j)>0$ and $p_{m}(j, i)>0$.

For $i, j \in S$ we introduce a binary relation $i \sim j$ when they intercommunicate. Then $\sim$ becomes an equivalence relation on $S$ :
(i) $i \sim i$;
(ii) $i \sim j \Longrightarrow j \sim i$;
(iii) $i \sim j, j \sim k \Longrightarrow i \sim k$.

In fact, (i) and (ii) are obvious by definition, and (iii) is verified by the Chapman-Kolmogorov equation. Thereby the state space $S$ is classified into a disjoint set of equivalence classes. In each equivalence class any two states intercommunicate each other.

Definition 6.2.3 A state $i$ is called absorbing if $p_{i i}=1$. In particular, an absorbing state is a state which constitutes an equivalence class by itself.

Definition 6.2.4 A Markov chain is called irreducible if every state can be reached from every other state, i.e., if there is only one equivalence class of intercommunicating states.

Theorem 6.2.5 An irreducible Markov chain on a finite state space $S$ admits a unique stationary distribution $\pi=\left[\pi_{i}\right]$. Moreover, $\pi_{i}>0$ for all $i \in S$.

In fact, the proof owes to the following two facts:
(1) For an irreducible Markov chain the following assertions are equivalent:
(i) it admits a stationary distribution;
(ii) every state is positive recurrent.

In this case the stationary distribution $\pi$ is unique and given by

$$
\pi_{i}=\frac{1}{\mathbf{E}\left(T_{i} \mid X_{0}=i\right)}, \quad i \in S
$$

(2) Every state of an irreducible Markov chain on a finite state space is positive recurrent (Theorem 7.3.2).

### 6.3 Convergence

Example 6.3.1 (2-state Markov chain) We recall Examples 5.3.5 and 6.1.2. If $p+q>0$, the distribution of the above Markov chain converges to the unique stationary distribution. Consider the case of $p=q=1$, i.e., the transition matrix becomes

$$
P=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]
$$

The stationary distribution is unique. But for a given initial distribution $\pi(0)$ it is not necessarily true that $\pi(n)$ converges to the stationary distribution.

Roughly speaking, we need to avoid the periodic transition in order to have the convergence to a stationary distribution.

Definition 6.3.2 For a state $i \in S$,

$$
\operatorname{GCD}\left\{n \geq 1 ; P\left(X_{n}=i \mid X_{0}=i\right)>0\right\}
$$

is called the period of $i$. (When the set in the right-hand side is empty, the period is not defined.) A state $i \in S$ is called aperiodic if its period is one.

Theorem 6.3.3 For an irreducible Markov chain, every state has a common period.
Theorem 6.3.4 Let $\pi$ be a stationary distribution of an irreducible Markov chain on a finite state space (It is unique, see Theorem 6.2.5). If $\left\{X_{n}\right\}$ is aperiodic, for any $j \in S$ we have

$$
\lim _{n \rightarrow \infty} P\left(X_{n}=j\right)=\pi_{j}
$$

Problem 10 Find all stationary distributions of the Markov chain determined by the transition diagram below. Then discuss convergence of distributions.


Problem 11 Let $\left\{X_{n}\right\}$ be the Markov chain introduced in Example 5.2.7:


For $n=1,2, \ldots$ let $H_{n}$ denote the probability of starting from H and terminating at D at $n$-step. Similarly, for $n=1,2, \ldots$ let $S_{n}$ denote the probability of starting from S and terminating at D at $n$-step.
(1) Show that $\left\{H_{n}\right\}$ and $\left\{S_{n}\right\}$ satisfies the following linear system:

$$
\left\{\begin{array}{l}
H_{n}=a H_{n-1}+b S_{n-1}, \\
S_{n}=p H_{n-1}+r S_{n-1},
\end{array} \quad n \geq 2 ; \quad H_{1}=0, \quad S_{1}=q\right.
$$

(2) Let $H$ and $S$ denote the life times starting from the state H and S , respectively. Solving the linear system in (1), prove the following identities for the mean life times:

$$
\mathbf{E}[H]=\sum_{n=1}^{\infty} n H_{n}=\frac{b+p+q}{b q}, \quad \mathbf{E}[S]=\sum_{n=1}^{\infty} n S_{n}=\frac{b+p}{b q}
$$

### 6.4 Page Rank

The hyperlinks among $N$ websites give rise to a digraph (directed graph) $G$ on $N$ vertices. It is natural to consider a Markov chain on $G$, which is defined by the transition matrix $P=\left[p_{i j}\right]$, where

$$
p_{i j}= \begin{cases}\frac{1}{\operatorname{deg} i} & \text { if } i \rightarrow j \\ 0, & \text { if } i \nrightarrow j \text { and } i \neq j \\ 1, & \operatorname{deg} i=0 \text { and } j=i\end{cases}
$$

where $\operatorname{deg} i=|\{j ; i \rightarrow j\}|$ is the out-degree of $i$.



There exists a stationary state but not necessarily unique. Taking $0 \leq d \leq 1$ we modify the transition matrix:

$$
Q=\left[q_{i j}\right], \quad q_{i j}=d p_{i j}+\epsilon, \quad \epsilon=\frac{1-d}{N}
$$

If $0 \leq d<1$, the Markov chain determined by $Q$ has necessarily a unique stationary distribution. Choosing a suitable $d<1$, we may understand the stationary distribution $\pi=[\pi(i)]$ as the page rank among the websites. In real application $d$ should not be close to 0 and $d \approx 0.85$ is often taken.

Problem 12 Consider the page rank introduced above.
(1) Let $\pi(i)$ be the page rank of a site $i$. Show that $\pi(i)$ satisfies the following relation

$$
\pi(i)=\frac{1-d}{N}+d \sum_{j: j \rightarrow i} \frac{\pi(j)}{\operatorname{deg} j}
$$

and explain the meaning.
(2) Show more examples of the page rank and discuss the role of sites which have no hyperlinks, that is, $\operatorname{deg} i=0$ (in terms of $P=\left[p_{i j}\right]$ such sites correspond to absorbing states).

## 7 Topics in Markov Chains I: Recurrence

### 7.1 Recurrence

Definition 7.1.1 Let $i \in S$ be a state. Define the first hitting time or first passage time to $i$ by

$$
T_{i}=\inf \left\{n \geq 1 ; X_{n}=i\right\} .
$$

If $\left\{n \geq 1 ; X_{n}=i\right\}$ is an empty set, we define $T_{i}=\infty$. A state $i$ is called recurrent if $P\left(T_{i}<\infty \mid X_{0}=i\right)=1$.
It is called transient if $P\left(T_{i}=\infty \mid X_{0}=i\right)>0$.
Theorem 7.1.2 A state $i \in S$ is recurrent if and only if

$$
\sum_{n=0}^{\infty} p_{n}(i, i)=\infty .
$$

If a state $i$ is transient, we have

$$
\sum_{n=0}^{\infty} p_{n}(i, i)<\infty \quad \text { and } \quad \sum_{n=0}^{\infty} p_{n}(i, i)=\frac{1}{1-P\left(T_{i}<\infty \mid X_{0}=i\right)} .
$$

Proof We first put

$$
\begin{aligned}
p_{n}(i, j) & =P\left(X_{n}=j \mid X_{0}=i\right), \quad n=0,1,2, \ldots, \\
f_{n}(i, j) & =P\left(T_{j}=n \mid X_{0}=i\right)=P\left(X_{1} \neq j, \ldots, X_{n-1} \neq j, X_{n}=j \mid X_{0}=i\right), \quad n=1,2, \ldots .
\end{aligned}
$$

$p_{n}(i, j)$ is nothing else but the $n$ step transition probability. On the other hand, $f_{n}(i, j)$ is the probability that the Markov chain starts from $i$ and reach $j$ first time after $n$ step. Dividing the set of sample paths from $i$ to $j$ in $n$ steps according to the number of steps after which the path reaches $j$ for the first time, we obtain

$$
\begin{equation*}
p_{n}(i, j)=\sum_{r=1}^{n} f_{r}(i, j) p_{n-r}(j, j), \quad i, j \in S, \quad n=1,2, \ldots \tag{7.1}
\end{equation*}
$$

We next introduce the generating functions:

$$
G_{i j}(z)=\sum_{n=0}^{\infty} p_{n}(i, j) z^{n}, \quad F_{i j}(z)=\sum_{n=1}^{\infty} f_{n}(i, j) z^{n} .
$$

In view of (7.1) we see easily that

$$
\begin{equation*}
G_{i j}(z)=p_{0}(i, j)+F_{i j}(z) G_{j j}(z) . \tag{7.2}
\end{equation*}
$$

Setting $i=j$ in (7.2), we obtain

$$
G_{i i}(z)=1+F_{i i}(z) G_{i i}(z) \quad \Longleftrightarrow \quad G_{i i}(z)=\frac{1}{1-F_{i i}(z)} .
$$

On the other hand, since

$$
G_{i i}(1)=\sum_{n=0}^{\infty} p_{n}(i, i), \quad F_{i i}(1)=\sum_{n=1}^{\infty} f_{n}(i, i)=P\left(T_{i}<\infty \mid X_{0}=i\right)
$$

we see that two conditions $F_{i i}(1)=1$ and $G_{i i}(1)=\infty$ are equivalent. The second statement is readily clear.

