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A DISCRETE-TIME POPULATION DYNAMICS MODEL FOR THE INFORMATION SPREAD UNDER THE EFFECT OF SOCIAL RESPONSE[a]

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In this paper, we construct and analyze a mathematically reasonable and simplest population dynamics model based on Mark Granovetter's idea for the spread of a matter (rumor, innovation, psychological state, etc.) in a population. The model is described by a one-dimensional difference equation. Individual threshold values with respect to the decision-making on the acceptance of a spreading matter are distributed throughout the population ranging from low (easily accepts it) to high (hardly accepts). Mathematical analysis on our model with some general threshold distributions (uniform; monotonically decreasing/increasing; unimodal) shows that a critical value necessarily exists for the initial frequency of acceptors. Only when the initial frequency of acceptors is beyond the critical, the matter eventually spreads over the population. Further, we give the mathematical results on how the equilibrium acceptor frequency depends on the nature of threshold distribution.

Keywords: Information Spread; Population Dynamics; Difference Equation; Threshold Model; Collective Behavior.

1. Introduction

Mark Granovetter[1] considered a model on the collective behavior for circumstances where people have to make one of two distinct choices such that the merit or demerit in each choice depends on the number of individuals who decide for or against it. When the number of individuals who have taken the decision reaches a threshold,

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the advantages of taking the decision begin to outweigh the disadvantages for a given individual. For instance, a radical who is capable of single-handedly starting a riot can be said to have a threshold of 0% as they are able to riot even if nobody else toes that line. On the other hand, a conservative might have a threshold close to 100% depending on their level of reluctance to join a riot. The principle of threshold is analogous to credulity and vulnerability in the spread of rumors and diseases, respectively. For some background works on this idea, see Refs. 2–4.

Granovetter and Soong[5] emphasized the reality of heterogeneity in collective behavior as opposed to the earlier simplifying assumption of homogeneous individuality and mixing in the adoption and spread of ideas. They showed the importance of threshold models as lying in the not-so-simple connection between individual choices and overall steady results. The work also refers to the importance of bandwagon effect in which people adopt a new concept because a certain number of people are into it and a snob effect in which some people drop the idea once a certain number of people sign up.[6–8] In such a case, there are two threshold values: one minimum inspiring the bandwagon and one maximum leading to snobbish behavior.

As a sociological concept, the Granovetter model has some similarities with the idea of behavioral contagion in psychology and the cultural phenomenon of bandwagon effect. In order to make sense of the concept of social influence, Watts and Dodds[9] reasoned that it can be viewed as a result of decision-making based on a series of binary possibilities. Such threshold models find application in areas like diffusion of innovation, public protests, migration, voting, market trends, international relations and information spread.[10–17]

Various mathematical and computational approaches abound in the literature for studying collective behavior.[18–24] The application of concepts to the study of collective behavior in social systems was seen to be fast emerging.[15,21,25–31] Castellano *et al.*[32] highlighted the relevance of statistical physics to other areas of learning apart from physics. Assuming a network that is random and non-finite with weak connections, Whitney[33] tried to understand diffusion (of information or innovation) on the network using generating functions. The theory proposed is based on a threshold rule which ensures that a node only changes state after a fraction of nearby nodes, surpassing a particular limit, have previously flipped over. Akhmetzhanov *et al.*[34] extended the Granovetter model to consider a network of individuals in a square lattice with each one having a state and a specified threshold for change in behavior. A utility-psychological threshold model based on Granovetter's threshold model was introduced by Ref. 25. The critical shift in phase of group behavior is studied by taking into account rational utility and psychological thresholds under the influence of space and intensity of social network.[16,35,36] We find other interesting approaches and methods in Refs. 26, 27 and 37–44.

Previous models[1,5,25,45] belonging to what could be called *Granovetter's threshold model* were to describe the conceptual process of the spread of a matter in a population, and they may not be regarded as reasonable population dynamics models to govern the temporal variation of the number/frequency of such acceptors of a matter spreading in a population. Actually, the basic Granovetter's threshold model is not necessarily for the explicit temporal variation of population dynamics, while it is usually described as a sequence of acceptor frequencies which is used to be regarded as representing the process in the spread of a matter in a population. Nor is what is called *Linear Threshold Model* (LTM) specifically for the dynamics of a spreading matter on network,[30,46–48] which could be regarded as a descendent line of Granovetter's threshold model. Such previous models lacked the proper definition of time evolution about the number/frequency, because they focused on the outcome of the spreading and its relation to the heterogeneous nature/structure of population as the matrix through which a matter spreads (refer to the discussion in Ref. 48 and references therein). As another line of mathematical models related to the LTM in a mathematical sense, we could find theoretical works on what is called *opinion dynamics* too.[49,50]

As far as we have studied literature of previous works, curiously we could not find any standard rational population dynamics model described as some recurrence relations with a mathematically reasonable modeling for Granovetter's threshold model, whereas it has been described like a discrete-time process as mentioned above. Thus it must be worth being derived because it could serve a basic model for the theoretical consideration in a variety of information spread contexts within a population or community. For an important example, the epidemic dynamics in a community or communities significantly depends on the collective behaviors within the population in which such a transmissible disease spreads.[29,31,51–54] It is necessary and valuable to have a reasonable population dynamics model to consider the nature of temporal change in social environment accompanied by temporal variations in their subpopulation sizes regarding the acceptance of a spreading matter.

In this paper, we are going to revisit Granovetter's idea of collective behavior, and refine the mathematical modeling to derive a mathematically accurate and reasonable population dynamics model which gives the temporal variation of number/frequency of individuals who have accepted a matter spreading in a population. We will construct our population dynamics model with a recurrence relation, that is, a difference equation, following reasonable assumptions as well as Dansu and Seno[55] did for their model with an ordinary differential equation. Differently from their continuous-time population dynamics model, our model is a kind of discrete-time population dynamics model, which corresponds straightforward to Granovetter's threshold model rather than any continuous-time model. We shall mathematically investigate our model in order to consider the nature of the population dynamics governed by it, focusing on how the nature of threshold distribution affects the result of the spread of a matter in a population. Mathematical results obtained for our population dynamics model will show that its mathematical features qualitatively correspond to those of the continuous-time model in Ref. 55 with respect to the temporal variation of acceptor frequency and its equilibrium value.

2. Population Dynamics Modeling

2.1. *Assumptions*

As the idea by Granovetter,[1] we consider here the dichotomous decision-making on a matter spreading in a population, which results in whether an individual accepts the matter or not, with the following primary assumptions:

- Each individual has his/her own criterion for the decision-making on a matter spreading in the population.
- Each individual's criterion is independent of any other's.

It is essential to assume that the individual decision-making follows his/her own criterion. With respect to the criterion, Granovetter used a simplest idea such that the criterion is more likely to be satisfied as the frequency of *acceptors*, that is, the number of individuals who have made the decision to accept the matter gets larger. Since Granovetter[1] considered the emergence of a riot in a population for example, he called the acceptors *instigators*, whereas we shall set an assumption in a more general sense without calling them so. Actually, such the acceptor frequency/number itself could be hardly recognized by each individual in the considered population.

We assume that not the acceptor frequency itself but the social environment reflecting the acceptor frequency affects the decision-making by each individual (non-acceptor) with respect to the acceptance of a matter spreading in the population. Such an influence by the social environment could become effective by receiving some pieces of information on the spreading matter through mass media, SNS (e.g., X, Facebook, WhatsApp), daily sight, etc. For example as a matter spreading in a population, we may consider a daily custom (e.g., wearing a mask, hand-washing), a fashion, or a rumor. Thus, each individual could not recognize the acceptor frequency itself but get such information influenced by the acceptors. That is, the decision-making significantly depends on the information obtained from the social environment around each individual. We shall call here the influence of such information *social response* according to a matter spreading in a population. Inversely such information could be regarded as an embodiment of the social response. Then Granovetter's idea is newly described in our modeling as follows (Fig. 1):

- The criterion for the decision-making is more likely to be satisfied as the social response becomes stronger.

The social response could be defined more widely, and then it may be regarded as working to suppress the decision-making, though we shall not assume such a suppressive aspect of the social response in this paper. It is introduced here as a definitive factor to determine whether an individual accepts the matter or not, as described above and below.

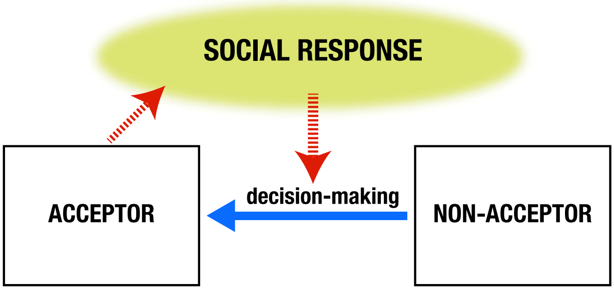

Fig. 1. Schematic relation of the decision-making and social response according to the acceptance of a matter spreading in a population.

With the above primary assumptions, Granovetter's *threshold model* follows the below assumptions for its modeling on the dichotomous decision-making:

(H1) Each individual makes the dichotomous decision about the acceptance of a matter by the threshold value according to the strength of social response;
(H2) The threshold value for each individual is unchanged under the considered population dynamics;
(H3) The strength of social response is proportional to the acceptor frequency;
(H4) A non-accepter may become accepter only when the strength of social response is beyond own threshold value;
(H5) The threshold value in general varies between individuals;
(H6) Each decision-making is independent of the past experience about the spreading matter;
(H7) Any accepter never returns to non-accepter.

In this paper, we are going to construct a population dynamics model about a matter spreading in a population, following these assumptions similar to those in Ref. 55.

Assumption (H1) distinctly indicates that the criterion for the decision-making is introduced as a threshold value about the strength of social response. Assumption (H2) is an assumption on the population dynamics considered here. We may regard the time scale for the population dynamics as sufficiently smaller than that for the temporal change of the threshold value. The threshold value of an individual may temporally change due to a transition of the individuality for some reason. However such a transition of the individuality must generally take a long-term process. With the assumption (H2), the dynamics considered here on the spread of a matter in a population is assumed to be sufficiently faster than such a transition of the individuality.

Assumption (H3) is to theoretically introduce the positive correlation of the strength of social response to the acceptor frequency in the population. Assumption (H4) defines the threshold value for the strength of social response with respect to the decision-making of each individual. Assumption (H5) indicates the inhomogeneity of threshold values among members of the population. Then we may regard the threshold value of each individual as a factor of the individuality. Assumption (H6) simplifies the way of decision-making, ignoring any effect of the past experiences about the matter spreading in the population.

Assumption (H7) indicates that we are going to focus on the process of information spread. From this assumption, *the number of acceptors never decreases in the population dynamics* considered in this paper. This may be regarded as an assumption to consider the dynamics for the spread of a matter only in a period such that every accepter remains to be an acceptor in the population, any transition of the acceptor to the other state which could affect the spread of a matter being negligible. Thus our model would be regarded as for an early stage of the spread of a matter at which such a process of discarding or losing the accepted matter could be negligible. As an example, we may consider an innovation for the daily life or a cultural aspect like a household appliance or agricultural technology, which is kept long-term once it becomes settled in the community life. As another example, we may consider a population of communities, companies, institutes, or countries instead of individuals, where the spreading matter could be a way of systemization, methodology, administration, etc.[11,17,28]

For a consistency in the reasonability of our modeling for a population dynamics with these assumptions, we add the following assumption:

(H8) Any demographic change is negligible in the spread of a considered matter.

The distribution of threshold values in the population could change by the demographic change with birth, death, and migration. In our mathematical modeling described in the following part, the distribution of threshold values is given and temporally unchanged in accordance with the assumption (H2). This means that we consider the population dynamics of a time scale shorter than the time scale of the demographic change which could affect the threshold distribution. In other words, the spread of a matter is sufficiently faster than the demographic change. The assumption (H8) sets the population size constant independently of time under the considered population dynamics for the spread of a matter.

2.2. *Decision-making rule*

Following the assumption (H3) given in the previous section, let denote the strength of social response by αP with the acceptor frequency in the population P $(0 \leq P \leq 1)$ and a positive constant α. The strength of social response may depend not only on the nature of spreading matter but also on the cultural and social environment/background of the population. The parameter α reflects such factors with respect to the spread of the matter in the population. A large α would indicate the social tendency to have a strong interest in the spreading matter.

Now let ξ denote the threshold value for an individual. Then from the assumptions (H1) and (H4), *the individual may make the decision to accept the matter only if* $\xi \leq \alpha P$, *and otherwise the individual never makes the decision or alternatively denies/disregards the matter*.

From this rule of decision-making, if there is an individual with non-positive ξ, such an individual necessarily accepts the matter independently of how strong the social response is. Such individuals may accept the matter even when there is no acceptor of the matter in the population. Hence a specific situation must be set up to have such individuals about the spread of a matter in the population. Here, it does not seem reasonable to take into account such a specific situation in our modeling for the population dynamics about the temporal variation of the number of acceptors.

In contrast, if ξ is greater than α, such an individual never accepts the matter independently of how strong the social response is, because the above modeling means that the supremum of the strength of social response is given by α which is formally the strength of social response when every individual accepts the matter in the population, that is, when $P = 1$. Thus such individuals are out of the population dynamics about the spreading matter, since they never accept the matter.

Therefore, for a simplicity to exclude such individuals with the threshold value non-positive or greater than α whose consequent states are trivial as explained in the above, we are going to consider hereafter a population which consists of only individuals with the threshold value positive and less than α: $\xi \in (0, \alpha)$. In such a population, the decision-making to accept the spreading matter becomes possible only if the above condition is satisfied.