### 7.2 Random Walks on Lattices

Example 7.2.1 (random walk on $\mathbb{Z}$ ) Since the random walk starting from the origin 0 returns to it only after even steps, for recurrence we only need to compute the sum of $p_{2 n}(0,0)$. We start with the obvious result:

$$
p_{2 n}(0,0)=\frac{(2 n)!}{n!n!} p^{n} q^{n}, \quad p+q=1
$$

Then, using the Stirling formula:

$$
\begin{equation*}
n!\sim \sqrt{2 \pi n}\left(\frac{n}{e}\right)^{n} \tag{7.3}
\end{equation*}
$$

we obtain

$$
p_{2 n}(0,0) \sim \frac{1}{\sqrt{\pi n}}(4 p q)^{n}
$$

Hence,

$$
\sum_{n=0}^{\infty} p_{2 n}(0,0) \begin{cases}<\infty, & p \neq q \\ =\infty, & p=q=1 / 2\end{cases}
$$

Consequently, one-dimensional random walk is transient if $p \neq q$, and it is recurrent if $p=q=1 / 2$.
Remark 7.2.2 Let $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$ be sequences of positive numbers. We write $a_{n} \sim b_{n}$ if $\lim _{n \rightarrow \infty} a_{n} / b_{n}=1$. In this case, there exist two constant numbers $c_{1}>0$ and $c_{2}>0$ such that $c_{1} a_{n} \leq b_{n} \leq c_{2} a_{n}$. Hence $\sum_{n=1}^{\infty} a_{n}$ and $\sum_{n=1}^{\infty} b_{n}$ converge or diverge at the same time.

Example 7.2.3 (random walk on $\mathbb{Z}^{2}$ ) Obviously, the random walk starting from the origin 0 returns to it only after even steps. Therefore, for recurrence we only need to compute the sum of $p_{2 n}(0,0)$. For twodimensional random walk we need to consider two directions along with $x$-axis and $y$-axis. We see easily that

$$
p_{2 n}(0,0)=\sum_{i+j=n} \frac{(2 n)!}{i!i!j!j!}\left(\frac{1}{4}\right)^{2 n}=\frac{(2 n)!}{n!n!}\left(\frac{1}{4}\right)^{2 n} \sum_{i+j=n} \frac{n!n!}{i!i!j!j!}=\binom{2 n}{n}\left(\frac{1}{4}\right)^{2 n} \sum_{i=0}^{n}\binom{n}{i}^{2}
$$

Employing the formula for the binomial coefficients:

$$
\begin{equation*}
\sum_{i=0}^{n}\binom{n}{i}^{2}=\binom{2 n}{n} \tag{7.4}
\end{equation*}
$$

which is a good exercise for the readers, we obtain

$$
p_{2 n}(0,0)=\binom{2 n}{n}^{2}\left(\frac{1}{4}\right)^{2 n} \sim \frac{1}{\pi n}
$$

Then we have

$$
\sum_{n=1}^{\infty} p_{2 n}(0,0)=\infty
$$

which means that two-dimensional random walk is recurrent.
Example 7.2.4 (random walk on $\mathbb{Z}^{3}$ ) Let us consider the isotropic random walk in 3-dimension. As there are three directions, say, $x, y, z$-axis, we have

$$
p_{2 n}(0,0)=\sum_{i+j+k=n} \frac{(2 n)!}{i!i!j!j!k!k!}\left(\frac{1}{6}\right)^{2 n}=\frac{(2 n)!}{n!n!}\left(\frac{1}{6}\right)^{2 n} \sum_{i+j+k=n} \frac{n!n!}{i!i!j!j!k!k!}=\binom{2 n}{n}\left(\frac{1}{6}\right)^{2 n} \sum_{i+j+k=n}\left(\frac{n!}{i!j!k!}\right)^{2}
$$

We note the following two facts. First,

$$
\begin{equation*}
\sum_{i+j+k=n} \frac{n!}{i!j!k!}=3^{n} . \tag{7.5}
\end{equation*}
$$

Second, the maximum value $M_{n}=\max _{i+j+k=n} \frac{n!}{i!j!k!}$ is attained when $\frac{n}{3}-1 \leq i, j, k \leq \frac{n}{3}+1$ so

$$
M_{n} \sim \frac{3 \sqrt{3}}{2 \pi n} 3^{n}
$$

by the Stirling formula. Then we have

$$
p_{2 n}(0,0) \leq\binom{ 2 n}{n}\left(\frac{1}{6}\right)^{2 n} 3^{n} M_{n} \sim \frac{3 \sqrt{3}}{2 \pi \sqrt{\pi}} n^{-3 / 2}
$$

Therefore.

$$
\sum_{n=1}^{\infty} p_{2 n}(0,0)<\infty
$$

which implies that the random walk is not recurrent (i.e., transient).

### 7.3 Positive Recurrence and Null Recurrence

If a state $i$ is recurrent, i.e., $P\left(T_{i}<\infty \mid X_{0}=i\right)=1$, the mean recurrent time is defined:

$$
\mathbf{E}\left(T_{i} \mid X_{0}=i\right)=\sum_{n=1}^{\infty} n P\left(T_{i}=n \mid X_{0}=i\right)
$$

The state $i$ is called positive recurrent if $\mathbf{E}\left(T_{i} \mid X_{0}=i\right)<\infty$, and null recurrent otherwise.
Theorem 7.3.1 The states in an equivalence class are all positive recurrent, or all null recurrent, or all transient. In particular, for an irreducible Markov chain, the states are all positive recurrent, or all null recurrent, or all transient.

Theorem 7.3.2 For an irreducible Markov chain on a finite state space $S$, every state is positive recurrent.
Example 7.3.3 The mean recurrent time of the one-dimensional isotropic random walk is infinity, i.e., the one-dimensional isotropic random walk is null recurrent. The proof will be given in Section ??.

Problem 13 Let $\left\{X_{n}\right\}$ be a Markov chain described by the following transition diagram:

where $p>0$ and $q>0$. For a state $i \in S$ let $T_{i}=\inf \left\{n \geq 1 ; X_{n}=i\right\}$ be the first hitting time to $i$.
(1) Calculate

$$
P\left(T_{0}=1 \mid X_{0}=0\right), \quad P\left(T_{0}=2 \mid X_{0}=0\right), \quad P\left(T_{0}=3 \mid X_{0}=0\right), \quad P\left(T_{0}=4 \mid X_{0}=0\right)
$$

(2) Find $P\left(T_{0}=n \mid X_{0}=0\right)$ and calculate

$$
\sum_{n=1}^{\infty} P\left(T_{0}=n \mid X_{0}=0\right), \quad \sum_{n=1}^{\infty} n P\left(T_{0}=n \mid X_{0}=0\right)
$$

## 8 Topics in Markov Chains II: Absorption

### 8.1 Absorbing States

A state $i$ is called absorbing if $p_{i i}=1$ and $p_{i j}=0$ for all $j \neq i$. Once a Markov chain hits an absorbing state, it stays thereat forever.

Let us consider a Markov chain on a finite state space $S$ with some absorbing states. We set

$$
S=S_{a} \cup S_{0}
$$

where $S_{a}$ denotes the set of absorbing states and $S_{0}$ the rest. According to the above partition, the transition matrix is written as

$$
P=\left[\begin{array}{ccccccc}
1 & & & & 0 & \cdots & 0 \\
& 1 & & & 0 & \cdots & 0 \\
& & \ddots & & \vdots & \ddots & \vdots \\
& & & 1 & 0 & \cdots & 0 \\
& & & & & & \\
& & * & & & * &
\end{array}\right]=\left[\begin{array}{cc}
I & 0 \\
S & T
\end{array}\right]
$$

Then

$$
P^{n}=\left[\begin{array}{ll}
I & 0 \\
S & T
\end{array}\right]^{n}=\left[\begin{array}{cc}
I & 0 \\
S_{n} & T^{n}
\end{array}\right]
$$

where $S_{1}=S$ and $S_{n}=S_{n-1}+T^{n-1} S$. To avoid inessential tediousness we assume the following condition (C1) For any $i \in S_{0}$ there exist $j \in S_{a}$ and $n \geq 1$ such that $\left(P^{n}\right)_{i j}>0$.

In other words, the Markov chain starting from $i \in S_{0}$ has a positive probability of absorption. Since $S$ is finite by assumption, the $n$ in (C1) is chosen independently of $i \in S_{0}$. Hence (C1) is equivalent to the following
(C2) There exists $N \geq 1$ such that for any $i \in S_{0}$ there exist $j \in S_{a}$ with $\left(P^{N}\right)_{i j}>0$.
Lemma 8.1.1 Notations and assumptions being as above, $\lim _{n \rightarrow \infty} T^{n}=0$.
Proof We see from the obvious relation

$$
1=\sum_{j \in S}\left(P^{N}\right)_{i j}=\sum_{j \in S_{0}}\left(P^{N}\right)_{i j}+\sum_{j \in S_{a}}\left(P^{N}\right)_{i j}
$$

and condition (C2) that

$$
\sum_{j \in S_{0}}\left(P^{N}\right)_{i j}<1, \quad i \in S_{0}
$$

Note that for $i, j \in S_{0}$ we have $\left(P^{N}\right)_{i j}=\left(T^{N}\right)_{i j}$. We choose $\delta<1$ such that

$$
\sum_{j \in S_{0}}\left(T^{N}\right)_{i j} \leq \delta<1 \quad i \in S_{0}
$$

Now let $i \in S_{0}$ and $n \geq N$. We see that

$$
\sum_{j \in S_{0}}\left(T^{n}\right)_{i j}=\sum_{j, k \in S_{0}}\left(T^{n-N}\right)_{i k}\left(T^{N}\right)_{k j}=\sum_{k \in S_{0}}\left(T^{n-N}\right)_{i k} \sum_{j \in S_{0}}\left(T^{N}\right)_{k j} \leq \delta \sum_{k \in S_{0}}\left(T^{n-N}\right)_{i k}=\delta \sum_{j \in S_{0}}\left(T^{n-N}\right)_{i j} .
$$

Repeating this procedure, we have

$$
\sum_{j \in S_{0}}\left(T^{n}\right)_{i j} \leq \delta^{k} \sum_{j \in S_{0}}\left(T^{n-k N}\right)_{i j} \leq \delta^{k} \sum_{j \in S}\left(P^{n-k N}\right)_{i j} \leq \delta^{k}
$$

where $0 \leq n-k N<N$. Therefore,

$$
\lim _{n \rightarrow \infty} \sum_{j \in S_{0}}\left(T^{n}\right)_{i j}=0
$$

from which we have $\lim _{n \rightarrow \infty}\left(T^{n}\right)_{i j}=0$ for all $i, j \in S_{0}$.

Remark 8.1.2 It is shown that every state $i \in S_{0}$ is transient.
Theorem 8.1.3 Let $\pi_{0}=[\alpha \beta]$ be the initial distribution (according to $S=S_{a} \cup S_{0}$ ). Then the limit distribution is given by

$$
\left[\alpha+\beta S_{\infty} 0\right], \quad \text { where } \quad S_{\infty}=(I-T)^{-1} S
$$

Proof The limit distribution is given by

$$
\lim _{n \rightarrow \infty} \pi_{0} P^{n}=\lim _{n \rightarrow \infty}[\alpha \beta]\left[\begin{array}{cc}
I & 0 \\
S_{n} & T^{n}
\end{array}\right]=\lim _{n \rightarrow \infty}\left[\alpha+\beta S_{n} \beta T^{n}\right] .
$$

We see from Lemma 8.1.1 that

$$
\lim _{n \rightarrow \infty} \beta T^{n}=0
$$

On the other hand, since $S_{n}=S_{n-1}+T^{n-1} S$ we have

$$
S_{n}=\left(I+T+T^{2}+\cdots+T^{n-1}\right) S
$$

and

$$
(I-T) S_{n}=\left(I-T^{n}\right) S
$$

Hence

$$
\lim _{n \rightarrow \infty} S_{n}=\lim _{n \rightarrow \infty}(I-T)^{-1}\left(I-T^{n}\right) S=(I-T)^{-1} S
$$

which shows the result.

Example 8.1.4 Consider the Markov chain given by the transition diagram, which is a random walk with absorbing barriers.


The transition matrix is given by

$$
P=\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
q & 0 & 0 & p \\
0 & p & q & 0
\end{array}\right]=\left[\begin{array}{cc}
I & 0 \\
S & T
\end{array}\right], \quad S=\left[\begin{array}{cc}
q & 0 \\
0 & p
\end{array}\right], \quad T=\left[\begin{array}{ll}
0 & p \\
q & 0
\end{array}\right]
$$

Then

$$
S_{\infty}=(I-T)^{-1} S=\frac{1}{1-p q}\left[\begin{array}{cc}
q & p^{2} \\
q^{2} & p
\end{array}\right]
$$

Suppose that the initial distribution is given by $\pi_{0}=[\alpha \beta \gamma \delta]$. Then the limit distribution is

$$
\left[\begin{array}{llll}
\alpha+\frac{q \gamma+q^{2} \delta}{1-p q} & \beta+\frac{p^{2} \gamma+p \delta}{1-p q} & 0 & 0
\end{array}\right]
$$

In particular, if the Markov chain starts at the state 3, setting $\pi_{0}=\left[\begin{array}{lll}0 & 0 & 1\end{array}\right]$, we obtain the limit distribution

$$
\left[\begin{array}{llll}
\frac{q}{1-p q} & \frac{p^{2}}{1-p q} & 0 & 0
\end{array}\right],
$$

which means that the Markov chain is absorbed in the states 1 or 2 at the ratio $q: p^{2}$.
Problem 14 Following Example 8.1.4, study the Markov chain given by the following transition diagram, where $p+q=1$.


### 8.2 Gambler's Ruin

We consider a random walk with absorbing barriers at $-A$ and $B$, where $A>0$ and $B>0$. This is a Markov chain on the state space $S=\{-A,-A+1, \ldots, B-1, B\}$ with the transition diagram as follows:


We are interested in the absorbing probability, i.e.,

$$
\begin{aligned}
& R=P\left(X_{n}=-A \text { for some } n=1,2, \ldots\right)=P\left(\bigcup_{n=1}^{\infty}\left\{X_{n}=-A\right\}\right), \\
& S=P\left(X_{n}=B \text { for some } n=1,2, \ldots\right)=P\left(\bigcup_{n=1}^{\infty}\left\{X_{n}=B\right\}\right)
\end{aligned}
$$

Note that the events in the right-hand sides are not the unions of disjoint events.
A sample path is shown in the following picture:


A key idea is to introduce a similar random walk starting at $k,-A \leq k \leq B$, which is denoted by $X_{n}^{(k)}$. Then the original one is $X_{n}=X_{n}^{(0)}$. Let $R_{k}$ and $S_{k}$ be the probabilities that the random walk $X_{n}^{(k)}$ is absorbed at $-A$ and $B$, respectively. We wish to find $R=R_{0}$ and $S=S_{0}$.

Lemma 8.2.1 $\left\{R_{k} ;,-A \leq k \leq B\right\}$ fulfills the following difference equation:

$$
\begin{equation*}
R_{k}=p R_{k+1}+q R_{k-1}, \quad R_{-A}=1, \quad R_{B}=0 \tag{8.1}
\end{equation*}
$$

Similarly, $\left\{S_{k} ;,-A \leq k \leq B\right\}$ fulfills the following difference equation:

$$
\begin{equation*}
S_{k}=p S_{k+1}+q S_{k-1}, \quad S_{-A}=0, \quad S_{B}=1 \tag{8.2}
\end{equation*}
$$

Theorem 8.2.2 Let $A \geq 1$ and $B \geq 1$. Let $\left\{X_{n}\right\}$ be the random walk with absorbing barriers at $-A$ and $B$, and with right-move probability $p$ and left-move probability $q(p+q=1)$. Then the probabilities that $\left\{X_{n}\right\}$ is absorbed at the barriers are given by

$$
\begin{gathered}
P\left(X_{n}=-A \text { for some } n\right)= \begin{cases}\frac{(q / p)^{A}-(q / p)^{A+B}}{1-(q / p)^{A+B}}, & p \neq q, \\
\frac{B}{A+B}, & p=q=\frac{1}{2}\end{cases} \\
P\left(X_{n}=B \text { for some } n\right)= \begin{cases}\frac{1-(q / p)^{A}}{1-(q / p)^{A+B}}, & p \neq q, \\
\frac{A}{A+B}, & p=q=\frac{1}{2} .\end{cases}
\end{gathered}
$$

In particular, the random walk is absorbed at the barriers at probability 1.
An interpretation of Theorem 8.2.2 gives the solution to the gambler's ruin problem. Two players A and B toss a fair coin by turns. Let $A$ and $B$ be their allotted points when the game starts. They exchange 1 point after each trial. This game is over when one of the players loses all the allotted points and the other gets $A+B$ points. We are interested in the probability of each player's win. For each $n \geq 0$ define $X_{n}$ in such a way that the allotted point of A at time $n$ is given by $A+X_{n}$. Then $\left\{X_{n}\right\}$ becomes a random walk with absorbing barrier at $-A$ and $B$. It then follows from Theorem 8.2.2 that the winning probability of A and B are given by

$$
\begin{equation*}
P(A)=\frac{A}{A+B}, \quad P(B)=\frac{B}{A+B} \tag{8.3}
\end{equation*}
$$

respectively. As a result, they are proportional to the initial allotted points. For example, if $A=1$ and $B=100$, we have $P(A)=1 / 101$ and $P(B)=100 / 101$, which sounds that almost no chance of A's win.

In a fair bet the recurrence is guaranteed by Theorem ??. Even if one has much more losses than wins, continuing the game one will be back to the zero balance. However, in reality there is a barrier of limited money. (8.3) tells the effect of the barrier.