2.3. *Threshold distribution*

In this section, we introduce a threshold distribution in the considered population, following the assumption (H5). It is now given by the following modeling with the distribution function of threshold value ξ in the population:

$$F(x) = \mathrm{Prob}(\xi \leq x) = \int_{-\infty}^{x} f(\xi)\, d\xi, \tag{2.1}$$

where the value of distribution function $F(x) \in [0, 1]$ means the frequency of individuals who have the threshold value ξ equal to or less than x, or the probability that an arbitrarily chosen individual has the threshold value ξ equal to or less than x. The function $f(\xi)$ is what is called *the density distribution function for the threshold* ξ. Since $\xi \in (0, \alpha)$ as assumed in the previous section, these functions satisfy the following nature:

$$F(x) = \begin{cases} 0 & \text{for } x \leq 0; \\ 1 & \text{for } x \geq \alpha, \end{cases} \qquad f(\xi) = \begin{cases} 0 & \text{for } \xi \leq 0; \\ \text{non-negative} & \text{for } \xi \in (0, \alpha); \\ 0 & \text{for } \xi \geq \alpha. \end{cases} \tag{2.2}$$

The frequency of individuals who have the threshold value in the range $[\xi, \xi + d\xi]$ can be now denoted mathematically by $f(\xi)d\xi$.

With the size of considered population N which is constant independently of time by the assumption (H8), the value $NF(x)$ means the number of individuals who have the threshold value less than x. The number of individuals who have the threshold value in the range $[\xi, \xi + d\xi]$ can be denoted by $Nf(\xi)d\xi$ as well. The population is now characterized by the distribution function $F(x)$ or $f(\xi)$ with respect to the threshold ξ about the decision-making for the spreading matter. In our modeling, since the individuality is represented by the threshold value, the density distribution function f can be regarded as an expression of the heterogeneity of individuality in the population.

2.4. *Non-acceptor and acceptor frequencies at time t*

Now for a mathematical convenience to the following description on our modeling, let us introduce the frequencies of acceptors and non-acceptors in the population by P_t and U_t at time t. Since every individual must be alternatively a non-acceptor or acceptor from our assumptions given in Sec. 2.1, we have $U_t + P_t = 1$ for any time t. Then, to make our modeling clearer, we can denote them in the following manner from (2.2):

$$P_t = \int_{-\infty}^{\infty} p(\xi, t)\, d\xi = \int_{0}^{\alpha} p(\xi, t)\, d\xi; \quad U_t = \int_{-\infty}^{\infty} u(\xi, t)\, d\xi = \int_{0}^{\alpha} u(\xi, t)\, d\xi, \tag{2.3}$$

where $p(\xi, t)$ and $u(\xi, t)$ are the density distribution functions of acceptors and non-acceptors for the threshold value $\xi \in \mathbb{R}$, respectively, at time t. They necessarily take non-negative values for any $\xi \in \mathbb{R}$ and $t \geq 0$. The frequencies of acceptors and non-acceptors who have the threshold value in the range $[\xi, \xi + d\xi]$ at time t can be now denoted mathematically by $p(\xi, t)d\xi$ and $u(\xi, t)d\xi$, respectively. Thus, from the definitions of $f(\xi)$, $p(\xi, t)$, and $u(\xi, t)$, it holds that $p(\xi, t) + u(\xi, t) = f(\xi)$ for any $\xi \in \mathbb{R}$ and $t \geq 0$.

2.5. *Initial acceptors*

No initial acceptor results in no information spread. Setting aside the decision-making rule given in Sec. 2.2, we introduce the initial frequency of acceptors $P_0 > 0$ as

$$P_0 = \int_{-\infty}^{\infty} p(\xi, 0)\, d\xi = \int_{0}^{\alpha} p(\xi, 0)\, d\xi = \int_{0}^{\alpha} \varphi_0(\xi) f(\xi)\, d\xi, \tag{2.4}$$

where $p(\xi, 0)d\xi = \varphi_0(\xi)f(\xi)d\xi$ means the fraction of individuals who are given as the initial acceptors with the threshold value in $(\xi, \xi + d\xi)$. The value of the introduced function $\varphi_0(\xi)$ of ξ means the fraction of the initial acceptors in the subpopulation which has the threshold value in $(\xi, \xi + d\xi)$. In other words, $\varphi_0(\xi)$ could be regarded as giving the probability that an individual with the threshold value ξ is chosen as an initial acceptor, $0 \leq \varphi_0(\xi) \leq 1$. The number of initial acceptors is then denoted as NP_0.

For a mathematical convenience, we may formally put $\varphi_0(\xi) = 0$ for any $\xi \notin (0, \alpha)$, because no individual has the threshold value out of the range $(0, \alpha)$ as already assumed. Besides, from the mathematical reasonability, we have

$$0 < P_0 \leq \int_{-\infty}^{\infty} f(\xi)\, d\xi = \int_0^{\alpha} f(\xi)\, d\xi = 1,$$

taking account of (2.2) about the threshold distribution function $f(\xi)$.

We have $u(\xi, 0) = f(\xi) - p(\xi, 0) = \{1 - \varphi_0(\xi)\}f(\xi)$ at the same time as the threshold density distribution function of the initial non-acceptors, and then the initial frequency of non-acceptors $U_0 = 1 - P_0$ can be given as

$$U_0 = \int_0^{\alpha} u(\xi, 0)\, d\xi = \int_0^{\alpha} \{f(\xi) - p(\xi, 0)\}\, d\xi = \int_0^{\alpha} \{1 - \varphi_0(\xi)\}f(\xi)\, d\xi.$$

2.6. *Non-acceptors capable of the decision-making at time t*

From the assumption (H7) in Sec. 2.1, any acceptor keeps being an acceptor. Thus, the acceptor subpopulation contains all initial accepters at any time t. Since the initial acceptor is assumed to be chosen independently of the decision-making rule introduced in Sec. 2.2, some of the initial acceptors may have the threshold value greater than αP_t. As illustratively shown in Fig. 2, the frequency of such initial acceptors can be given by

$$P_0[\xi > \alpha P_t] := \int_{\alpha P_t}^{\alpha} p(\xi, 0)\, d\xi = \int_{\alpha P_t}^{\alpha} \varphi_0(\xi) f(\xi)\, d\xi. \tag{2.5}$$

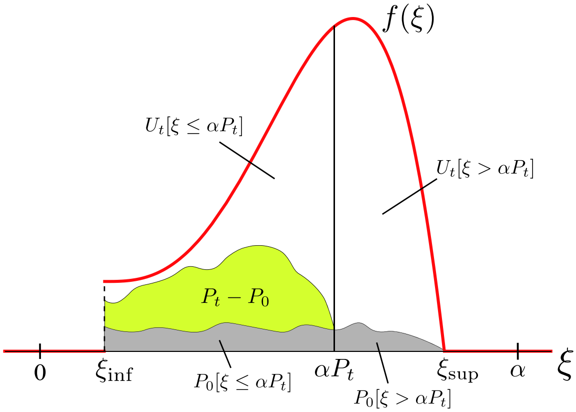

Fig. 2. Schematic figure to illustrate the frequencies of initial acceptors, acceptors and non-acceptors in time step t, where the threshold distribution function $f(t)$ is positive only in $(\xi_{\inf}, \xi_{\sup}) \subseteq (0, \alpha)$. The filled regions are of acceptors, and the blank region is of non-acceptors.

Now, let us define two subpopulations of acceptors at time t according to the threshold value:

$$P_t[\xi > \alpha P_t] := \int_{\alpha P_t}^{\alpha} p(\xi, t)\, d\xi; \quad P_t[\xi \le \alpha P_t] := \int_0^{\alpha P_t} p(\xi, t)\, d\xi, \tag{2.6}$$

where the former is the frequency of acceptors who have the threshold value greater than αP_t at time t, and the latter is that of those who have the threshold value less than or equal to αP_t at time t. Thus, we have $P_t \equiv P_t[\xi > \alpha P_t] + P_t[\xi \le \alpha P_t]$ for any time t in accordance with the definition of P_t given by (2.3).

From the assumptions (H4) and (H7) about the transition from non-acceptor to acceptor, the number of acceptors never decreases as already mentioned in Sec. 2.1. Hence the frequency of acceptors before time t must not be beyond P_t. Besides, as indicated in Fig. 2, the acceptors at time t with the threshold value greater than αP_t must consist of only initial acceptors because any individual with the threshold value greater than αP_t cannot satisfy the condition until time t for the decision-making rule given in Sec. 2.2. Thus, from (2.5) and (2.6), the following equation must hold:

$$P_t[\xi > \alpha P_t] = P_0[\xi > \alpha P_t]. \tag{2.7}$$

On the other hand, the number of individuals who have the threshold value equal to or less than αP_t is given by $NF(\alpha P_t)$, in which the individuals except for the initial acceptors count $NF(\alpha P_t) - NP_0[\xi \le \alpha P_t]$, where

$$P_0[\xi \le \alpha P_t] := \int_0^{\alpha P_t} p(\xi, 0)\, d\xi$$

denotes the initial frequency of acceptors who have the threshold value equal to or less than αP_t, being in accordance with the definition of $P_0[\xi > \alpha P_t]$ given by (2.5). We have $P_0 \equiv P_0[\xi > \alpha P_t] + P_0[\xi \le \alpha P_t]$ for any time t in accordance with the definition of P_0 given by (2.4). The individuals in $NF(\alpha P_t) - NP_0[\xi \le \alpha P_t]$ were initially non-acceptors who have the threshold value equal to or less than αP_t at time t. Some of them have become acceptors while others are still non-acceptors until time t since the process to change a non-acceptor to acceptor takes a certain time in the population dynamics.

For the same reason, the non-acceptor subpopulation at time t consists of two kinds of individuals: Ones who have the threshold value beyond αP_t, and the others who have the threshold value equal to or less than αP_t, which correspond to the non-acceptor frequencies $U_t[\xi > \alpha P_t]$ and $U_t[\xi \le \alpha P_t]$, respectively, (see Fig. 2):

$$U_t[\xi > \alpha P_t] := \int_{\alpha P_t}^{\alpha} u(\xi, t)\, d\xi; \quad U_t[\xi \le \alpha P_t] := \int_0^{\alpha P_t} u(\xi, t)\, d\xi, \tag{2.8}$$

where we have $U_t \equiv U_t[\xi > \alpha P_t] + U_t[\xi \le \alpha P_t]$ for any time t in accordance with the definition of U_t given by (2.3). The subpopulation corresponding to $U_t[\xi \le \alpha P_t]$ consists of individuals who have not yet made the decision even with the threshold value equal to or less than αP_t at time t. From the definitions of (2.1) and (2.8) now we have also the equality

$$F(\alpha P_t) = U_t[\xi \leq \alpha P_t] + P_t[\xi \leq \alpha P_t], \tag{2.9}$$

which holds for any time t (see Fig. 2). This equality follows the assumption that every individual is either a non-acceptor or acceptor.

Only non-acceptors of $NU_t[\xi \leq \alpha P_t]$ are capable of the decision-making to become the acceptor before the next time step $t+1$, while some of them remain non-acceptors even at time $t+1$. This is because the decision-making needs an occasion for it, and such an occasion does not necessarily arise for a non-acceptor in the unit time step. In the following section, we are going to introduce a reasonable modeling about the availability of such an occasion for the decision-making following the assumptions.

2.7. *Recruitment of acceptors in the unit time step*

The decision-making to become an acceptor is possible only if its occasion arises for a non-acceptor. In accordance with the assumptions (H1) and (H3), we now assume that such an occasion is more likely to arise as the social response is stronger, so that it depends on the acceptor frequency in the population.

First, we introduce the probability $B(P) \in [0, 1)$ that a non-acceptor gets an occasion for the decision-making in the unit time step when the acceptor frequency is P in the population. This probability is applied only for the non-acceptor who may make the decision with a threshold value equal to or less than the strength of social response, following the decision-making rule given in Sec. 2.2. As already mentioned above and in the earlier part of Sec. 2.1 to make the assumption (H3), this probability could be assumed to have a positive correlation to the acceptor frequency, since the larger frequency P leads to the higher availability of the information about the spreading matter in the population. Thus, we reasonably assume that $B(P)$ is non-decreasing and almost everywhere differentiable for $P \in (0, 1)$. In addition, since we are going to consider the spread of a matter driven by a population dynamics itself, we assume also that $B(0) = 0$ and $B(P) \in (0, 1)$ for $P \in (0, 1)$.

Next, since some non-acceptors may not make the decision even when such an occasion arises, we additionally introduce the probability $\gamma \in (0, 1]$ that the non-acceptor makes the decision in the unit time step when the occasion arises. In this paper, we give the probability γ as a constant independent of the threshold value ξ and any other factors in the population dynamics. With probability $1 - \gamma$, a non-acceptor does not make the decision to accept the matter in the unit time step even though the non-acceptor has a threshold value equal to or less than the strength of social response. Such a case may be regarded as a certain situation in which the individual could not care about the matter for some reason. If $\gamma = 1$, the non-acceptor necessarily makes the decision in the unit time step when the occasion arises. The probabilities $B(P)$ and γ may depend on the nature of the spreading matter and the social environment in the population.

It is essential to introduce these probabilities for the reasonable mathematical modeling of a population dynamics to describe the temporal variation of a matter spreading in a community, which can be mathematically regarded as the same idea as one used in Ref. 48. As indicated by Refs. 30 and 48, many previous threshold models with Granovetter's idea are constructed with the deterministic decision-making in a sense: Any non-acceptor becomes acceptor at the moment that the condition for the decision-making is satisfied. Although some models with such a mathematical simplification would be useful to illustrate the significance of a threshold distribution about the spread of a matter, it must be disruptive for the reasonable modeling of a population dynamics about the spread of a matter. For this reason, we shall introduce these probabilities for our reasonable modeling, while it will be revealed by our analysis that the detail of the probabilities is not relevant to the principal nature of the spread of a matter. Note that our introduction of these probabilities are not actually for any stochastic modeling, but for making a reasonable deterministic model of a population dynamics, as classic population dynamics models did so.[56]

The number of new acceptors in the unit time step from t to $t+1$ is expressed as $NP_{t+1} - NP_t$. Since the number of non-acceptors with the threshold value equal to or less than αP_t at time t is given by $NU_t[\xi \leq \alpha P_t]$ from the definition of (2.8), we can have the following equality with the probabilities introduced above:

$$NP_{t+1} - NP_t = \gamma B(P_t) NU_t[\xi \leq \alpha P_t] \geq 0, \tag{2.10}$$

which shows the recruitment of acceptors in the unit time step.