It is also interesting to know the expectation of the number of coin tosses until the game is over.
Theorem 8.2.3 Let $\left\{X_{n}\right\}$ be the same as in Theorem 8.2.2. The expected life time of this random walk until absorption is given by

$$
\begin{cases}\frac{A}{q-p}-\frac{A+B}{q-p} \frac{1-(q / p)^{A}}{1-(q / p)^{A+B}}, & p \neq q, \\ A B, & p=q=\frac{1}{2}\end{cases}
$$

Proof Let $Y_{k}$ be the life time of a random walk starting from the position $k(-A \leq k \leq B)$ at time $n=0$ until absorption．In other words，

$$
Y_{k}=\min \left\{j \geq 0 ; X_{j}^{(k)}=-A \text { または } X_{j}^{(k)}=B\right\} .
$$

We wish to compute $\mathbf{E}\left(Y_{0}\right)$ ．We see by definition that

$$
\begin{equation*}
\mathbf{E}\left(Y_{-A}\right)=\mathbf{E}\left(Y_{B}\right)=0 \tag{8.4}
\end{equation*}
$$

For $-A<k<B$ we have

$$
\begin{equation*}
\mathbf{E}\left(Y_{k}\right)=\sum_{j=1}^{\infty} j P\left(Y_{k}=j\right) \tag{8.5}
\end{equation*}
$$

In a similar manner as in the proof of Theorem 8．2．2 we note that

$$
\begin{equation*}
P\left(Y_{k}=j\right)=p P\left(Y_{k+1}=j-1\right)+q P\left(Y_{k-1}=j-1\right) \tag{8.6}
\end{equation*}
$$

Inserting（8．6）into（8．5），we obtain

$$
\begin{align*}
\mathbf{E}\left(Y_{k}\right) & =p \sum_{j=1}^{\infty} j P\left(Y_{k+1}=j-1\right)+q \sum_{j=1}^{\infty} j P\left(Y_{k-1}=j-1\right) \\
& =p \mathbf{E}\left(Y_{k+1}\right)+q \mathbf{E}\left(Y_{k-1}\right)+1 \tag{8.7}
\end{align*}
$$

Thus， $\mathbf{E}\left(Y_{k}\right)$ is the solution to the difference equation（8．7）with boundary condition（8．4）．This difference equation is solved in a standard manner and we find

$$
\mathbf{E}\left(Y_{k}\right)= \begin{cases}\frac{A+k}{q-p}-\frac{A+B}{q-p} \frac{1-(q / p)^{A+k}}{1-(q / p)^{A+B}}, & p \neq q \\ (A+k)(B-k), & p=q=\frac{1}{2}\end{cases}
$$

Setting $k=0$ ，we obtain the result．
If $p=q=1 / 2$ and $A=1, B=100$ ，the expected life time is $A B=100$ ．The gambler A is much inferior to $B$ in the amount of funds（as we have seen already，the probability of A＇s win is just $1 / 101$ ），however，the expected life time until the game is over is 100 ，which sounds longer than one expects intuitively．Perhaps this is because the gambler cannot quit gambling．

Problem 15 （A bold gambler）In each game a gambler wins the dollars he bets with probability $p$ ，and loses with probability $q=1-p$ ．The goal of the gambler is to get 5 dollars．His strategy is to bet the difference between 5 dollars and what he has．Let $X_{n}$ be the amount he has just after $n$th bet．

（1）Analyze the Markov chain $\left\{X_{n}\right\}$ with initial condition $X_{0}=1$ ．
（2）Compare with the steady gambler discussed in this section，who bets just 1 dollar in each game．

## 9 Galton-Watson Branching Processes

Consider a simplified family tree where each individual gives birth to offspring (children) and dies. The number of offsprings is random. We are interested in whether the family survives or not. A fundamental model was proposed by F. Galton in 1873 and basic properties were derived by Galton and Watson in their joint paper in the next year. The name "Galton-Watson branching process" is quite common in literatures after their paper, but it would be more fair to refer to it as "BGW process." In fact, Irénée-Jules Bienaymé studied the same model independently already in 1845.


### 9.1 Definition

Let $X_{n}$ be the number of individuals of the $n$th generation. Then $\left\{X_{n} ; n=0,1,2, \ldots\right\}$ becomes a discretetime stochasic process. We assume that the number of children born from each individual obeys a common probability distribution and is independent of individuals and of generation. Under this assupmtion $\left\{X_{n}\right\}$ becomes a Markov chain.

Let us find the transition probability. Let $Y$ be the number of children born from an individual and set

$$
P(Y=k)=p_{k}, \quad k=0,1,2, \ldots
$$

The sequence $\left\{p_{0}, p_{1}, p_{2}, \ldots\right\}$ describes the distribution of the number of children born from an individual. In fact, what we need is the condition

$$
p_{k} \geq 0, \quad \sum_{k=0}^{\infty} p_{k}=1 .
$$

We refer to $\left\{p_{0}, p_{1}, \ldots\right\}$ as the offspring distribution. Let $Y_{1}, Y_{2}, \ldots$ be independent identically distributed random variables, of which the distribution is the same as $Y$. Then, we define the transition probability by

$$
p(i, j)=P\left(X_{n+1}=j \mid X_{n}=i\right)=P\left(\sum_{k=1}^{i} Y_{k}=j\right), \quad i \geq 1, \quad j \geq 0,
$$

and

$$
p(0, j)= \begin{cases}0, & j \geq 1, \\ 1, & j=0 .\end{cases}
$$

Clearly, the state 0 is an absorbing one. The above Markov chain $\left\{X_{n}\right\}$ over the state space $\{0,1,2, \ldots\}$ is called the Galton-Watson branching process with offspring distribution $\left\{p_{k} ; k=0,1,2, \ldots\right\}$.

For simplicity we assume that $X_{0}=1$. When $p_{0}+p_{1}=1$, the famility tree is reduced to just a path without branching so the situation is much simpler (Problem 16). We will focus on the case where

$$
p_{0}+p_{1}<1, \quad p_{2}<1, \quad \ldots, \quad p_{k}<1, \quad \ldots
$$

In the next section on we will always assume the above conditions.
Problem 16 (One-child policy) Consider the Galton-Watson branching process with offspring distribution satisfying $p_{0}+p_{1}=1$. Calculate the probabilities

$$
q_{1}=P\left(X_{1}=0\right), \quad q_{2}=P\left(X_{1} \neq 0, X_{2}=0\right), \quad \ldots, \quad q_{n}=P\left(X_{1} \neq 0, \ldots, X_{n-1} \neq 0, X_{n}=0\right), \quad \ldots
$$

and find the extinction probability

$$
P\left(\bigcup_{n=1}^{\infty}\left\{X_{n}=0\right\}\right)=P\left(X_{n}=0 \text { occurs for some } n \geq 1\right)
$$

### 9.2 Generating Functions

Let $\left\{X_{n}\right\}$ be the Galton-Watson branching process with offspring distribution $\left\{p_{k} ; k=0,1,2, \ldots\right\}$. Let $p(i, j)=P\left(X_{n+1}=j \mid X_{n}=i\right)$ be the transition probability. We assume that $X_{0}=1$.

Define the generating function of the offspring distribution by

$$
\begin{equation*}
f(s)=\sum_{k=0}^{\infty} p_{k} s^{k} \tag{9.1}
\end{equation*}
$$

The series in the right-hand side converges for $|s| \leq 1$. We set

$$
f_{0}(s)=s, \quad f_{1}(s)=f(s), \quad f_{n}(s)=f\left(f_{n-1}(s)\right)
$$

Lemma 9.2.1

$$
\begin{equation*}
\sum_{j=0}^{\infty} p(i, j) s^{j}=[f(s)]^{i}, \quad i=1,2, \ldots \tag{9.2}
\end{equation*}
$$

Proof By definition,

$$
p(i, j)=P\left(Y_{1}+\cdots+Y_{i}=j\right)=\sum_{\substack{k_{1}+\ldots+k_{i}=j \\ k_{1} \geq 0, \ldots, k_{i} \geq 0}} P\left(Y_{1}=k_{1}, \ldots, Y_{i}=k_{i}\right)
$$

Since $Y_{1}, \ldots, Y_{i}$ are independent, we have

$$
p(i, j)=\sum_{\substack{k_{1}+\ldots+k_{i}=j \\ k_{1} \geq 0, \ldots, k_{i} \geq 0}} P\left(Y_{1}=k_{1}\right) \cdots P\left(Y_{i}=k_{i}\right)=\sum_{\substack{k_{1}+\ldots+k_{i}=j \\ k_{1} \geq 0, \ldots, k_{i} \geq 0}} p_{k_{1}} \cdots p_{k_{i}} .
$$

Hence,

$$
\sum_{j=0}^{\infty} p(i, j) s^{j}=\sum_{j=0}^{\infty} \sum_{\substack{k_{1}+\ldots+k_{i}=j \\ k_{1} \geq 0, \ldots, k_{i} \geq 0}} p_{k_{1}} \cdots p_{k_{i}} s^{j}=\sum_{k_{1}=0}^{\infty} p_{k_{1}} s^{k_{1}} \cdots \sum_{k_{i}=0}^{\infty} p_{k_{i}} s^{k_{i}}=[f(s)]^{i}
$$

which proves the assertion.

Lemma 9.2.2 Let $p_{n}(i, j)$ be the $n$-step transition probability of the Galton-Watson branching process. We have

$$
\begin{equation*}
\sum_{j=0}^{\infty} p_{n}(i, j) s^{j}=\left[f_{n}(s)\right]^{i}, \quad i=1,2, \ldots \tag{9.3}
\end{equation*}
$$

Proof We prove the assertion by induction on $n$. First note that $p_{1}(i, j)=p(i, j)$ and $f_{1}(s)=f(s)$ by definition. For $n=1$ we need to show that

$$
\begin{equation*}
\sum_{j=0}^{\infty} p(i, j) s^{j}=[f(s)]^{i}, \quad i=1,2, \ldots \tag{9.4}
\end{equation*}
$$

Which was shown in Lemma 9.2.1. Suppose that $n \geq 1$ and the claim (9.3) is valid up to $n$. Using the Chapman-Kolmogorov identity, we see that

$$
\sum_{j=0}^{\infty} p_{n+1}(i, j) s^{j}=\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p(i, k) p_{n}(k, j) s^{j}
$$

Since

$$
\sum_{j=0}^{\infty} p_{n}(k, j) s^{j}=\left[f_{n}(s)\right]^{k}
$$

by assumption of induction, we obtain

$$
\sum_{j=0}^{\infty} p_{n+1}(i, j) s^{j}=\sum_{k=0}^{\infty} p(i, k)\left[f_{n}(s)\right]^{k}
$$