2.8. *Temporal variation of acceptor frequency*

From (2.10), we obtain the following recurrence relation to generate the temporal sequence of acceptor frequencies $\{P_t\}$ with $P_0 \in (0, 1]$ given by (2.4):

$$P_{t+1} = \Psi(P_t) := P_t + \gamma B(P_t) U_t[\xi \leq \alpha P_t]. \tag{2.11}$$

As already mentioned in accordance with the assumption (H7), the sequence $\{P_t\}$ must be non-decreasing, which is consistently expressed by the non-negative right side of (2.10). For a mathematical formality and convenience in the subsequent arguments, we put this nature here as the following lemma.

Lemma 2.1. *The sequence* $\{P_t\}$ *generated by* (2.11) *is non-decreasing for any* $P_0 \in (0, 1)$.

Then we can find the following lemma which shows a mathematical result in accordance with the non-decreasing monotonicity of sequence $\{P_t\}$ generated by the recurrence relation (2.11).

Lemma 2.2. *The function* $\Psi(P_t)$ *defining the recurrence relation* (2.11) *is increasing in terms of* $P_t \in [P_0, 1]$.

The proof is given in Appendix A in which we use the following equality derived from the definitions of F, U_t, P_t, and their relations given by (2.7) and (2.9):

$$\begin{aligned} 0 \le U_t[\xi \le \alpha P_t] &= F(\alpha P_t) - P_t[\xi \le \alpha P_t] \\ &= F(\alpha P_t) - \{P_t - P_t[\xi > \alpha P_t]\} \\ &= F(\alpha P_t) - \{P_t - P_0[\xi > \alpha P_t]\} \\ &= F(\alpha P_t) - \left\{P_t - \int_{\alpha P_t}^{\alpha} \varphi_0(\xi) f(\xi)\, d\xi\right\}, \end{aligned} \tag{2.12}$$

for any time $t \ge 0$ (refer also to Fig. 2). The result of Lemma 2.2 will be used as an essential feature of the recurrence relation to generate the sequence $\{P_t\}$ in a proof for its convergence to a value in $(0, 1)$ about some specific cases considered later in this paper.

Substituting (2.12) for (2.10), we finally obtain the following recurrence relation to generate the temporal sequence of acceptor frequencies $\{P_t\}$ with $P_0 \in (0, 1]$ given by (2.4):

$$P_{t+1} = \Psi(P_t) = \{1 - \gamma B(P_t)\}P_t + \gamma B(P_t)\left\{F(\alpha P_t) + \int_{\alpha P_t}^{\alpha} \varphi_0(\xi) f(\xi)\, d\xi\right\}, \tag{2.13}$$

where $0 < \gamma \le 1$, and $B(P)$ is non-decreasing and almost everywhere differentiable for $P \in (0, 1)$, satisfying $B(0) = 0$, $B(P) \in (0, 1)$ for $P > 0$. The functions F and f are defined by (2.1) and (2.2). The probability $\varphi_0(\xi) \in [0, 1]$ is introduced by (2.4) in Sec. 2.5. This recurrence relation gives our discrete-time population dynamics model for the spread of a matter in the considered population.

3. Mathematical Nature of Generic Model

As the mathematical reasonability of the population dynamics model (2.13), we can show the following nature of the sequence $\{P_t\}$ generated by (2.13) (Appendix B).

Lemma 3.1. *For the sequence* $\{P_t\}$ *generated by* (2.13), *it holds that* $P_t \in [P_0, 1)$ *for any* $t \ge 0$ *and any* $P_0 \in (0, 1)$. *If* $P_0 = 0$, *then* $P_t \equiv 0$, *while, if* $P_0 = 1$, *then* $P_t \equiv 1$.

This lemma indicates the invariance of interval $[0, 1]$ with respect to the dynamical system defined by (2.13). Lemmas 2.1 and 3.1 confirm the reasonability of our mathematical modeling to construct the recurrence relation (2.13) as the population dynamics model for the spread of a matter in a population, in accordance with the assumptions given in Sec. 2.1.

Consequently, Lemmas 2.1 and 3.1 have mathematically shown that the sequence $\{P_t\}$ is monotone and bounded. Hence, we obtain the following general result from the Monotone Sequence Theorem.

Theorem 3.1. *The sequence* $\{P_t\}$ *generated by* (2.13) *is non-decreasing and converges to a value* $P^* \in [P_0, 1]$ *for any* $P_0 \in (0, 1)$ *as* $t \to \infty$.

4. Further Modeling Setup

4.1. *Additional assumptions*

For the further discussion on the spread of a matter in a population, we shall add two assumptions for our model (2.13). First let us assume that the initial acceptors are chosen randomly with a common probability *independent of the threshold value*. Then we put $\varphi_0(\xi) = \varphi_0$ with a constant $\varphi_0 \in (0, 1)$. From (2.4), we have the initial frequency of acceptors $P_0 = \varphi_0$.

Second, following the assumption about the probability $B(P)$ that a non-acceptor gets an occasion for the decision-making when the acceptor frequency is P in the population, we put $B(P) = bP$ with a constant $b \in (0, 1)$, satisfying the nature given in Sec. 2.7.

From (2.13), we are going to consider hereafter the following recurrence relation as our population dynamics model to give the temporal sequence of acceptor frequencies $\{P_t\}$:

$$P_{t+1} = \Psi(P_t) = [1 + \gamma b\{\varphi_0 - P_t + (1 - \varphi_0)F(\alpha P_t)\}]P_t. \tag{4.1}$$

To consider how the spread of a matter depends on the heterogeneity of individuality in a population, we shall consider hereafter the threshold distribution defined as

$$f(\xi) = \begin{cases} 0 & \text{for } \xi \in (-\infty, \xi_{\text{inf}}]; \\ f_+(\xi) & \text{for } \xi \in (\xi_{\text{inf}}, \xi_{\text{sup}}); \\ 0 & \text{for } \xi \in [\xi_{\text{sup}}, \infty), \end{cases} \tag{4.2}$$

where $0 \le \xi_{\text{inf}} < \xi_{\text{sup}} \le \alpha$ (see Fig. 2). The function $f_+(\xi)$ is continuous, positive, and almost everywhere differentiable in $(\xi_{\text{inf}}, \xi_{\text{sup}})$, satisfying

$$\int_{\xi_{\text{inf}}}^{\xi_{\text{sup}}} f_+(\xi)\, d\xi = 1.$$

The formula (4.2) matches the definition of density distribution function f in (2.2). When $\xi_{\text{inf}} > 0$ or $\xi_{\text{sup}} < \alpha$, we have $F(x) = 0$ for any $x \in (0, \xi_{\text{inf}}]$ and $F(x) = 1$ for any $x \in [\xi_{\text{sup}}, \alpha)$ since no individual has the threshold value less than ξ_{inf} or greater than ξ_{sup}. As assumed in the last part of Sec. 2.2, we consider only the threshold distribution in $(\xi_{\text{inf}}, \xi_{\text{sup}})$ where $0 \le \xi_{\text{inf}} < \xi_{\text{sup}} \le \alpha$.

4.2. *The least initial frequency of acceptors for the spread of a matter*

We find the following lemma about the existence of the least initial frequency of acceptors for the spread of a matter.

Lemma 4.1. *If* $P_0 = \varphi_0 \leq \theta_{\text{inf}} := \xi_{\text{inf}}/\alpha$, *the acceptor frequency remains the initial frequency* P_0 *with no increase at any time step. Otherwise, it temporally increases.*

Thus there exists the least initial frequency of acceptors to make the matter spread in the population, that is now given by θ_{inf}. For the initial frequency of acceptors not beyond it, the spread of the matter cannot occur at all, while it can occur for the initial frequency of acceptors beyond it. As seen in the subsequent sections, it depends on the nature of threshold distribution how widely the matter can spread in the population.

The proof of Lemma 4.1 is straightforward. If $\alpha P_0 = \alpha\varphi_0 \leq \xi_{\text{inf}}$, we have $F(\alpha P_0) = 0$. Then, from (4.1), we can get

$$P_1 = \Psi(P_0) = [1 + \gamma b\{\varphi_0 - P_0 + (1 - \varphi_0)F(\alpha P_0)\}]P_0 = P_0.$$

Otherwise with $\alpha P_0 > \xi_{\text{inf}}$, we have $F(\alpha P_0) > 0$ from the continuity of density distribution function f. In this case, we can get

$$P_1 = \Psi(P_0) = \{1 + \gamma b(1 - \varphi_0)F(\alpha P_0)\}P_0 > P_0.$$

Then, from Lemma 2.1, the acceptor frequency increases. These arguments lead to Lemma 4.1.

4.3. *The complete spread of a matter*

In contrast to the result of Lemma 4.1, we can find the following lemma about the spread of a matter.

Lemma 4.2. *If* $P_0 = \varphi_0 \geq \theta_{\text{sup}} := \xi_{\text{sup}}/\alpha$, *the acceptor frequency monotonically increases toward* 1.

The condition that $P_0 = \varphi_0 \geq \theta_{\text{sup}}$ is a sufficient condition for such the complete spread of a matter in the population. Here note that $0 \leq \theta_{\text{inf}} < \theta_{\text{sup}} \leq 1$ because $0 \leq \xi_{\text{inf}} < \xi_{\text{sup}} \leq \alpha$ as assumed in Sec. 4.1.

The proof of Lemma 4.2 is straightforward. If $P_0 = \varphi_0 \geq \theta_{\text{sup}}$, we have $F(\alpha P_0) = 1$. Then, since $F(\alpha P_t) \geq F(\alpha P_0)$ for $P_t \geq P_0$, we have $F(\alpha P_t) = 1$ for any $t \geq 0$. Thus, from (4.1), we can get

$$P_{t+1} = \Psi(P_t) = [1 + \gamma b(1 - P_t)]P_t \tag{4.3}$$

for any $t \geq 0$. From Lemma 3.1, $\Psi(P_t)$ is always greater than P_t and less than 1, that is, $P_t < P_{t+1} < 1$ for any $P_t \in [P_0, 1) \subseteq [\theta_{\text{sup}}, 1)$. Hence, from Theorem 3.1 and Lemma 4.1, we can find that $P_t \to 1$ as $t \to \infty$.

This lemma indicates that a sufficiently large initial frequency of acceptors necessarily causes the complete spread of a matter over the population. It may be seen that the satisfaction of such a condition would be unrealistic, because it is hardly possible in general to establish such a large number of *initial* acceptors. However, the condition is more likely to be satisfied for a sufficiently small ξ_{sup} or a sufficiently large α. A sufficiently small ξ_{sup} means that the threshold is distributed in a range of sufficiently small value, which indicates that people tends to easily accept the spreading matter. A sufficiently large α means that a small frequency of acceptors could cause a strong social response, where the spreading matter would be very attractive to people even when the matter initially spreads only in a small subpopulation. In this sense, Lemma 4.2 could be regarded as giving a sufficient condition for the threshold distribution and the nature of social response to cause the complete spread of a matter over a population.

4.4. *Population dynamics to determine the final spread*

Taking account of Lemmas 4.1 and 4.2, we note it non-trivial whether the spread of a matter results in the complete or a partial spread in the population for $P_0 = \varphi_0 \in (\theta_{\text{inf}}, \theta_{\text{sup}})$. With the recurrence relation (4.1) and the threshold distribution (4.2) for $\varphi_0 \in (\theta_{\text{inf}}, \theta_{\text{sup}})$, the population dynamics model can be now rewritten as

$$P_{t+1} = \Psi(P_t) = \begin{cases} [1 + \gamma b \mathscr{G}(P_t; \varphi_0)] P_t & \text{for } P_t \in [\varphi_0, \theta_{\text{sup}}); \\ [1 + \gamma b (1 - P_t)] P_t & \text{for } P_t \in [\theta_{\text{sup}}, 1], \end{cases} \tag{4.4}$$

where

$$\mathscr{G}(P; \varphi_0) := \varphi_0 - P + (1 - \varphi_0) \int_{\xi_{\text{inf}}}^{\alpha P} f_+(\xi)\, d\xi. \tag{4.5}$$

Note that $\mathscr{G}(P_t; \varphi_0) = U_t[\xi \leq \alpha P_t]$ of (2.12) in Sec. 2.7.

Now, we find the following lemma (Appendix C).

Lemma 4.3. *For* $P_0 = \varphi_0 \in (\theta_{\text{inf}}, \theta_{\text{sup}})$, *if the equation* $\mathscr{G}(P; \varphi_0) = 0$ *in terms of* P *has no root in* $(\varphi_0, \theta_{\text{sup}})$, *then* $P_t \to 1$ *as* $t \to \infty$.

Thus, a partial spread of a matter could occur only when the equation $\mathscr{G}(P; \varphi_0) = 0$ in terms of P has a root in $(\varphi_0, \theta_{\text{sup}})$.

In the subsequent sections, we further investigate how the final spread of a matter in a population could depend on the nature of threshold distribution, analyzing the population dynamics model with the recurrence relation (4.1) and the threshold distribution (4.2).

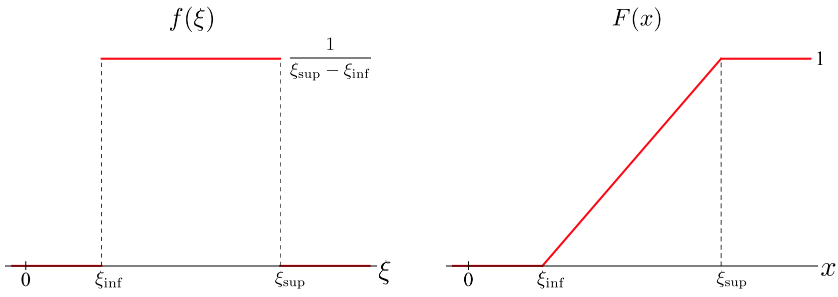

Fig. 3. The uniform threshold distribution defined by (4.2) with (5.1).

5. Dependence on Heterogeneity of Individuality

5.1. *Uniform threshold distribution*

In this section, we consider the uniform threshold distribution given by (4.2) with

$$f_+(\xi) = \frac{1}{\xi_{\sup} - \xi_{\inf}}. \tag{5.1}$$

See Fig. 3. In this case, we have the following result on the spread of a matter in the population (Appendix D).

Theorem 5.1. *For the uniform threshold distribution* (4.2) *with* (5.1), *the acceptor frequency* P_t *monotonically approaches* 1 *as time passes if* $P_0 = \varphi_0 > \theta_{\inf}$, *while it remains the initial frequency,* $P_t \equiv \varphi_0$, *if* $\varphi_0 \leq \theta_{\inf}$.

The latter case has been already shown by Lemma 4.1, while the former case indicates that the matter finally spreads over the whole population if the initial frequency of acceptors is beyond the critical value $\theta_{\inf} = \xi_{\inf}/\alpha$. Numerical examples of the sequence $\{P_t\}$ are given in Fig. 4. The result of the spread of a matter is *all-or-none* in such the population with a uniform threshold distribution.

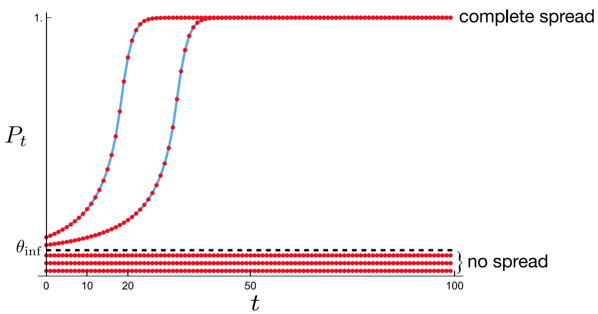

Fig. 4. Numerical examples of the sequence $\{P_t\}$ generated by (4.1) for the uniform threshold distribution (4.2) with (5.1). Sequences for different initial frequencies of acceptors are shown together. $P_0 = \varphi_0 = 0.02,\ 0.05,\ 0.08,\ 0.12,\ 0.15$; $\alpha = 1.0$; $\gamma b = 0.5$; $\xi_{\inf} = 0.1$; $\xi_{\sup} = 0.6$; $\theta_{\inf} := \xi_{\inf}/\alpha = 0.1$.

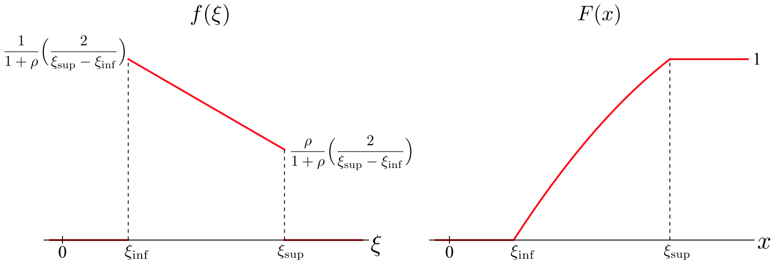

Fig. 5. A linear example of monotonically decreasing threshold distribution for (4.2) with $0 \le \rho < 1$.

5.2. *Monotonically decreasing threshold distribution*

For the monotonically decreasing threshold distribution (4.2) with a monotonically decreasing function $f_+(\xi)$ (as a linear example, see Fig. 5), we can get the following result (Appendix E).

Theorem 5.2. *For the monotonically decreasing threshold distribution* (4.2), *the acceptor frequency* P_t *monotonically approaches* 1 *as time passes for the initial frequency of acceptors such that* $P_0 = \varphi_0 > \theta_{\inf}$, *while it remains the initial frequency,* $P_t \equiv \varphi_0$, *for* $\varphi_0 \le \theta_{\inf}$.

This result is qualitatively the same as Theorem 5.1, and it is valid independently of the detail of the monotonically decreasing function $f_+(\xi)$ satisfying the condition given in Sec. 4.1. Consequently the result of the spread of a matter is all-or-none in the population with a monotonically decreasing threshold distribution, the same as that with a uniform threshold distribution.

5.3. *Monotonically increasing threshold distribution*

For the monotonically increasing threshold distribution (4.2) with a monotonically increasing function $f_+(\xi)$ (as a linear example, see Fig. 6), we can get the following result (Appendix F).

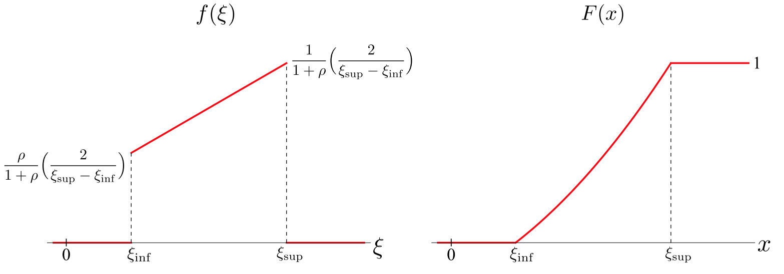

Fig. 6. A linear example of monotonically increasing threshold distribution for (4.2) with $0 \le \rho < 1$.

Theorem 5.3. *For the monotonically increasing threshold distribution* (4.2), *the acceptor frequency* P_t *has the following behavior as time passes: If*

$$\alpha f_+^{\inf} := \lim_{\xi \to \xi_{\inf}+0} \alpha f_+(\xi) < \frac{1}{1-\theta_{\inf}}, \tag{5.2}$$

then

$$\begin{cases} P_t \equiv P_0 = \varphi_0 & \text{for } \varphi_0 \in [0, \theta_{\inf}]; \\ P_t \to P^* & \text{for } \varphi_0 \in (\theta_{\inf}, \varphi_0^c] \subset (\theta_{\inf}, \theta_{\sup}); \\ P_t \to 1 & \text{for } \varphi_0 \in (\varphi_0^c, 1], \end{cases} \tag{5.3}$$

where P^* *is uniquely determined as the smallest root* $P = P^* \in (\varphi_0, \theta_{\sup})$ *of the equation* $\mathscr{G}(P; \varphi_0) = 0$ *in terms of* P, *and*

$$\varphi_0^c := 1 - \frac{1}{\alpha f_+(\alpha P_c)} \tag{5.4}$$

with the unique root $P = P_c \in (\theta_{\inf}, \theta_{\sup})$ *of the equation*

$$Q(P) := 1 - (1-P)\alpha f_+(\alpha P) - \int_{\xi_{\inf}}^{\alpha P} f_+(\xi)\, d\xi = 0. \tag{5.5}$$

Unless the condition (5.2) *is satisfied, then*

$$\begin{cases} P_t \equiv P_0 = \varphi_0 & \text{for } \varphi_0 \in [0, \theta_{\inf}]; \\ P_t \to 1 & \text{for } \varphi_0 \in (\theta_{\inf}, 1]. \end{cases}$$

In the latter case where the condition (5.2) is unsatisfied, the qualitative consequence of the spread of a matter is all-or-none as well as that for the uniform or monotonically decreasing threshold distribution, which has been shown by Theorems 5.1 and 5.2. On the other hand, in the former case where the condition (5.2) is satisfied, there is a range of the initial frequency of acceptors with which the matter

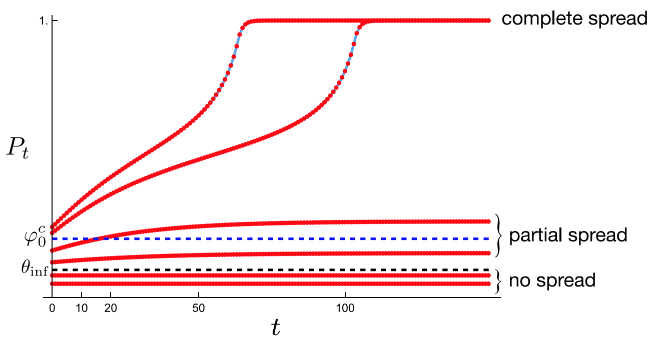

Fig. 7. Numerical examples of the sequence $\{P_t\}$ generated by (4.1) with the linearly increasing threshold distribution given by Fig. 6. Sequences for different initial frequencies of acceptors are shown together. $P_0 = \varphi_0 = 0.05, 0.08, 0.127, 0.170, 0.233, 0.255$; $\alpha = 1.0$; $\gamma b = 0.5$; $\xi_{\inf} = \theta_{\inf} = 0.1$; $\xi_{\sup} = 0.9$; $\rho = 0.3$; $\varphi_0^c = 0.212$; $\rho_c = 0.8$.

spreads only up to a certain portion of the population and cannot spread over it. In other words, there exists a critical value for the initial frequency of acceptors φ_0^c such that the matter successfully spreads over the population with and only with $\varphi_0 > \varphi_0^c$. Numerical examples are shown in Fig. 7 with a linearly increasing threshold distribution defined in Fig. 6.

For the initial frequency of acceptors $\varphi_0 \in (\theta_{\text{inf}}, \varphi_0^c]$ under the condition (5.2), we find the following result on the equilibrium acceptor frequency P^* (Appendix G).

Corollary 5.1. *For the monotonically increasing threshold distribution* (4.2) *satisfying the condition* (5.2), *the equilibrium acceptor frequency P^* is continuous and monotonically increasing in terms of $\varphi_0 \in (\theta_{\text{inf}}, \varphi_0^c]$, where $P^* \to \theta_{\text{inf}}$ as $\varphi_0 \to \theta_{\text{inf}} + 0$, and $P^* = P_c \in (\theta_{\text{inf}}, \theta_{\text{sup}}) \subset (0, 1)$ for $\varphi_0 = \varphi_0^c$.*

The result of Corollary 5.1 with Theorem 5.1 indicates that the equilibrium acceptor frequency P^* is continuous and monotonically increasing in terms of $\varphi_0 \in (0, \varphi_0^c]$, while there necessarily exists a discontinuous jump about the equilibrium value of acceptor frequency at $\varphi_0 = \varphi_0^c$ since $P_t \to 1$ for $\varphi_0 \in (\varphi_0^c, 1]$. From the definition of P_c in Theorem 5.3, the size of the jump at $\varphi_0 = \varphi_0^c$ can be given as $\Delta P_c := 1 - P_c$.

Linearly increasing threshold distribution

For the linearly increasing threshold distribution defined in Fig. 6 with (4.2), the condition (5.2) becomes

$$\frac{\rho}{1+\rho} < \frac{1}{2}\left(1 - \frac{1-\theta_{\text{sup}}}{1-\theta_{\text{inf}}}\right),$$

that is,

$$\rho < \rho_c := \frac{(1-\theta_{\text{inf}}) - (1-\theta_{\text{sup}})}{(1-\theta_{\text{inf}}) + (1-\theta_{\text{sup}})}. \tag{5.6}$$

Since the distribution becomes uniform as $\rho \to 1$, the condition (5.6) indicates that a partial spread of the matter occurs only when the monotonically increasing threshold distribution is sufficiently biased to the higher value. In fact, from the mean threshold value $\overline{\xi}$ now given as

$$\overline{\xi} = \int_{\xi_{\text{inf}}}^{\xi_{\text{sup}}} \xi f(\xi)\, d\xi = \frac{1}{3}\left\{\left(1 + \frac{1}{1+\rho}\right)\xi_{\text{sup}} + \left(2 - \frac{1}{1+\rho}\right)\xi_{\text{inf}}\right\},$$

which is monotonically decreasing in terms of ρ, the condition (5.6) is equivalent to $\overline{\xi} > \overline{\xi}_c := \overline{\xi}|_{\rho=\rho_c}$. Thus, in a population which has a sufficiently large mean threshold value, that is, which has a sufficiently conservative characteristics according to the acceptance of the spreading matter, only a partial spread may be resulted in even when the spread proceeds.

The specific value P_c can be explicitly derived by solving the equation $Q(P) = 0$ of (5.5):

$$P_c = 1 - \sqrt{\left\{1 - \theta_{\text{inf}} + \frac{1+\rho}{1-\rho}(\theta_{\text{sup}} - \theta_{\text{inf}})\right\}(1 - \theta_{\text{sup}})}, \tag{5.7}$$

and subsequently we can get the critical value for the initial frequency of acceptors by (5.4) with (5.7):

$$\varphi_0^c = 1 - \frac{1-\rho^2}{2}\left\{1 - \theta_{\text{inf}} + \frac{\rho}{1-\rho}(\theta_{\text{sup}} - \theta_{\text{inf}}) + 1 - P_c\right\}. \tag{5.8}$$

Numerically obtained φ_0-dependences of the equilibrium acceptor frequency for different values of ρ are shown in Fig. 8. The condition (5.2), that is, (5.6) is satisfied in Figs. 8(a) and 8(b), while it is not in Fig. 8(c).

As shown in Theorem 5.3, when $\rho \geq \rho_c$, the φ_0-dependence of the equilibrium acceptor frequency has the qualitatively same nature as for the uniform threshold distribution which was shown in Theorem 5.1. Figure 8(c) shows such an example. In accordance with what is already mentioned before, the threshold distribution becomes nearer to the uniform distribution as ρ gets larger.

Figure 9 numerically shows the ρ-dependence of φ_0^c and P_c for the linearly increasing threshold distribution defined in Fig. 6 with (4.2). Both of them are monotonically decreasing in terms of ρ, and commonly take the value θ_{inf} for $\rho = \rho_c$. This nature can be easily found from (5.6)–(5.8). Hence for the linearly increasing threshold distribution defined in Fig. 6 with (4.2), if we mathematically define the critical value for the initial frequency of acceptors φ_0^c as follows: φ_0^c is given by (5.8) with (5.7) for $\rho < \rho_c$, while $\varphi_0^c = \theta_{\text{inf}}$ for $\rho \geq \rho_c$, then the φ_0-dependence of the equilibrium acceptor frequency can be generally expressed as (5.3) for $\rho \in [0, 1]$.