The right-hand side coincides with (9.4) where $s$ is replaced by $f_{n}(s)$. Consequently, we come to

$$
\sum_{j=0}^{\infty} p_{n+1}(i, j) s^{j}=\left[f\left(f_{n}(s)\right)\right]^{i}=\left[f_{n+1}(s)\right]^{i}
$$

which proves the claim for $n+1$.
Since $X_{0}=1$,

$$
P\left(X_{n}=j\right)=P\left(X_{n}=j \mid X_{0}=1\right)=p_{n}(1, j)
$$

In particular,

$$
P\left(X_{1}=j\right)=P\left(X_{1}=j \mid X_{0}=1\right)=p_{1}(1, j)=p(1, j)=p_{j}
$$

Theorem 9.2.3 Assume that the mean value of the offspring distribution is finite:

$$
m=\sum_{k=0}^{\infty} k p_{k}<\infty
$$

Then we have

$$
\mathbf{E}\left[X_{n}\right]=m^{n}
$$

Proof Differentiating (9.1), we obtain

$$
\begin{equation*}
f^{\prime}(s)=\sum_{k=0}^{\infty} k p_{k} s^{k-1}, \quad|s|<1 \tag{9.5}
\end{equation*}
$$

Letting $s \rightarrow 1-0$, we have

$$
\lim _{s \rightarrow 1-0} f^{\prime}(s)=m
$$

On the other hand, setting $i=1$ in (9.3), we have

$$
\begin{equation*}
\sum_{j=0}^{\infty} p_{n}(1, j) s^{j}=f_{n}(s)=f_{n-1}(f(s)) \tag{9.6}
\end{equation*}
$$

Differentiating both sides, we come to

$$
\begin{equation*}
f_{n}^{\prime}(s)=\sum_{j=0}^{\infty} j p_{n}(1, j) s^{j-1}=f_{n-1}^{\prime}(f(s)) f^{\prime}(s) \tag{9.7}
\end{equation*}
$$

Letting $s \rightarrow 1-0$, we have

$$
\lim _{s \rightarrow 1-0} f_{n}^{\prime}(s)=\sum_{j=0}^{\infty} j p_{n}(1, j)=\lim _{s \rightarrow 1-0} f_{n-1}^{\prime}(f(s)) \lim _{s \rightarrow 1-0} f^{\prime}(s)=m \lim _{s \rightarrow 1-0} f_{n-1}^{\prime}(s)
$$

Therefore,

$$
\lim _{s \rightarrow 1-0} f_{n}^{\prime}(s)=m^{n}
$$

which means that

$$
\mathbf{E}\left(X_{n}\right)=\sum_{j=0}^{\infty} j P\left(X_{n}=j\right)=\sum_{j=0}^{\infty} j p_{n}(1, j)=m^{n}
$$

In conclusion, the mean value of the number of individuals in the $n$th generation, $\mathbf{E}\left(X_{n}\right)$, decreases and converges to 0 if $m<1$ and diverges to the infinity if $m>1$, as $n \rightarrow \infty$. It stays at a constant if $m=1$. We are thus suggested that extinction of the family occurs when $m<1$.

### 9.3 Extinction Probability

The event $\left\{X_{n}=0\right\}$ means that the family died out until the $n$th generation. So

$$
q=P\left(\bigcup_{n=1}^{\infty}\left\{X_{n}=0\right\}\right)
$$

is the probability of extinction of the family. Note that the events in the right-hand side is not mutually exclusive but

$$
\left\{X_{1}=0\right\} \subset\left\{X_{2}=0\right\} \subset \cdots \subset\left\{X_{n}=0\right\} \subset \ldots
$$

Therefore, it holds that

$$
\begin{equation*}
q=\lim _{n \rightarrow \infty} P\left(X_{n}=0\right) \tag{9.8}
\end{equation*}
$$

If $q=1$, this family almost surely dies out in some generation. If $q<1$, the survival probability is positive $1-q>0$. We are interested in whether $q=1$ or not.

Lemma 9.3.1 Let $f(s)$ be the generating function of the offspring distribution, and set $f_{n}(s)=f\left(f_{n-1}(s)\right)$ as before. Then we have

$$
q=\lim _{n \rightarrow \infty} f_{n}(0)
$$

Therefore, $q$ satisfies the equation:

$$
\begin{equation*}
q=f(q) \tag{9.9}
\end{equation*}
$$

Proof It follows from Lemma 9.2.2 that

$$
f_{n}(s)=\sum_{j=0}^{\infty} p_{n}(1, j) s^{j}
$$

Hence,

$$
f_{n}(0)=p_{n}(1,0)=P\left(X_{n}=0 \mid X_{0}=1\right)=P\left(X_{n}=0\right)
$$

where the last identity is by the assumption of $X_{0}=1$. The assertion is now straightforward by combining (9.8). The second assertion follows since $f(s)$ is a continuous function on $[0,1]$.

Lemma 9.3.2 Assume that the offspring distribution satisfies the conditions:

$$
p_{0}+p_{1}<1, \quad p_{2}<1, \quad \ldots, \quad p_{k}<1, \quad \ldots .
$$

Then the generating function $f(t)$ verifies the following properties.
(1) $f(s)$ is increasing, i.e., $f\left(s_{1}\right) \leq f\left(s_{2}\right)$ for $0 \leq s_{1} \leq s_{2} \leq 1$.
(2) $f(s)$ is strictly convex, i.e., if $0 \leq s_{1}<s_{2} \leq 1$ and $0<\theta<1$ we have

$$
f\left(\theta s_{1}+(1-\theta) s_{2}\right)<\theta f\left(s_{1}\right)+(1-\theta) f\left(s_{2}\right)
$$

Proof (1) is apparent since the coefficient of the power series $f(s)$ is non-negative. (2) follows by $f^{\prime \prime}(s)>0$.

Lemma 9.3.3 (1) If $m \leq 1$, we have $f(s)>s$ for $0 \leq s<1$.
(2) If $m>1$, there exists a unique $s$ such that $0 \leq s<1$ and $f(s)=s$.

Lemma 9.3.4 $f_{1}(0) \leq f_{2}(0) \leq \cdots \rightarrow q$.
Theorem 9.3.5 The extinction probability $q$ of the Galton-Watson branching process as above coincides with the smallest $s$ such that

$$
s=f(s), \quad 0 \leq s \leq 1
$$

Moreover, if $m \leq 1$ we have $q=1$, and if $m>1$ we have $q<1$.
The Galton-Watson branching process is called subcritical, critical and supercritical if $m<1, m=1$ and $m>1$, respectively. The survival is determined only by the mean value $m$ of the offspring distribution. The situation changes dramatically at $m=1$ and, following the terminology of statistical physics, we call it phase transition.

Problem 17 Let $b, p$ be constant numbers such that $b>0,0<p<1$ and $b+p<1$. Suppose that the offspring distribution given by

$$
\begin{aligned}
& p_{k}=b p^{k-1}, \quad k=1,2, \ldots, \\
& p_{0}=1-\sum_{k=1}^{\infty} p_{k}
\end{aligned}
$$

(1) Find the generating function $f(s)$ of the off-spring distribution.
(2) Set $m=1$ and find $f_{n}(s)$.

Problem 18 Show your own model based on the Galton-Watson branching process with $m=0.72$ (this is motivated by the total fertility rate of Japan in 2016, that is, 1.44 ). Then, by computer simulation or by numerical computation, estimate the extinction probability of your model.

## 10 Poisson Processes

Let $T \subset \mathbf{R}$ be an interval. A family of random variables $\{X(t) ; t \in T\}$ is called a continuous time stochastic process. We often consider $T=[0,1]$ and $T=[0, \infty)$. As $X(t)$ is a random variable for each $t \in T$, it has another variable $\omega \in \Omega$. When we need to explicitly refer to $\omega$, we write $X(t, \omega)$ or $X_{t}(\omega)$. For fixed $\omega \in \Omega$, the function

$$
t \mapsto X(t, \omega)
$$

is called a sample path of the stochastic process $\{X(t)\}$. It is the central idea of stochastic processes that a random evolution in the real world is expressed by a single sample path selected randomly from all the possible sample paths.

The most fundamental continuous time stochastic processes are the Poisson process and the Brownian motion (Wiener process). In the recent study of mathematical physics and mathematical finance, a kind of composition of these two processes, called the Lévy process (or additive process), has received much attention.

### 10.1 Heuristic Introduction

Let us imagine that the number of objects changes as time goes on. The number at time $t$ is modelled by a random variable $X_{t}$ and we wish to construct a stochastic process $\left\{X_{t}\right\}$. In this case $X_{t}$ takes values in $\{0,1,2, \ldots\}$. In general, such a stochastic process is called a counting process.

There are many different variations of randomness and so wide variations of counting processes. We below consider the simple situation as follows: We focus an event $E$ which occurs repeatedly at random as time goes on. For example,
(i) alert of receiving an e-mail;
(ii) telephone call received by a call center;
(iii) passengers making a queue at a bus stop;
(iv) customers visiting a shop;
(v) occurrence of defect of a machine;
(vi) traffic accident at a corner;
(vii) radiation from an atom.

Let fix a time origin as $t=0$. We count the number of occurrence of the event $E$ during the time interval $[0, t]$ and denote it by $X_{t}$. Let $t_{1}, t_{2}, \ldots$ be the time when $E$ occurs, see Fig. 10.1.


Figure 10.1: Recording when the event $E$ occurs

There are two quantities which we measure.
(i) The number of occurrence of $E$ up to time $t$, say, $X_{t}$. Then $\left\{X_{t} ; t \geq 0\right\}$ becomes a counting process.
(ii) The waiting time of the $n$th occurrence after the $(n-1)$ th occurrence, say, $T_{n}$. Here $T_{1}$ is defined to be the waiting time of the first occurrence of $E$ after starting the observation. Then $\left\{T_{n} ; n=1,2, \ldots\right\}$ is a sequence of random variables taking values in $[0, \infty)$.