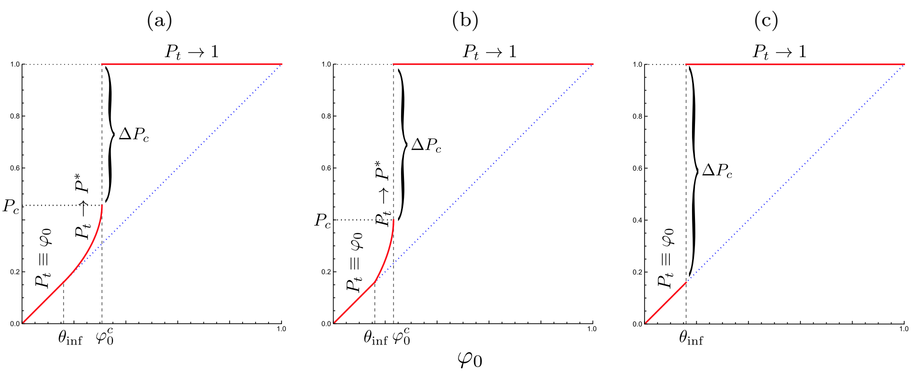

Fig. 8. φ_0-dependence of the equilibrium acceptor frequency for the linearly increasing threshold distribution given by Fig. 6 with (4.2). Numerically drawn with (a) $\rho = 0.0$ ($P_c = 0.456$; $\varphi_0^c = 0.308$); (b) $\rho = 0.2$ ($P_c = 0.400$; $\varphi_0^c = 0.232$); (c) $\rho = 0.7$, and commonly $\theta_{\text{inf}} = 0.16$; $\theta_{\text{sup}} = 0.8$; $\rho_c = 0.615$.

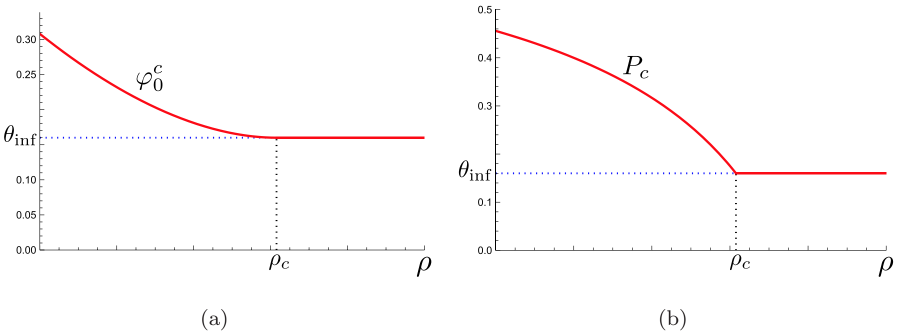

Fig. 9. ρ-dependence of φ_0^c and P_c for the linearly increasing threshold distribution given by Fig. 6 with (4.2). φ_0^c and P_c are given by (5.8) and (5.7). Numerically drawn with $\theta_{\rm inf} = 0.16$; $\theta_{\rm sup} = 0.8$; $\rho_c = 0.615$.

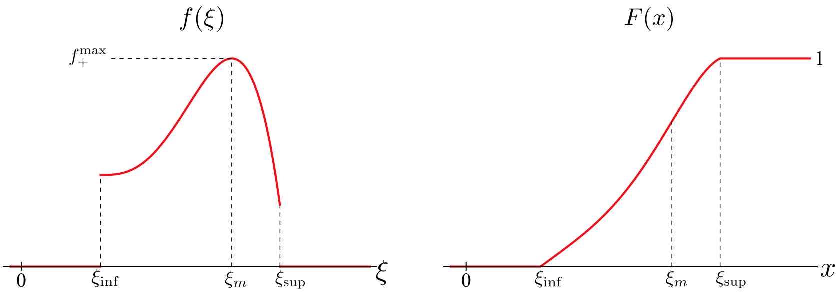

Fig. 10. An illustrative example of unimodal threshold distribution for (4.2).

5.4. *Unimodal threshold distribution*

As the most likely distribution for the threshold value, we may assume a unimodal threshold distribution like one illustrated in Fig. 10, for example, as a cut-off Gaussian distribution.[1,16] The frequency distribution $f_+(\xi)$ of (4.2) has the unique maximum at $\xi = \xi_m \in (\xi_{\rm inf}, \xi_{\rm sup})$ which is called the *mode* for ξ in statistical science.

For the unimodal threshold distribution, we can get the following theorem qualitatively corresponding to Theorem 5.3 for the monotonically increasing threshold distribution (Appendix H).

Theorem 5.4. *For the unimodal threshold distribution* (4.2), *the acceptor frequency* P_t *has the following behavior as time passes: If the condition* (5.2) *is satisfied, then*

$$\begin{cases} P_t \equiv P_0 = \varphi_0 & \text{for } \varphi_0 \in [0, \theta_{\rm inf}]; \\ P_t \to P^* & \text{for } \varphi_0 \in (\theta_{\rm inf}, \varphi_0^c] \subset (\theta_{\rm inf}, \theta_m) := (\theta_{\rm inf}, \xi_m/\alpha); \\ P_t \to 1 & \text{for } \varphi_0 \in (\varphi_0^c, 1], \end{cases} \tag{5.9}$$

where P^* *is uniquely determined as the smallest root* $P = P^* \in (\varphi_0, \theta_m)$ *of the equation* $\mathscr{G}(P; \varphi_0) = 0$, *and* $\varphi_0^c \in (\theta_{\inf}, \theta_m)$ *is given by* (5.4) *with the unique root* $P = P_c \in (\theta_{\inf}, \theta_m)$ *of Eq.* (5.5). *Unless the condition* (5.2) *is satisfied, we have*

$$\begin{cases} P_t \equiv P_0 = \varphi_0 & \textit{for } \varphi_0 \in [0, \theta_{\inf}]; \\ P_t \to 1 & \textit{for } \varphi_0 \in (\theta_{\inf}, 1]. \end{cases}$$

We note that the critical value φ_0^c is necessarily less than θ_m for the unimodal threshold distribution, while it is mathematically defined the same as for the monotonically increasing threshold distribution. For any initial frequency of acceptors equal to or greater than θ_m, the acceptor frequency monotonically increases toward 1 to make the complete spread of the matter in the population. The φ_0-dependence of the equilibrium acceptor frequency becomes qualitatively similar with those shown in Fig. 8 for the linearly increasing threshold distribution given by Fig. 6. As a result, such qualitative correspondence and difference indicate that the increasing part in the threshold distribution is essential to cause a partial spread of the matter.

Piecewise-linear unimodal threshold distribution

Let us consider the piecewise-linear unimodal threshold distribution defined in Fig. 11 with (4.2), where $0 \le \rho_{\inf} < 1$; $0 \le \rho_{\sup} < 1$;

$$\xi_m := (1-m)\xi_{\inf} + m\xi_{\sup}; \quad f_+^{\max} := \frac{1}{1 + m\rho_{\inf} + (1-m)\rho_{\sup}} \cdot \frac{2}{\xi_{\sup} - \xi_{\inf}}$$

with $0 < m < 1$. The condition (5.2) becomes

$$\rho_{\inf} < \frac{1 + (1-m)\rho_{\sup}}{2(1-\theta_{\inf})/(\theta_{\sup} - \theta_{\inf}) - m}. \tag{5.10}$$

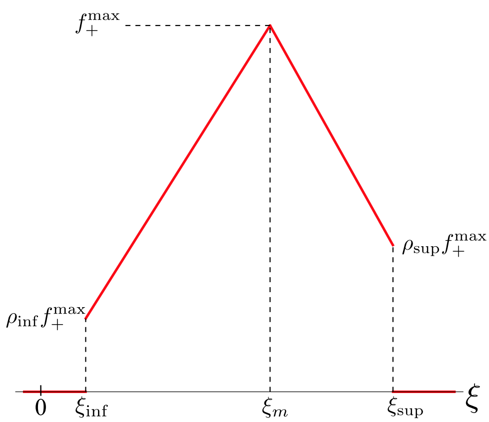

Fig. 11. A piecewise-linear unimodal threshold distribution (4.2).

The specific value P_c can be derived by solving the equation $Q(P) = 0$ of (5.5) for $P \in (\theta_{\mathrm{inf}}, \theta_m)$:

$$P_c = 1 - \sqrt{(1-\theta_{\mathrm{inf}})^2 - mK} \tag{5.11}$$

with

$$K := \frac{2(\theta_{\mathrm{sup}} - \theta_{\mathrm{inf}})}{(1-\rho_{\mathrm{inf}})\alpha f_+^{\mathrm{max}}}\{1 - (1-\theta_{\mathrm{inf}})\rho_{\mathrm{inf}}\alpha f_+^{\mathrm{max}}\},$$

and subsequently we can get the critical value for the initial frequency of acceptors $\varphi_0^c \in (\theta_{\mathrm{inf}}, \theta_m)$ by (5.4) with (5.11).

As shown in Fig. 12 obtained from the condition (5.10), a partial spread of the matter can occur only for a sufficiently small value of ρ_{inf}. Especially, if

$$\rho_{\mathrm{inf}} \geq \rho_{cc} := 1 - \frac{1-\theta_{\mathrm{sup}}}{1-\theta_{\mathrm{inf}}}, \tag{5.12}$$

such a partial spread does not occur for any initial frequency of acceptors φ_0, independently of the value m. On the other hand, if $\rho_{\mathrm{inf}} \leq \rho_{cc}/2$, a partial spread must occur for a range of initial frequency of acceptors φ_0, independently of the value m. This result implies that a partial spread of the matter can occur only when the unimodal threshold distribution is sufficiently biased to the higher value, the same as the result for the linearly increasing threshold distribution in the previous section.

As shown in Secs. 5.1 and 5.2, the uniform or monotonically decreasing threshold distribution does not have the case of a partial spread of the matter, that

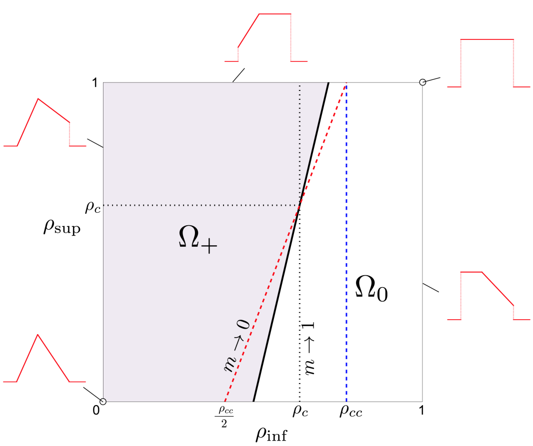

Fig. 12. $(\rho_{\mathrm{inf}}, \rho_{\mathrm{sup}})$-dependence of the occurrence of a partial spread of the matter for the piecewise-linear unimodal threshold distribution defined in Fig. 11, following (5.10) from the condition (5.2). For the region Ω_+, there is a range of the initial frequency of acceptors φ_0 with which $P_t \to P^* \in (0, 1)$ as $t \to \infty$. For the region Ω_0, such a partial spread of the matter never occurs, while P_t alternatively remains $P_0 = \varphi_0$ or approaches 1 as $t \to \infty$. The boundary line between Ω_0 and Ω_+ is determined by m, θ_{inf} and θ_{sup} from (5.10). The critical values ρ_c and ρ_{cc} are, respectively, defined by (5.6) and (5.12).

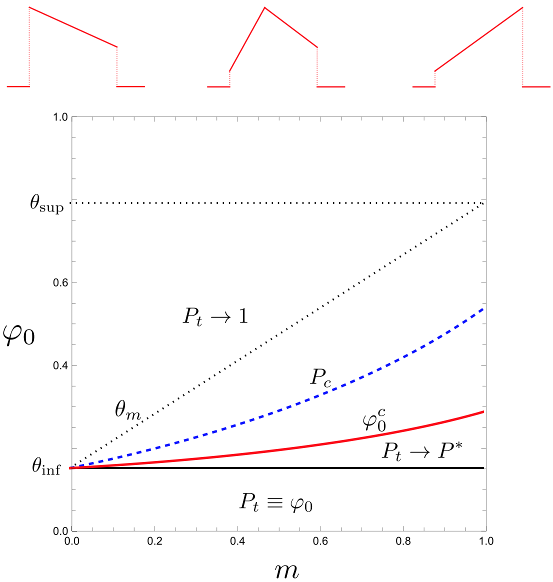

Fig. 13. (m, φ_0)-dependence of the spread of a matter for the piecewise-linear unimodal threshold distribution defined in Fig. 11. Numerically drawn by (5.10), (5.11), and (5.4) with $\theta_{\rm inf} = 0.16$; $\theta_{\rm sup} = 0.8$; $\rho_{\rm inf} = 0.2$; $\rho_{\rm sup} = 0.5$; $\rho_c = 0.615$; $\rho_{cc} = 0.762$.

is, the result of the matter spread is all-or-none in the population. We note that the result shown in Fig. 12 for the piecewise-linear unimodal threshold distribution indicates its dependence on the characteristics of unimodal threshold distribution, corresponding with the results for the uniform or monotonically decreasing threshold distribution. The monotonically decreasing threshold distribution in a wider sense ($\rho_{\rm inf} = 1$) does not cause any partial spread. Neither does the unimodal distribution near to the uniform distribution (sufficiently large $\rho_{\rm inf}$ and $\rho_{\rm sup}$).

In Fig. 13, it is numerically indicated that the critical value for the initial frequency of acceptors φ_0^c is monotonically increasing in terms of m. That is, it is indicated that the complete spread of a matter becomes harder as the unimodal threshold distribution is biased to the larger value. On the other hand, since the specific value P_c is monotonically increasing in terms of m, the size of the partial spread P^* becomes larger as the threshold distribution is biased to the larger value. Consequently, for the unimodal threshold distribution biased to the larger value, the complete spread of a matter becomes harder, while the partial spread becomes relatively more successful when it occurs.

6. Concluding Remarks

In this paper, we constructed the discrete-time population dynamics model which accurately corresponds to Granovetter's threshold model in mathematical sense, and analyzed it, focusing on how the nature of threshold distribution affects the result of the spread of a matter in a population. We have obtained the analytical results for the general uniform, monotonically decreasing, monotonically increasing, and unimodal threshold distributions. For all of them, we find the critical value for the initial frequency of acceptors beyond which the matter finally spreads over the population. With the initial frequency of acceptors not beyond it, the matter cannot spread over the population at all or may spread only in part.

Such a nature in the population dynamics could be mathematically regarded as a bistability structure. However, accurately, that of our model is not the case of bistability in the standard sense. As explicitly shown in our model formula of recurrence relation (2.13) or (4.1), the dynamical system itself is defined for each initial frequency of acceptors (or its distribution). This is a special feature of our mathematical model given by (2.13) or (4.1). In other words of the discrete dynamical system theory with recurrence relation, the return map itself depends on the initial value, and it gets changed for different initial values. Hence it is not possible to investigate such a bistable-like nature with a fixed return map for our model, for example, making use of the cobwebbing method which is usually a useful method to understand the dynamical nature of one dimensional discrete-time dynamical system with a recurrence relation.[56–58] In the analyses on our model, we used the cobwebbing method only in part for the return map with a given initial frequency of acceptors, and showed that the acceptor frequency must monotonically increase toward an equilibrium value less than or equal to 1 (Lemma F.6 in Appendix F). Nevertheless, we are confident that our model is the standard rational population dynamics model accurately corresponding to Granovetter's threshold model.

Those results obtained about our discrete-time population dynamics model may be regarded as qualitatively the same as those for the continuous-time model in Ref. 55. It could be regarded as consistent because the essential assumptions for the modeling are correspondent with each other. In a general sense, the discrete-time dynamics with a difference equation could have some mathematically specific features different from the corresponding continuous-time dynamics with an ordinary differential equation. Our discrete-time dynamics model is not the case.

In our modeling, we assumed a population which consists of only members who have threshold value ξ less than α, as described in Sec. 2.2. However, it is likely that there are some individuals unconcerned about the matter spreading in the population. If we consider a strategic spread of a matter in a population, for example, for a commercial purpose, it would be programmed with the presumption that there are such individuals. Then the complete success of the spread of such a matter means its spread over all individuals who have a possibility to accept the matter, that is, all targeted ones. In our modeling, such individuals unconcerned about the spreading matter could be assumed to have their threshold values greater than α. Our model would be applicable even to such a case by focusing on the maximal subpopulation of individuals who have a possibility to accept the spreading matter. In other

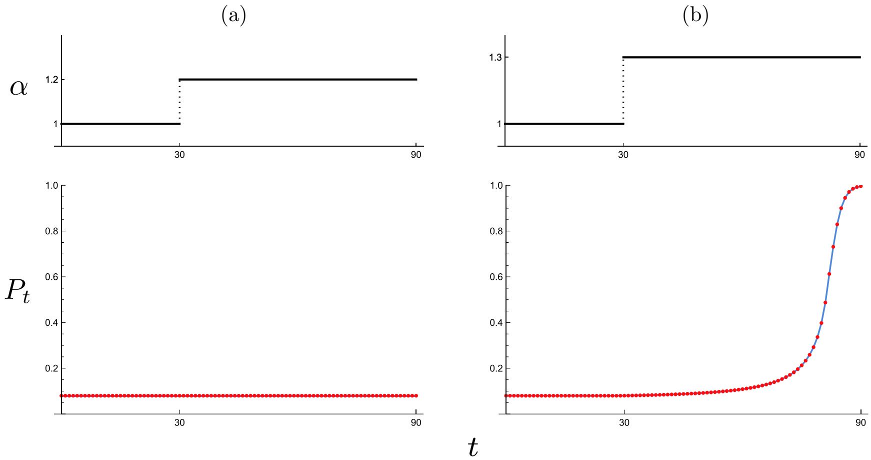

Fig. 14. A numerical example of the sequence $\{P_t\}$ generated by (4.4) for the uniform threshold distribution (5.1) with a shift of the cultural/social environment. The parameter value α is shifted from 1.0 to a larger value (a) 1.2; (b) 1.3 after $t = 30$. The spread of a matter does not occur in (a) even after the shift, while it occurs in (b). Commonly, $P_0 = \varphi_0 = 0.08$; $\gamma b = 0.5$; $\xi_{\text{sup}} = 0.6$; $\xi_{\text{inf}} = 0.1$.

words, we may consider a population dynamics in such a subpopulation, excluding those who are unconcerned about the spreading matter. In such a modeling, the parameters α and b may be related to the proportion of those unconcerned about the spreading matter, because the strength of social response could be weaker or the chance to get the occasion for the decision-making could become rarer as the proportion of unconcerned individuals gets larger.[59] On the other hand, if we consider a spreading matter interesting only for a specific subcommunity/class/group, the spread may depend on the communication only among its members. Actually, like an OTAKU community today, the communication would follow a specifically strong link among the members through SNS, etc.[60–62] The population dynamics model presented in this paper may be reasonably applicable for such a case of the information spread in a specific subcommunity/class/group.

The social response to a spreading matter is the important factor in our modeling. The strength of its effect on the decision-making is given by αP with the acceptor frequency P in the population, where α is the parameter reflecting the nature of spreading matter and the cultural and social environment in the population. Although the parameter α was given as a constant independent of time in our model, it may alter correspondingly to a change of social environment, for example, due to a social riot, an economical crash, an outbreak of communicable disease, etc. A change of social environment may suppress or stimulate the consciousness to a spreading matter, even if the threshold distribution is kept unchanged. Our results on the model (4.1) imply that such a change of the value α could have a significant effect on the spread of a matter.

Figure 14 gives a numerical example with our model (4.4) for the uniform threshold distribution (5.1) when the parameter α shifts up after a certain time step. If the shift is sufficiently small like Fig. 14(a), the spread of the matter is not affected but remains as it was at the phase of no spread. In contrast, if it is sufficiently large like Fig. 14(b), it changes the phase after a while to increase toward the complete spread in the population. Such a phase shift could be regarded as a dynamical nature of our model following the bifurcation structure about the equilibrium acceptor frequency which has a discontinuous gap at the critical threshold value $\theta_{\rm inf}$ or $\varphi_0^{\rm c}$ for the initial frequency of acceptors. The shift of the value α changes the critical threshold value in our model. Then the acceptor frequency could be derived to the other phase even for the same initial frequency of acceptors, as exemplified by Fig. 14(b).

Now, we note that our model (4.1) is valid only for the case where the parameter α does not change to become smaller in the population dynamics, because the modeling especially in Sec. 2.6 is not applicable for such a temporal changing α. To modify and generalize our model in order to be applied for such a population dynamics about the spread of a matter in a population, it would be insufficient to modify just the recurrence relation (2.13) or (4.1), and instead it must be necessary to reconsider the modeling itself, taking into account the temporal variation of the threshold distribution in the acceptor and non-acceptor subpopulations.

As assumed in Sec. 2.1, our population dynamics model follows only the decision on whether an individual accepts a spreading matter or not, and does not have any possibility to discard the accepted matter. When there exists a possibility to discard the accepted matter, it would happen that the spread of a matter shrinks to become disappeared in the population. We know many examples of such temporal spread, shrinkage, and disappearance of a matter like fashion, technology, rumor, and custom. To introduce a discard process in our modeling, it would be reasonable to introduce another threshold according to it, because the decision-making on the discard could be done with the reason different from that about the acceptance. In such a modeling, we must take account of the frequency of individuals who have discarded the matter, since they would be assumed to have no contribution to the social response according to the acceptance of the matter by the individual who has not yet made its decision, or assumed to become stiflers to suppress the trend to accept the spreading matter. Since the threshold distributions according to the acceptance and discard of the matter would be generally different from each other, we would have to reasonably set up and model the recruitment of new acceptors in unit time step. Therefore, although it would be distinctly possible to proceed such an extended modeling, the prospective mathematical structure of the constructed model would become more complicated in a mathematical sense. Such a direction of mathematical modeling for the population dynamics with respect to the spread of a matter in a population would be interesting, whereas it certainly requires an amount of reasonable arguments enough to make a reasonable mathematical model.

Despite such toughness of reasonable modeling, we note that Granovetter's threshold model is frequently used with the cobwebbing method to conceptually illustrate the dynamics of spreading a matter, even for the case where the spread of a matter could shrink to become disappeared in a population.[1,5,16] However, it should be noted that the recurrence relation used in such an interpretation of the dynamics would be mathematically inaccurate and sometimes incorrect. Indeed, the recurrence relation used in not a few literatures referring to Granovetter's threshold model is (2.13) with $\gamma B(P_t) \equiv 1$ and $\varphi_0(\xi) \equiv 0$: $P_{t+1} = F(\alpha P_t)$.[1,5,16,45] The former setup of $\gamma B(P_t) \equiv 1$ matches the conceptual modeling by Granovetter[1,5] in which all non-acceptors who satisfy the condition synchronously become acceptors in each time step. The latter of $\varphi_0(\xi) \equiv 0$ may be regarded as a mathematical approximation with the assumption that the initial acceptors are very few in comparison to the total number of individuals. It is clear by our modeling and analysis that the recurrence relation with these specific setups is hardly acceptable as a population dynamics model to govern the temporal change of acceptor frequency. Moreover, it is obviously irrational to discuss the dependence of the outcome of a matter spreading on the initial acceptor frequency with such a formula as the model. Further it must be improper to interpret the effect of the discard of accepted matter by the acceptor with the cobwebbing method for such the recurrence relation, because the population dynamics could not be reasonably given by it as we mentioned above. We could not discuss the reasonable modeling here furthermore, and would like to do it elsewhere in future.

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Appendix A. Proof of Lemma 2.2

From (2.12), $U_t[\xi \leq \alpha P_t]$ is continuous and differentiable in terms of $P_t \in [P_0, 1]$ because so are $F(\alpha P_t)$ and the integral of $\varphi_0(\xi)f(\xi)$ over $[\alpha, \alpha P_t]$. Since $B(P) \in [0,1)$ is differentiable for almost every $P \in (0,1)$, we have

$$\frac{d\Psi(P_t)}{dP_t} = 1 + \gamma\frac{dB(P_t)}{dP_t}U_t[\xi \leq \alpha P_t] + \gamma B(P_t)\frac{\partial U_t[\xi \leq \alpha P_t]}{\partial P_t}$$

for almost every $P_t \in (0,1)$ from (2.11), and

$$\frac{\partial U_t[\xi \leq \alpha P_t]}{\partial P_t} = \alpha f(\alpha P_t) - \{1 + \alpha\varphi_0(\alpha P_t)f(\alpha P_t)\}$$

from (2.12). Then from these derivatives, we obtain

$$\begin{aligned}\frac{d\Psi(P_t)}{dP_t} &= 1 - \gamma B(P_t) + \gamma\frac{dB(P_t)}{dP_t}U_t[\xi \leq \alpha P_t] \\ &\quad + \gamma B(P_t)\{1 - \varphi_0(\alpha P_t)\}\alpha f(\alpha P_t) > 0 \qquad \text{(A.1)}\end{aligned}$$

for almost every $P_t \in (0,1)$, since $U_t[\xi \leq \alpha P_t] \geq 0$, $\gamma B(P_t) \in [0,1)$, $dB(P_t)/dP_t \geq 0$, $\varphi_0(\alpha P_t) \in [0,1]$, and $f(\alpha P_t) > 0$ for $\alpha P_t \in (0, \alpha)$ from their definitions. Hence from the almost everywhere positive P_t-derivative of $\Psi(P_t)$ and the non-decreasing feature of $B(P_t)$ in terms of $P_t \in [P_0, 1]$, we can get the result of Lemma 2.2.

Appendix B. Proof of Lemma 3.1

First, for $P_0 = 0$, we have $\varphi_0(\xi) = 0$, $F(0) = 0$, and $B(0) = 0$ from the definitions and assumptions. Thus, from the recurrence relation (2.13), we can easily find that $P_1 = 0$ when $P_0 = 0$. Hence the mathematical induction shows that $P_t = 0$ for any $t \geq 0$ if $P_0 = 0$.

Next, for $P_0 > 0$, we find from (2.10) that, if $U_0[\xi \leq \alpha P_0] > 0$,

$$P_1 - P_0 = \gamma B(P_0)U_0[\xi \leq \alpha P_0] > U_0[\xi \leq \alpha P_0] \leq U_0 = 1 - P_0,$$

and

$$P_1 = P_0 + \gamma B(P_0)U_0[\xi \leq \alpha P_0] \geq P_0 > 0.$$

As a result, we have $P_1 \in [P_0, 1)$ when $U_0[\xi \leq \alpha P_0] > 0$ for $P_0 > 0$. If $U_0[\xi \leq \alpha P_0] = 0$, we have $P_1 = P_0$ from (2.10) as well. Thus, we have $P_1 \in [P_0, 1)$ for $P_0 > 0$.

Now, let us assume that it holds that $P_t \in [P_0, 1)$ for some $t \geq 0$. Then, since $U_t[\xi \leq \alpha P_t] \in [0,1]$ and $\gamma B(P_t) \in (0,1)$, we find from (2.10) that, if $U_t[\xi \leq \alpha P_t] > 0$,

$$P_{t+1} = P_t + \gamma B(P_t)U_t[\xi \leq \alpha P_t] > P_t > 0,$$

and

$$P_{t+1} - P_t = \gamma B(P_t)U_t[\xi \leq \alpha P_t] < U_t[\xi \leq \alpha P_t] \leq U_t = 1 - P_t.$$

As a result, we have $P_{t+1} \in [P_0, 1)$ when $U_t[\xi \leq \alpha P_t] > 0$ for $P_t \in [P_0, 1)$. If $U_t[\xi \leq \alpha P_t] = 0$, we have $P_{t+1} = P_t \in [P_0, 1)$ from (2.10) as well. Therefore, we result in $P_{t+1} \in [P_0, 1)$ Consequently, from the mathematical induction, $P_t \in [P_0, 1)$ for all $t > 0$.

Further, if $P_0 = 1$, we have $U_0 = 1 - P_0 = 0$, and $U_t = 0$ for any $t \geq 0$ from the definition. Hence, since $U_t[\xi \leq \alpha P_t] = 0$, we have $P_{t+1} = P_t$ by the above arguments. Lastly these arguments prove the lemma.