Figure 10.2: The counting process and waiting times

We will introduce heuristically a stochastic process $\left\{X_{t}\right\}$ from the viewpoint of (i). It is convenient to start with discrete time approximation. Fix $t>0$ and divide the time interval $[0, t]$ into $n$ small intervals. Let

$$
\Delta t=\frac{t}{n}
$$

be the length of the small intervals and number from the time origin in order.


We assume the following conditions on the occurrence of the event $E$ :
(1) There exists a constant $\lambda>0$ such that
$P(E$ occurs just once in a small time interval of length $\Delta t)=\lambda \Delta t+o(\Delta t)$,
$P(E$ does not occur in a mall time interval of length $\Delta t)=1-\lambda \Delta t+o(\Delta t)$,
$P(E$ occurs more than once in a small time interval of length $\Delta t)=o(\Delta t)$.
(2) Occurrence of $E$ in disjoint time intervals is independent.

Some more accounts. Let us imagine the alert of receiving an e-mail. That
$P(E$ occurs more than once in a small time interval of length $\Delta t)=o(\Delta t)$
means that two occurrences of the event $E$ is always separated. That
$P(E$ occurs just once in a small time interval of length $\Delta t)=\lambda \Delta t+o(\Delta t)$
means that when $\Delta t$ is small the probability of occurrence of $E$ in a time interval is proportional to the length of the time interval. We understand from (2) that occurrence of $E$ is independent of the past occurrence.

Let $Z_{i}$ denote the number of occurrence of the event $E$ in the $i$ th time interval. Then $Z_{1}, Z_{2}, \ldots, Z_{n}$ become a sequence of independent random variables with an identical distribution such that

$$
P\left(Z_{i}=0\right)=1-\lambda \Delta t+o(\Delta t), \quad P\left(Z_{i}=1\right)=\lambda \Delta t+o(\Delta t), \quad P\left(Z_{i} \geq 2\right)=o(\Delta t)
$$

The number of occurrence of $E$ during the time interval $[0, t]$ is given by

$$
\sum_{i=1}^{n} Z_{i}
$$

The length $\Delta t$ is introduced for a technical reason and is not essential in the probability model so letting $\Delta t \rightarrow 0$ or equivalently $n \rightarrow \infty$, we define $X_{t}$ by

$$
\begin{equation*}
X_{t}=\lim _{\Delta t \rightarrow 0} \sum_{i=1}^{n} Z_{i} \tag{10.1}
\end{equation*}
$$

Although the limit does require mathematical justification, we obtain heuristically a continuous time stochastic process $\left\{X_{t}\right\}$, which gives the number of occurrence of the event $E$ up to time $t$. This is called a Poisson process with parameter $\lambda>0$. A Poisson process belongs to the calss of continuous time Markov chains.

Theorem 10.1.1 A Poisson process $\left\{X_{t} ; t \geq 0\right\}$ satisfies the following properties:
(1) (counting process) $X_{t}$ takes vales in $\{0,1,2, \ldots\}$;
(2) $X_{0}=0$;
(3) (monotone increasing) $X_{s} \leq X_{t}$ for $0 \leq s \leq t$;
(4) (independent increment) if $0 \leq t_{1}<t_{2}<\cdots<t_{k}$, then

$$
X_{t_{2}}-X_{t_{1}}, \quad X_{t_{3}}-X_{t_{2}}, \quad \ldots, \quad X_{t_{k}}-X_{t_{k-1}}
$$

are independent;
(5) (stationarity) for $0 \leq s<t$ and $h \geq 0$, the distributions of $X_{t+h}-X_{s+h}$ and $X_{t}-X_{s}$ are identical;
(6) $X_{t}$ obeys the Poisson distribution with parameter $\lambda t$.
(7) there exists a constant $\lambda>0$ such that

$$
P\left(X_{h}=1\right)=\lambda h+o(h), \quad P\left(X_{h} \geq 2\right)=o(h)
$$

Proof (1) Since $X_{t}$ obeys the Poisson distribution with parameter $\lambda t$, it takes values in non-negative integers almost surely.
(2) Obvious by definition.
(3) Let $s=m \Delta t, t=n \Delta t, m<n$. Then we have obviously

$$
X_{s}=\lim _{\Delta t \rightarrow 0} \sum_{i=1}^{m} Z_{i} \leq \lim _{\Delta t \rightarrow 0} \sum_{i=1}^{n} Z_{i}=X_{t}
$$

(4) Suppose $t_{1}=n_{1} \Delta t, \ldots, t_{k}=n_{k} \Delta t$ with $n_{1}<\cdots<n_{k}$. Then we have

$$
X_{t_{2}}-X_{t_{1}}=\lim _{\Delta t \rightarrow 0} \sum_{i=1}^{n_{2}} Z_{i}-\lim _{\Delta t \rightarrow 0} \sum_{i=1}^{n_{1}} Z_{i}=\lim _{\Delta t \rightarrow 0} \sum_{i=n_{1}+1}^{n_{2}} Z_{i}
$$

In other words, $X_{t_{2}}-X_{t_{1}}$ is the sum of $Z_{i}$ 's corresponding to the small time intervals contained in [ $t_{2}, t_{1}$ ). Hence, $X_{t_{2}}-X_{t_{1}}, \ldots, X_{t_{k}}-X_{t_{k-1}}$ are the sums of $Z_{i}$ 's and there is no common $Z_{i}$ appearing in the summands. Since $\left\{Z_{i}\right\}$ are independent, so are $X_{t_{2}}-X_{t_{1}}, \ldots, X_{t_{k}}-X_{t_{k-1}}$.
(5) Since $X_{t+h}-X_{s+h}$ and $X_{t}-X_{s}$ are defined from the sums of $Z_{i}$ 's and the numbers of the terms coincide. Therefore the distributions are the same.
(6) We note that

$$
P\left(\sum_{i=1}^{n} Z_{i}=k\right)=\binom{n}{k}(\lambda \Delta t)^{k}(1-\lambda \Delta t)^{n-k}+o(\Delta t)
$$

In other words, $X_{t}$ obeys apporoximately $B(n, \lambda \Delta t)$. In view of $\Delta t=t / n$ we let $n$ tend to the infinity and obtain

$$
P\left(X_{t}=k\right)=\lim _{\Delta t \rightarrow 0} \frac{(\lambda t)^{k}}{k!} \frac{n(n-1) \ldots(n-k+1)}{n^{k}}\left(1-\frac{\lambda t}{n}\right)^{n-k}=\frac{(\lambda t)^{k}}{k!} e^{-\lambda t}
$$

This proves the assertion.
(7) Since $X_{h}$ obeys the Poisson distribution with parameter $\lambda h$, we have

$$
\begin{aligned}
& P\left(X_{h}=0\right)=e^{-\lambda h}=1-\lambda h+\cdots=1-\lambda h+o(h) \\
& P\left(X_{h}=1\right)=\lambda h e^{-\lambda h}=\lambda h(1-\lambda h+\ldots)=\lambda h+o(h)
\end{aligned}
$$

Therefore we have

$$
P\left(X_{h} \geq 2\right)=1-P\left(X_{h}=0\right)-P\left(X_{h}=1\right)=o(h)
$$

Remark 10.1.2 The essence of the above argument in (7) is the Poisson's law of small numbers which says that the binomial distribution $B(n, p)$ is approximated by Poisson distribution with parameter $\lambda=n p$ when $n$ is large and $p$ is small. The following table shows the distributions of $B(100,0.02)$ and the Poisson distribution with parameter $\lambda=2$.

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | $\cdots$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Binomial | 0.1326 | 0.2707 | 0.2734 | 0.1823 | 0.0902 | 0.0353 | 0.0114 | $\cdots$ |
| Poisson | 0.1353 | 0.2707 | 0.2707 | 0.1804 | 0.0902 | 0.0361 | 0.0120 | $\cdots$ |

Example 10.1.3 The average number of customers visiting a certain service gate is two per minute. Using the Poisson model, calculate the following probabilities.
(1) The probability that no customer visits during the first two minutes after the gate opens.
(2) The probability that no customer visits during a time interval of two minutes.
(3) The probability that no customer visits during the first two minutes after the gate opens and that two customers visit during the next one minute.