Appendix C. Proof of Lemma 4.3

We can easily find from the definition of $\mathscr{G}(P_0; \varphi_0)$ by (4.5) that $\mathscr{G}(P_0; \varphi_0) = \mathscr{G}(\varphi_0; \varphi_0) > 0$; and $\mathscr{G}(\theta_{\sup}; \varphi_0) = 1 - \theta_{\sup} \geq 0$. Hence, when there is no root of the equation $\mathscr{G}(P; \varphi_0) = 0$ in terms of $P \in [\varphi_0, \theta_{\sup}) \subset (\theta_{\inf}, \theta_{\sup})$, we have $\mathscr{G}(P; \varphi_0) > 0$ for $P \in [\varphi_0, \theta_{\sup}) \subset (\theta_{\inf}, \theta_{\sup})$.

From (4.4), since $\mathscr{G}(P; \varphi_0) > 0$ for any $P \in [\varphi_0, \theta_{\sup}) \subset (\theta_{\inf}, \theta_{\sup})$, we have $P_{t+1} > P_t$ for $P_t \in [\varphi_0, \theta_{\sup}) \subset (\theta_{\inf}, \theta_{\sup})$. Thus P_t monotonically approaches $\theta_{\sup}$ as t increases. When $\theta_{\sup} = 1$, it means from Lemma 3.1 that $P_t \to 1$ as $t \to \infty$. When $\theta_{\sup} < 1$, P_t becomes larger than $\theta_{\sup}$ for a certain $t > 0$ since $P_{t+1} > P_t$ for $P_t \in [\varphi_0, \theta_{\sup}) \subset (\theta_{\inf}, \theta_{\sup})$ with $\mathscr{G}(\theta_{\sup}; \varphi_0) > 0$. Then from (4.4), we can find that $P_t \to 1$ as $t \to \infty$ with the same arguments for (4.3) in Sec. 4.3.

Appendix D. Proof of Theorem 5.1

The population dynamics model (4.4) for the initial frequency of acceptors $P_0 = \varphi_0 > \theta_{\inf} := \xi_{\inf}/\alpha$ becomes

$$\begin{aligned} P_{t+1} &= \Psi(P_t) \\ &= \begin{cases} \left[1 + \gamma b\left\{\varphi_0 - P_t + (1-\varphi_0)\dfrac{\alpha P_t - \xi_{\inf}}{\xi_{\sup} - \xi_{\inf}}\right\}\right] P_t & \text{for } P_t \in [\varphi_0, \theta_{\sup}); \\ [1 + \gamma b(1 - P_t)]P_t & \text{for } P_t \in [\theta_{\sup}, 1] \end{cases} \end{aligned} \tag{D.1}$$

with the uniform threshold distribution given by (4.2) and (5.1). From Lemma 4.1, we have $P_t > P_0 = \varphi_0 > \theta_{\rm inf}$ for $t > 0$. Note that $F(\alpha P_t) = 1$ for $\alpha P_t \geq \xi_{\rm sup}$ because of the definition of f in (5.1).

Then, since $1 - \varphi_0 = 1 - P_0 < 1 - \theta_{\rm inf}$, we have

$$\begin{aligned}\varphi_0 - P_t + (1-\varphi_0)\frac{\alpha P_t - \xi_{\rm inf}}{\xi_{\rm sup} - \xi_{\rm inf}} &= 1 - P_t - \frac{1-\varphi_0}{\theta_{\rm sup} - \theta_{\rm inf}}(\theta_{\rm sup} - P_t) \\ &> 1 - P_t - \frac{1-\theta_{\rm inf}}{\theta_{\rm sup} - \theta_{\rm inf}}(\theta_{\rm sup} - P_t) \\ &= \frac{1-\theta_{\rm sup}}{\theta_{\rm sup} - \theta_{\rm inf}}(P_t - \theta_{\rm inf}) \geq 0\end{aligned}$$

for $P_t \in [\varphi_0, \theta_{\rm sup}) \subset (\theta_{\rm inf}, \theta_{\rm sup})$. Therefore $\Psi(P_t)$ of (D.1) is necessarily greater than P_t for any $P_t \in [\varphi_0, \theta_{\rm sup})$. Further, for $P_t \in [\theta_{\rm sup}, 1)$, $\Psi(P_t)$ is necessarily greater than P_t. Consequently, we find that $P_{t+1} > P_t$ for any $P_t \in [\varphi_0, 1)$. Hence from Theorem 3.1, we conclude that $P_t \to 1$ as $t \to \infty$ for $P_0 = \varphi_0 \in (\theta_{\rm inf}, 1]$. The latter case in Theorem 5.1 simply follows Lemma 4.1.

Appendix E. Proof of Theorem 5.2

Since we have Lemmas 2.1, 3.1, Theorem 3.1, and Lemma 4.1, it is sufficient to show that $P_t \to 1$ as $t \to \infty$ for $P_0 = \varphi_0 \in (\theta_{\rm inf}, 1)$. Now let us consider the root of the equation $\mathscr{G}(P;\varphi_0) = 0$ in terms of $P \in [\varphi_0, \theta_{\rm sup})$ with $\varphi_0 \in (\theta_{\rm inf}, \theta_{\rm sup})$, where $\mathscr{G}(P;\varphi_0)$ is defined by (4.5) for $P \in (\theta_{\rm inf}, \theta_{\rm sup})$. The equation $\mathscr{G}(P;\varphi_0) = 0$ is equivalent to

$$\frac{P - \varphi_0}{1 - \varphi_0} = \int_{\xi_{\rm inf}}^{\alpha P} f_+(\xi)\, d\xi. \tag{E.1}$$

With the monotonically decreasing $f_+(\xi) > 0$ for $\xi \in (\xi_{\rm inf}, \xi_{\rm sup})$, the right side of (E.1) is monotonically increasing and *concave* in terms of $P \in [\varphi_0, \theta_{\rm sup})$, since its derivative is positive and monotonically decreasing in terms of $P \in [\varphi_0, \theta_{\rm sup})$. Besides, the right side is positive for $P = \varphi_0$ and becomes 1 for $P = \theta_{\rm sup}$. The left side of (E.1) is a line with the slope $1/(1-\varphi_0)$, which takes 0 for $P = \varphi_0$, and $(\theta_{\rm sup} - \varphi_0)/(1-\varphi_0) \leq 1$ for $P = \theta_{\rm sup} \leq 1$. Thus, the right side of (E.1) is greater than the left side for $P = \varphi_0$, and greater than or equal to the left side for $P = \theta_{\rm sup}$, respectively. From this result, we find that the right side of (E.1) is greater than the left side for any $P \in [\varphi_0, \theta_{\rm sup})$, because the right side is monotonically increasing and concave while the left side is a line.

Therefore the equation $\mathscr{G}(P;\varphi_0) = 0$ has no root in terms of $P \in [\varphi_0, \theta_{\rm sup}) \subset (\theta_{\rm inf}, \theta_{\rm sup})$. Since $\mathscr{G}(P_0;\varphi_0) = \mathscr{G}(\varphi_0;\varphi_0) > 0$ and $\mathscr{G}(\theta_{\rm sup};\varphi_0) = 1 - \theta_{\rm sup} \geq 0$, we find that $\mathscr{G}(P;\varphi_0) > 0$ for $P \in [\varphi_0, \theta_{\rm sup}) \subset (\theta_{\rm inf}, \theta_{\rm sup})$. Hence the recurrence relation (4.4) indicates that $P_{t+1} > P_t$ for $P_t \in [\varphi_0, \theta_{\rm sup})$ about $P_0 = \varphi_0 \in (\theta_{\rm inf}, \theta_{\rm sup})$.

Then, from the same arguments as those in the last part of Appendix D, we have the result of Theorem 5.2.

Appendix F. Proof of Theorem 5.3

The same as the proof of Theorem 5.2 in Appendix E, it is sufficient to investigate the case of $P_0 = \varphi_0 \in (\theta_{\inf}, \theta_{\sup}) \subset (0, 1]$, and let us consider the root for the equation $\mathscr{G}(P; \varphi_0) = 0$ in terms of $P \in [\varphi_0, \theta_{\sup}) \subset (\theta_{\inf}, \theta_{\sup})$. The equation $\mathscr{G}(P; \varphi_0) = 0$ is equivalent to (E.1) in Appendix E.

With the monotonically increasing $f_+(\xi) > 0$ for $\xi \in (\xi_{\inf}, \xi_{\sup})$, the right side of (E.1) is monotonically increasing and *convex* in terms of $P \in [\varphi_0, \theta_{\sup})$, since its derivative is positive and monotonically increasing in terms of $P \in [\varphi_0, \theta_{\sup})$. The left side of (E.1) is a line with the slope $1/(1 - \varphi_0)$ in terms of P, which becomes 0 for $P = P_0 = \varphi_0$ and $(\theta_{\sup} - \varphi_0)/(1 - \varphi_0) \leq 1$ for $P = \theta_{\sup}$. As already shown in the last part of Appendix E, it holds that $\mathscr{G}(\varphi_0; \varphi_0) > 0$ and $\mathscr{G}(\theta_{\sup}; \varphi_0) \geq 0$, so that the right side of (E.1) is greater than the left side for $P = \varphi_0$, and greater than or equal to the left side for $P = \theta_{\sup}$, respectively.

Now, suppose that the equation $\mathscr{G}(P; \varphi_0) = 0$ has a root in terms of $P \in [\varphi_0, \theta_{\sup}) \subset (\theta_{\inf}, \theta_{\sup})$. For a monotonically increasing threshold distribution, the equation $\mathscr{G}(P; \varphi_0) = 0$, equivalently (E.1), may have at most two roots since the right side of (E.1) is continuous, monotonically increasing and convex in terms of $P \in [\varphi_0, \theta_{\sup})$. If the equation has two roots P^* and P^{**} in $[\varphi_0, \theta_{\sup})$ such that $\varphi_0 < P^* < P^{**} < \theta_{\sup}$, then the left side of (E.1) is greater than the right side only for $P \in (P^*, P^{**})$. This is because $\mathscr{G}(P_0; \varphi_0) = \mathscr{G}(\varphi_0; \varphi_0) > 0$ and $\mathscr{G}(\theta_{\sup}; \varphi_0) \geq 0$ as shown in the last part of Appendix E. If there is only one root P^* such that $\varphi_0 < P^* < \theta_{\sup}$, then the left side of (E.1) is less than the right side for $P \in (\varphi_0, \theta_{\sup}) \backslash \{P^*\}$ or it is greater than the right side only for $P \in (P^*, \theta_{\sup}$ with $\mathscr{G}(\theta_{\sup}; \varphi_0) = 0$. The latter case occurs only if $\theta_{\sup} = 1$.

From the nature of each side of (E.1), a root $P^* \in (\varphi_0, \theta_{\sup})$ exists if and only if the right side of (E.1) has the slope $1/(1-\varphi_0)$ at a unique value $P = P^\dagger \in (\varphi_0, \theta_{\sup})$, and becomes less than or equal to the left side for $P = P^\dagger$. That is, from the derivative of the right side of (E.1) in terms of P, we have the following lemma.

Lemma F.1. *The equation* $\mathscr{G}(P; \varphi_0) = 0$ *in terms of* P *has a root in* $(\varphi_0, \theta_{\sup}) \subset (\theta_{\inf}, \theta_{\sup})$ *if and only if the equation*

$$\alpha f_+(\alpha P) = \frac{1}{1 - \varphi_0} \tag{F.1}$$

has a root $P = P^\dagger \in (\varphi_0, \theta_{\sup})$ *such that*

$$\frac{P^\dagger - \varphi_0}{1 - \varphi_0} \geq \int_{\xi_{\inf}}^{\alpha P^\dagger} f_+(\xi)\, d\xi. \tag{F.2}$$

If exists, such a $P^\dagger$ is uniquely determined by Eq. (F.1) for given $P_0 = \varphi_0 \in (\theta_{\inf}, \theta_{\sup})$, and it is monotonically increasing in terms of φ_0. This is because the right side of (F.1) is monotonically increasing in terms of $\varphi_0 \in (\theta_{\rm inf}, \theta_{\rm sup})$ while the left side is monotonically increasing in terms of $P \in (\theta_{\rm inf}, \theta_{\rm sup})$.

To find the condition for the existence of such a root $P^\dagger$ of (F.1) satisfying (F.2), we first show the following lemma.

Lemma F.2. *Suppose that there exists a root* $P = P^\dagger \in (\varphi_0, \theta_{\rm sup})$ *for Eq.* (F.1) *with* $\varphi_0 \in (\theta_{\rm inf}, \theta_{\rm sup})$, *satisfying the condition* (F.2). *Then the condition* (5.2) *is necessarily satisfied.*

Proof. When the root $P = P^\dagger$ for (F.1) exists in $(\varphi_0, \theta_{\rm sup}) \subset (\theta_{\rm inf}, \theta_{\rm sup})$, we have $\varphi_0 = 1 - 1/\{\alpha f_+(\alpha P^\dagger)\}$, and then the condition (F.2) can be expressed equivalently as $Q(P^\dagger) \geq 0$ with $Q(P)$ defined by (5.5). Since $f_+(\alpha P)$ is continuous, monotonically increasing, and almost everywhere differentiable for $P \in (\theta_{\rm inf}, \theta_{\rm sup})$ from the definition, we find that $Q'(P) = -(1-P)\alpha^2 f'_+(\alpha P) < 0$ for $P \in (\theta_{\rm inf}, \theta_{\rm sup})$ where $f_+(\alpha P)$ is differentiable. Since $Q(P)$ is continuous in terms of $P \in (\theta_{\rm inf}, \theta_{\rm sup})$, we can hence find that $Q(P)$ is monotonically decreasing in terms of $P \in (\theta_{\rm inf}, \theta_{\rm sup})$. Besides we have

$$\lim_{P \to \theta_{\rm sup} - 0} Q(P) = -(1 - \theta_{\rm sup}) \lim_{\xi \to \xi_{\rm sup} - 0} \alpha f_+(\xi) \leq 0,$$

where the equality holds when and only when $\theta_{\rm sup} = 1$, because

$$\alpha f_+^{\rm sup} := \lim_{\xi \to \xi_{\rm sup} - 0} \alpha f_+(\xi) > 0$$

for the monotonically increasing threshold distribution $f_+(\xi)$ satisfying the conditions given in Sec. 4.1. Here, we note that $\alpha f_+^{\rm sup}$ is defined finite from the boundedness of $f_+(\xi)$ as assumed in Sec. 4.1.