Let $X_{t}$ be the number of visitors up to time $t$. By assumption $\left\{X_{t}\right\}$ is a Poisson process with parameter $\lambda=2$.
(1) We need to calculate $P\left(X_{2}=0\right)$. Since $X_{2}$ obeys the Poisson distribution with parameter $2 \lambda=4$, we have

$$
P\left(X_{2}=0\right)=\frac{4^{0}}{0!} e^{-4} \approx 0.018
$$

(2) Suppose that the time interval starts at $t_{0}$. Then the probability under discussion is given by $P\left(X_{t_{0}+2}-\right.$ $X_{t_{0}}=0$ ). By stationarity we have

$$
P\left(X_{t_{0}+2}-X_{t_{0}}=0\right)=P\left(X_{2}-X_{0}=0\right)=P\left(X_{2}=0\right)
$$

which coincides with (1).
(3) We need calculate the probability $P\left(X_{2}=0, X_{3}-X_{2}=2\right)$. Since $X_{2}$ and $X_{3}-X_{2}$ are independent,

$$
P\left(X_{2}=0, X_{3}-X_{2}=2\right)=P\left(X_{2}=0\right) P\left(X_{3}-X_{2}=2\right) .
$$

By stationarity we have

$$
=P\left(X_{2}=0\right) P\left(X_{1}=2\right)=\frac{4^{0}}{0!} e^{-4} \times \frac{2^{2}}{2!} e^{-2} \approx 0.00496 .
$$

Problem 19 Let $\left\{X_{t}\right\}$ be a Poisson process. Show that

$$
P\left(X_{s}=k \mid X_{t}=n\right)=\binom{n}{k}\left(\frac{s}{t}\right)^{k}\left(1-\frac{s}{t}\right)^{n-k}, \quad k=0,1, \ldots, n,
$$

for $0<s<t$. Then give an intuitive explanation of the above formula.
Problem 20 How often do you receive a message through internet? Check your iPhone (or other similar devises) and collect the data focussing on how many messages are received during one hour. Probably you must check the records during at least 1 week ( 168 hours).
(1) Make up a table that shows the relative frequency of the number of received messages per one hour.
(2) Calculate the mean value and variance of the number of messages received during one hour. (If they are close each other, your data should well fit to the Poisson distribution.)
(3) Suppose that your data obeys Poisson distribution. Using the Poisson model, find the probability that 0 messages arrive during 1 hour and 2 messages arrive during the next 2 hours.

### 10.2 Waiting Time

Let $\left\{X_{t} ; t \geq 0\right\}$ be a Poisson process with parameter $\lambda$. By definition $X_{0}=0$ and $X_{t}$ increases by one as time passes. Recall that the Poisson process counts the number of events occurring up to time $t$. First we set

$$
\begin{equation*}
T_{1}=\inf \left\{t \geq 0 ; X_{t} \geq 1\right\} . \tag{10.2}
\end{equation*}
$$

This is the waiting time for the first occurrence of the event $E$. Let $T_{2}$ be the waiting time for the second occurrence of the event $E$ after the first occurrence, i.e.,

$$
T_{2}=\inf \left\{t \geq 0 ; X_{t} \geq 2\right\}-T_{1} .
$$

In a similar fashion, we set

$$
\begin{equation*}
T_{n}=\inf \left\{t \geq 0 ; X_{t} \geq n\right\}-T_{n-1}, \quad n=2,3, \ldots \tag{10.3}
\end{equation*}
$$

Theorem 10.2.1 Let $\left\{X_{t}\right\}$ be a Poisson process with parameter $\lambda$. Define the waiting time $T_{n}$ by (10.2) and (10.3). Then, $\left\{T_{n} ; n=1,2, \ldots\right\}$ becomes a sequence of iid random variables, of which distribution is the exponential distribution with parameter $\lambda$. In particular, the waiting time for occurence of an event in the Poisson process obeys the exponential distribution with parameter $\lambda$.

Proof Set $t=n \Delta t$ and consider the approximation by refinement of the time interval. Recall that to each small time interval of length $\Delta t$ a random variable $Z_{i}$ is associated. Then we know that

$$
P\left(T_{1}>t\right)=\lim _{\Delta t \rightarrow 0} P\left(Z_{1}=\cdots=Z_{n}=0\right)=\lim _{\Delta t \rightarrow 0}(1-\lambda \Delta t)^{n}=\lim _{\Delta t \rightarrow 0}\left(1-\frac{\lambda t}{n}\right)^{n}=e^{-\lambda t} .
$$

Therefore,

$$
P\left(T_{1} \leq t\right)=1-e^{-\lambda t}=\int_{0}^{t} \lambda e^{-\lambda s} d s
$$

which shows that $T_{1}$ obeys the exponential distribution with parameter $\lambda$. The distributions of $T_{2}, T_{3}, \ldots$ are similar.

Remark 10.2.2 Let $\left\{X_{t}\right\}$ be a Poisson process with parameter $\lambda$. We know that $\mathbf{E}\left(X_{1}\right)=\lambda$, which means the average number of occurrence of the event during the unit time interval. Hence, it is expected that the average waiting time between two occurrences is $1 / \lambda$. Theorem 10.2 . 1 says that the waiting time obeys the exponential distribution with parameter $\lambda$ so its mean value is $1 / \lambda$. Thus, our rough consideration gives the correct answer.

Problem 21 Let $\left\{X_{t}\right\}$ be a Poisson process with parameter $\lambda$. The waiting time for $n$ occurrence of the events is defined by $S_{n}=T_{1}+T_{2}+\cdots+T_{n}$, where $T_{n}$ is given in Theorem 10.2.1. Calculate $P\left(S_{2} \leq t\right)$ and find the probability density function of $S_{2}$. [In general, $S_{n}$ obeys a gamma distribution.]

### 10.3 The Rigorous Definition of Poisson Processes

The "definition" of a Poisson process in (10.1) is intuitive and instructive for modeling random phenomena. However, strictly speaking, the argument is not sufficient to define a stochastic process $\left\{X_{t}\right\}$. For example, the probability space $(\Omega, \mathcal{F}, P)$ on which $\left\{X_{t}\right\}$ is defined is not at all clear.

We need to start with the waiting time $\left\{T_{n}\right\}$. First we prepare a sequence of iid random variables $\left\{T_{n} ; n=\right.$ $1,2, \ldots\}$, of which the distribution is the exponential distribution with parameter $\lambda>0$. Here the probability space $(\Omega, \mathcal{F}, P)$ is clearly defined. Next we set

$$
S_{0}=0, \quad S_{n}=T_{1}+\cdots+T_{n}, \quad n=1,2, \ldots,
$$

and for $t \geq 0$,

$$
X_{t}=\max \left\{n \geq 0 ; S_{n} \leq t\right\} .
$$

It is obvious that for each $t \geq 0, X_{t}$ is a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. In other words, $\left\{X_{t} ; t \geq 0\right\}$ becomes a continuous time stochastic process. This is called Poisson process with parameter $\lambda$ by definition.

Starting with the above definition one can prove the properties in mentioned Theorem 10.1.1.

### 10.4 Generating Function

Let $\left\{X_{t}\right\}$ be a Poisson process with parameter $\lambda>0$. For simplicity, we set

$$
p_{n}(t)=P\left(X_{t}=n\right), \quad t \geq 0, \quad n=0,1,2, \ldots .
$$

By observing the transition we have

$$
\begin{aligned}
& p_{n}(t+\Delta t)=\lambda \Delta t p_{n-1}(t)+(1-\lambda \Delta t) p_{n}(t), \quad n \geq 1, \\
& p_{0}(t+\Delta t)=(1-\lambda \Delta t) p_{0}(t) .
\end{aligned}
$$

Letting $\Delta t \rightarrow 0$, we have

$$
\begin{aligned}
& p_{n}^{\prime}(t)=\lambda p_{n-1}(t)-\lambda p_{n}(t), \quad n \geq 1, \\
& p_{0}^{\prime}(t)=-\lambda p_{0}(t) .
\end{aligned}
$$

The last equation is easily solved:

$$
p_{0}(t)=e^{-\lambda t},
$$

where the initial condition $p_{0}(0)=P\left(X_{0}=0\right)=0$ is taken into account.
Define the generating function by

$$
f(x, t)=\sum_{n=0}^{\infty} p_{n}(t) x^{n} .
$$

Then we have

$$
\begin{aligned}
\frac{\partial f}{\partial t} & =\sum_{n=0}^{\infty} p_{n}^{\prime}(t) x^{n}=-\lambda p_{0}(t)+\sum_{n=1}^{\infty}\left\{\lambda p_{n-1}(t) x^{n}-\lambda p_{n}(t) x^{n}\right\} \\
& =-\lambda \sum_{n=0}^{\infty} p_{n}(t) x^{n}+\lambda x \sum_{n=0}^{\infty} p_{n}(t) x^{n} \\
& =(-\lambda+\lambda x) f(x, t) .
\end{aligned}
$$

Moreover,

$$
f(0, t)=p_{0}(t)=e^{-\lambda t}
$$

Thus,

$$
f(x, t)=e^{(-\lambda+\lambda x) t}=e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!} x^{n},
$$

from which we have

$$
p_{n}(t)=e^{-\lambda t} \frac{(\lambda t)^{n}}{n!}
$$

## 11 Queueing Theory

### 11.1 Modeling Queues

In our daily life, we observe often waiting lines or queues of customers for services. Agner Krarup Erlang (1878-1929, Danish engineer at the Copenhagen Telephone Exchange) published in 1909 the paper entitled: The Theory of Probabilities and Telephone Conversations, which opened the door to the research field of queueing theory. Such a queue is modeled in terms of a system consisting of servers and a waiting room. Customers arriving at the system are served at once if there is an idle server. Otherwise, the customer waits for a vacant server in a waiting room. After being served, the customer leaves the system.


In most of the qeueing models, a customer arrives at random and the service time is also random. So we are interested in relevant statistics such as
(1) sojourn time (time of a customer staying in the system)
(2) waiting time (= sojourn time - service time)
(3) the number of customers in the system

Apparently, many different conditions may be introduced for the queueing system. In 1953, David G. Kendall introduced the so-called Kendall's notation

$$
A / B / c / K / m / Z
$$

for describing the characteristics of a queuing model, where
A: arrival process,
$B$ : service time distribution,
$c$ : number of servers,
$K$ : number of places in the system (or in the waiting room),
$m$ : calling population,
Z: queue's discipline or priority order, e.g., FIFO (First In First Out)
The first model analyzed by Erlang in 1909 was the $M / D / 1$ queue in Kendall's notation, where $M$ means that arrivals occur according to a Poisson process, and $D$ stands for deterministic (i.e., service time is not random but constant).

Most of queueing models are classified into four categories by the behavior of customers as follows:
（I）Delay models：customers wait in line until they can be served．
Example：$M / M / c$ queue，where
（i）customers arrive according to a Poisson process with rate $\lambda$ ；
（ii）there are $c$ servers and there is an infinite waiting space；
（iii）each customer requires an exponential service time with mean $1 / \mu$ ；
（iv）customers who upon arrival find all servers busy wait in line to be served．
（II）Loss models：customers leave the system when they find all servers busy upon arrival．
Example：Erlang＇s loss model $M / G / c / c$ ，where
（i）customers arrive according to a Poisson process with rate $\lambda$ ；
（ii）there are $c$ servers and the capacity of the system is limited to $c$ customers，i．e．，there is no waiting space；
（iii）each customer requires a generally distributed service time；
（iv）customers who upon arrival find all servers busy are rejected forever．
（III）Retrial models：customers who do not find an idle server upon arrival leave the system only tem－ porarily，and try to reenter some random time later．

Example：the Palm／Erlang－A queue，where
（i）customers arrive according to a Poisson process with rate $\lambda$ ；
（ii）there are $c$ servers and there is an infinite waiting space；
（iii）each customer requires an exponential service time with mean $1 / \mu$ ；
（iv）customers who upon arrival find all servers busy wait in line to be served；
（v）customers wait in line only an exponentially distributed time with mean $1 / \theta$（patience time）．
（IV）Abandonment models：customers waiting in line will leave the system before being served after their patience time has expired．

## References

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## $11.2 \quad M / M / 1$ Queue

This is the most fundamental model，which satisfies the following conditions：
（i）arrivals occur according to a Poisson process with parameter $\lambda$ ；
（ii）service times obey an exponential distribution with parameter $\mu$ ；
（iii）arrivals of customers and service times are independent；
（iii）the system contains a single server；
（iv）the size of waiting room is infinite；
（v）$($ FIFO $=$ First In First Out）customers are served from the front of the queue，i．e．，according to a first－come，first－served discipline．

Thus there are two parameters characterizing an $M / M / 1$ queue, that is, the parameter $\lambda>0$ for the Poisson arrival and the one $\mu>0$ for the exponential service. In other words, a customer arrives at the system with average time interval $1 / \lambda$ and the average service time is $1 / \mu$. In the queuing theory $\lambda$ is called the mean arrival rate and $\mu$ the mean service rate. Let $X(t)$ be the number of customers in the system at time $t$. It is the proved that $\{X(t) ; t \geq 0\}$ becomes a continuous time Markov chain on $\{0,1,2,3, \ldots\}$. In fact, the letter "M" stands for "Markov" or "memoryless".

Our main objective is

$$
p_{n}(t)=P(X(t)=n \mid X(0)=0)
$$

i.e., the probabbility of finding $n$ customers in the system at time $t>0$ subject to the initial condition $X(0)=0$. Let us consider the change of the system during the small time interval $[t, t+\Delta t]$. It is assumed that during the small time interval $\Delta t$ only one event happens, namely, a new customer arrives, a customer under service leaves the system, or nothing changes. The probabilities of these events are given by $\lambda \Delta t, \mu \Delta t$, $1-\lambda \Delta t-\mu \Delta t$.


Therefore, $P(X(t)=n \mid X(0)=0)$ fulfills the following equation:

$$
\begin{array}{rl}
P(X(t+\Delta t)=n \mid X(0)=0)=P & X(t+\Delta t)=n \mid X(t)=n-1) P(X(t)=n-1 \mid X(0)=0) \\
& +P(X(t+\Delta t)=n \mid X(t)=n) P(X(t)=n \mid X(0)=0) \\
& +P(X(t+\Delta t)=n \mid X(t)=n+1) P(X(t)=n+1 \mid X(0)=0) \\
=\lambda \Delta t & P(X(t)=n-1 \mid X(0)=0) \\
& +(1-\lambda \Delta t-\mu \Delta t) P(X(t)=n \mid X(0)=0) \\
& +\mu \Delta t P(X(t)=n+1 \mid X(0)=0) \\
P(X(t+\Delta t)=0 \mid X(0)=0)=(1-\lambda \Delta t) P(X(t)=0 \mid X(0)=0)+\mu \Delta t P(X(t)=1 \mid X(0)=0)
\end{array}
$$

Hence for $p_{n}(t)=P(X(t)=n \mid X(0)=0)$ we have

$$
\begin{align*}
& p_{n}^{\prime}(t)=\lambda p_{n-1}(t)-(\lambda+\mu) p_{n}(t)+\mu p_{n+1}(t), \quad n=1,2, \ldots,  \tag{11.1}\\
& p_{0}^{\prime}(t)=-\lambda p_{0}(t)+\mu p_{1}(t)
\end{align*}
$$

The initial condition is as follows:

$$
\begin{equation*}
p_{0}(0)=1, \quad p_{n}(0)=0 \quad \text { for } n \geq 1 \tag{11.2}
\end{equation*}
$$

Solving the linear system (??) with the initial condition (??) is not difficult with the help of linear algebra and spectral theory. However, the explicit solution is not so simple and is omitted. We only mention that most important characteristics are obtained from the explicit $p_{n}(t)$.

Here we focus on the equilibrium solution (limit transition probability), i.e.,

$$
p_{n}=\lim _{t \rightarrow \infty} p_{n}(t)
$$

whenever the limit exists. Since in the equilibrium the derivative of the left hand side of (??) is 0 , we have

$$
\begin{align*}
& \lambda p_{n-1}-(\lambda+\mu) p_{n}+\mu p_{n+1}=0 \quad n=1,2, \ldots,  \tag{11.3}\\
& -\lambda p_{0}+\mu p_{1}=0
\end{align*}
$$

A general solution to (??) is easily derived:

$$
p_{n}= \begin{cases}C_{1}+C_{2}\left(\frac{\lambda}{\mu}\right)^{n}, & \lambda \neq \mu, \\ C_{1}+C_{2} n, & \lambda=\mu\end{cases}
$$

Since $p_{n}$ gives a probability distribution, we have $p_{n} \geq 0$ and $\sum_{n=0}^{\infty} p_{n}=1$. This occurs only when $\lambda<\mu$ and we have

$$
p_{n}=\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{n}, \quad n=0,1,2, \ldots .
$$

This is the geometric distribution with parameter $\lambda / \mu$.
In queuing theory, the ratio of the mean arrival rate $\lambda$ and the mean service rate $\mu$ is called the utilization:

$$
\rho=\frac{\lambda}{\mu} .
$$

Utilization stands for how busy the system is. It was shown above that the number of customers in the system after long time obeys the geometric distribution with parameter $\rho$. If $\rho<1$, the system functions well. Otherwise, the queue will continue to grow as time goes on. After long time, i.e., in the equilibrium the number of customers in the system obeys the geometric distribution:

$$
(1-\rho) \rho^{n}, \quad n=0,1,2, \ldots
$$

In particular, the probability that the server is free is $1-\rho$ and the probability that the server is busy and the customer need to wait is $\rho$. This is the origin of the term utilization. Note also that the mean number of the customers in the system is given by

$$
\sum_{n=0}^{\infty} n p_{n}=\frac{\rho}{1-\rho}=\frac{\lambda}{\mu-\lambda}
$$

Example 11.2.1 There is an ATM, where each customer arrives with average time interval 5 minutes and spends 3 minutes in average for the service. Using an $M / M / 1$ queue, we know some statistical characteristics. We set

$$
\lambda=\frac{1}{5}, \quad \mu=\frac{1}{3}, \quad \rho=\frac{\lambda}{\mu}=\frac{3}{5} .
$$

Then the probability that the ATM is free is $p_{0}=1-\rho=\frac{2}{5}$. The probability that the ATM is busy but there is no waiting customer is

$$
p_{1}=\frac{2}{5} \times \frac{3}{5}=\frac{6}{25}
$$

Hence the probability that the ATM is busy and there is some waiting customers is

$$
1-p_{0}-p_{1}=1-\frac{2}{5}-\frac{6}{25}=\frac{9}{25}=0.36
$$

So, roughly speaking, a customer needs to make a queue once per three visits.

Problem $22(M / M / 1 / 1$ queue) There is a single server and no waiting space. Customers arrive according to the Poisson process with parameter $\lambda$, and their service time obeys the exponential distribution with parameter $\mu$. Let $Q(t)$ be the number of customers in the system at time $t$. In fact,

$$
Q(t)= \begin{cases}1, & \text { server is busy } \\ 0, & \text { server is idle }\end{cases}
$$

(1) Find

$$
\begin{aligned}
& p_{0}(t)=P(Q(t)=0 \mid Q(0)=0) \\
& p_{1}(t)=P(Q(t)=1 \mid Q(0)=0)
\end{aligned}
$$

by solving a linear system satisfied by those $p_{0}(t)$ and $p_{1}(t)$.
(2) Using the results in (1), calculate

$$
\bar{p}_{0}=\lim _{t \rightarrow \infty} p_{0}(t), \quad \bar{p}_{1}=\lim _{t \rightarrow \infty} p_{1}(t)
$$

(3) Find the mean number of customers in the system in the long time limit:

$$
\lim _{t \rightarrow \infty} \mathbf{E}[Q(t) \mid Q(0)=0] .
$$

### 11.3 Birth-and-Death Processes

The Markov process $X(t)$ appearing in the $M / M / 1$ queuing model is studied more generally within the framework of birth-and-death process.