Therefore when $Q(P^\dagger) \geq 0$ for $P^\dagger \in (\varphi_0, \theta_{\rm sup}) \subset (\theta_{\rm inf}, \theta_{\rm sup})$, it is necessary that

$$\lim_{P \to \theta_{\rm inf} + 0} Q(P) = 1 - (1 - \theta_{\rm inf}) \lim_{\xi \to \xi_{\rm inf} + 0} \alpha f_+(\xi) > 0,$$

because otherwise $Q(P) < 0$ for any $P \in (\theta_{\rm inf}, \theta_{\rm sup})$, which contradicts the existence of such a root $P = P^\dagger \in (\varphi_0, \theta_{\rm sup})$. This necessary condition becomes (5.2). □

On the other hand, we have the following fundamental nature of $f_+(\xi)$ as the density distribution function introduced by (4.2).

Lemma F.3. *For the density distribution function* (4.2) *with a monotonically increasing* $f_+(\xi)$, *it is satisfied that*

$$\alpha f_+^{\rm sup} > \frac{1}{1 - \theta_{\rm inf}}. \tag{F.3}$$

Proof. Since $f_+(\xi)$ is monotonically increasing, we have

$$\int_{\xi_{\inf}}^{\xi_{\sup}} f_+(\xi)\,d\xi < \int_{\xi_{\inf}}^{\xi_{\sup}} \sup_{\xi\in(\xi_{\inf},\xi_{\sup})} f_+(\xi)\,d\xi$$
$$= (\xi_{\sup}-\xi_{\inf})f_+^{\sup} = (\theta_{\sup}-\theta_{\inf})\alpha f_+^{\sup} \le (1-\theta_{\inf})\alpha f_+^{\sup}.$$

The left side becomes 1 from the definition of the density distribution function $f(\xi)$ by (4.2). This inequality proves the lemma. □

Then from Lemmas F.2 and F.3, we can obtain the following result on the existence of $P^\dagger \in (\varphi_0, \theta_{\sup})$.

Lemma F.4. *When and only when*

$$\alpha f_+^{\inf} \le \frac{1}{1-\theta_{\inf}}. \tag{F.4}$$

Equation (F.1) *has a unique root* $P = P^\dagger \in (\varphi_0, \theta_{\sup})$ *for every* $P_0 = \varphi_0 \in (\theta_{\inf}, \varphi_0^*) \subset (\theta_{\inf}, \theta_{\sup})$ *with*

$$\varphi_0^* := \min\left\{\theta_{\sup}, 1 - \frac{1}{\alpha f_+^{\sup}}\right\}. \tag{F.5}$$

For any $P_0 = \varphi_0 \ge \varphi_0^*$, *no such root* $P^\dagger$ *exists.*

Proof. About $\varphi_0 \in (\theta_{\inf}, \theta_{\sup})$, we now have

$$\frac{1}{1-\varphi_0} \in \left(\frac{1}{1-\theta_{\inf}},\ \frac{1}{1-\theta_{\sup}}\right)$$

for $\theta_{\sup} \in (\theta_{\inf}, 1) \subset (0,1)$. Mathematically, we can have $1/(1-\varphi_0) \in (1/(1-\theta_{\inf}), \infty)$ for $\theta_{\sup} = 1$. On the other hand, when the condition (F.4) is satisfied, we have

$$\alpha f_+^{\inf} \le \frac{1}{1-\theta_{\inf}} < \alpha f_+^{\sup},$$

taking account of Lemma F.3. Since $\alpha f_+(\alpha P) \in (\alpha f_+^{\inf}, \alpha f_+^{\sup})$, we therefore have a unique root $P^\dagger \in (\theta_{\inf}, \theta_{\sup})$ of Eq. (F.1) for every

$$\frac{1}{1-\varphi_0} \in \left(\frac{1}{1-\theta_{\inf}},\ \min\left\{\frac{1}{1-\theta_{\sup}},\ \alpha f_+^{\sup}\right\}\right)$$

about $\theta_{\sup} \in (\theta_{\inf}, 1)$, or for every

$$\frac{1}{1-\varphi_0} \in \left(\frac{1}{1-\theta_{\inf}},\ \alpha f_+^{\sup}\right)$$

about $\theta_{\sup} = 1$. This result means that we have a unique root $P^\dagger \in (\theta_{\inf}, \theta_{\sup})$ of Eq. (F.1) for every $\varphi_0 \in (\theta_{\inf}, \varphi_0^*)$ with φ_0^* given by (F.5) when the condition (F.4) is satisfied.

Inversely, when there is a unique root $P^\dagger \in (\theta_{\rm inf}, \theta_{\rm sup})$ of Eq. (F.1) for *every* $\varphi_0 \in (\theta_{\rm inf}, \varphi_0^*)$ with φ_0^* given by (F.5), the condition (F.4) must be satisfied. Unless the condition (F.4) is satisfied, we have

$$\alpha f_+^{\rm inf} > \frac{1}{1-\theta_{\rm inf}},$$

which means that there could be no root $P^\dagger \in (\theta_{\rm inf}, \theta_{\rm sup})$ of Eq. (F.1) for φ_0 sufficiently near and greater than $\theta_{\rm inf}$. This is contradictory. Hence when there is a unique root $P^\dagger \in (\theta_{\rm inf}, \theta_{\rm sup})$ of Eq. (F.1) for *every* $\varphi_0 \in (\theta_{\rm inf}, \varphi_0^*)$ with φ_0^* given by (F.5), the condition (F.4) must be satisfied. □

The result given by Lemma F.4 subsequently serves us to obtain the following considerable feature about the condition for the initial frequency of acceptors $P_0 = \varphi_0$ according to the existence of $P^\dagger$ satisfying the condition (F.2).

Lemma F.5. *When and only when the condition* (5.2) *is satisfied, there exists a certain value* $\varphi_0^{\rm c} \in (\theta_{\rm inf}, \varphi_0^*) \subset (\theta_{\rm inf}, \theta_{\rm sup})$ *such that Eq.* (F.1) *has a unique root* $P = P^\dagger \in (\varphi_0, \theta_{\rm sup})$ *which satisfies the condition* (F.2) *for every* $\varphi_0 \leq \varphi_0^{\rm c}$, *while no such root* $P^\dagger$ *exists for any* $\varphi_0 > \varphi_0^{\rm c}$.

Proof. When such a unique root $P = P^\dagger$ exists, we have

$$\lim_{P\to\theta_{\rm inf}+0} Q(P) > 0; \qquad \lim_{P\to\theta_{\rm sup}-0} Q(P) \leq 0,$$

where the equality holds when and only when $\theta_{\rm sup} = 1$, as shown in the proof of Lemma F.2 making use of the function $Q(P)$ defined by (5.5). Since the function $Q(P)$ is continuous and monotonically decreasing in terms of $P \in (\theta_{\rm inf}, \theta_{\rm sup})$, there exists a specific value $P_c \in (\theta_{\rm inf}, \theta_{\rm sup})$ for $\theta_{\rm sup} \in (\theta_{\rm inf}, 1)$ such that $Q(P_c) = 0$, $Q(P) > 0$ for $P < P_c$, and $Q(P) < 0$ for $P > P_c$. In contrast, for $\theta_{\rm sup} = 1$, $Q(P) > 0$ for any $P \in (\theta_{\rm inf}, \theta_{\rm sup}) = (\theta_{\rm inf}, 1)$, and $Q(\theta_{\rm sup}) = Q(1) = 0$. Let us consider hereafter the case of $\theta_{\rm sup} \in (\theta_{\rm sup}, 1)$.

When $1 - 1/\alpha f_+^{\rm sup} \leq \theta_{\rm sup}$, that is, when $1/(1-\varphi_0^*) = \alpha f_+^{\rm sup}$ from (F.5), we note that $P^\dagger \to \theta_{\rm sup} - 0$ as $\varphi_0 \to \varphi_0^* - 0$ because of Eq. (F.1) and the definition of $\alpha f_+^{\rm sup}$ given in (F.3). Then $Q(P^\dagger)$ becomes negative as $P^\dagger \to \theta_{\rm sup} - 0$ with $\varphi_0 \to \varphi_0^* - 0$, since $Q(P)$ becomes negative as $P \to \theta_{\rm sup} - 0$ as shown in the proof of Lemma F.2.

When $\varphi_0^* = \theta_{\rm sup}$ from (F.5) with $1/(1-\theta_{\rm sup}) < \alpha f_+^{\rm sup}$, we have $P^\dagger < \theta_{\rm sup}$ for Eq. (F.1) since this is the case where

$$\begin{aligned}\alpha f_+(\alpha P^\dagger) &= \frac{1}{1-\varphi_0} < \frac{1}{1-\theta_{\rm sup}} < \alpha f_+^{\rm sup} \\ &= \lim_{\xi\to\xi_{\rm sup}-0} \alpha f_+(\xi) = \lim_{P\to\theta_{\rm sup}-0} \alpha f_+(\alpha P).\end{aligned}$$

Then the left side of (F.2) becomes negative as $\varphi_0 \to \varphi_0^* - 0 = \theta_{\rm sup} - 0$ because $P^\dagger - \varphi_0 \to P^\dagger - \theta_{\rm sup} < 0$, while the right side is positive. This means that $Q(P^\dagger)$

becomes negative as $\varphi_0 \to \varphi_0^* - 0$. As a result, we find that $Q(P^\dagger)$ becomes negative as $\varphi_0 \to \varphi_0^* - 0$ for $\theta_{\sup} \in (\theta_{\inf}, 1)$.

On the other hand, from Lemma F.4, when the condition (5.2) is satisfied, we have a unique root of Eq. (F.1), $P^\dagger \in (\theta_{\inf}, \theta_{\sup})$ for every $\varphi_0 \in (\theta_{\inf}, \varphi_0^*)$ with φ_0^* given by (F.5). When the condition (5.2) is satisfied, $P^\dagger$ approaches a value in $(\theta_{\inf}, \theta_{\sup})$ as $\varphi_0 \to \theta_{\inf} + 0$, because $P^\dagger$ is continuous and monotonically increasing in terms of $\varphi_0 \in (\theta_{\inf}, \varphi_0^*)$:

$$P^\dagger_{\inf} := \lim_{\varphi_0 \to \theta_{\inf}+0} P^\dagger \in (\theta_{\inf}, \theta_{\sup})$$

with

$$\alpha f_+(\alpha P^\dagger_{\inf}) = \frac{1}{1-\theta_{\inf}}.$$

Then, for the monotonically increasing $f_+(\xi)$, we find that

$$\begin{aligned}\lim_{\varphi_0 \to \theta_{\inf}+0} \int_{\xi_{\inf}}^{\alpha P^\dagger} f_+(\xi)\, d\xi &= \int_{\alpha\theta_{\inf}}^{\alpha P^\dagger_{\inf}} f_+(\xi)\, d\xi \\ &< \int_{\alpha\theta_{\inf}}^{\alpha P^\dagger_{\inf}} f_+(\alpha P^\dagger_{\inf})\, d\xi \\ &= (\alpha P^\dagger_{\inf} - \alpha\theta_{\inf}) f_+(\alpha P^\dagger_{\inf}) = \frac{P^\dagger_{\inf} - \theta_{\inf}}{1-\theta_{\inf}}.\end{aligned}$$

Thus the condition (F.2) comes to hold as $\varphi_0 \to \theta_{\inf} + 0$. Hence $Q(P^\dagger)$ necessarily becomes positive with $P^\dagger \to P^\dagger_{\inf} \in (\theta_{\inf}, \theta_{\sup})$ as $\varphi_0 \to \theta_{\inf} + 0$.

Therefore, we find that there exists a unique $P_c \in (P^\dagger_{\inf}, \theta_{\sup}) \subset (\theta_{\inf}, \theta_{\sup})$ for $\theta_{\sup} \in (\theta_{\inf}, 1)$, as defined at the beginning of this proof such that $Q(P_c) = 0$, $Q(P) > 0$ for $P < P_c$, and $Q(P) < 0$ for $P > P_c$. Since $P^\dagger$ is monotonically increasing in terms of $\varphi_0 \in (\theta_{\inf}, \varphi_0^*)$, a unique value of $\varphi_0 \in (\theta_{\inf}, \varphi_0^*)$ is determined by (F.1) for $P = P_c \in (P^\dagger_{\inf}, \theta_{\sup})$, which is now denoted by φ_0^c and given by (5.4).

Now, we have $P^\dagger < P_c$ for $\varphi_0 < \varphi_0^c$, and $P^\dagger > P_c$ for $\varphi_0 > \varphi_0^c$. Thus, about $\theta_{\sup} \in (\theta_{\inf}, 1)$, we have $Q(P^\dagger) > 0$ for $\varphi_0 \in (\theta_{\inf}, \varphi_0^c)$, $Q(P^\dagger) < 0$ for $\varphi_0 \in (\varphi_0^c, \varphi_0^*)$, and $Q(P^\dagger) = Q(P_c) = 0$ for $\varphi_0 = \varphi_0^c$. Hence, the condition (F.2) holds for $\varphi_0 \in (\theta_{\inf}, \varphi_0^c]$ and does not for $\varphi_0 \in (\varphi_0^c, \varphi_0^*)$.

Inversely, when such a critical value φ_0^c exists with a value $P^\dagger$ satisfying the condition (F.2) as the root for Eq. (F.1), Lemma F.2 indicates that the condition (5.2) holds. Consequently these arguments prove the lemma. □

As for the temporal variation of P_t when a root $P^* \in (\varphi_0, \theta_{\sup})$ exists for the equation $\mathscr{G}(P; \varphi_0) = 0$, we can find the following lemma.

Lemma F.6. *For the monotonically increasing threshold distribution, if the equation* $\mathscr{G}(P; \varphi_0) = 0$ *has a root in* $(\varphi_0, \theta_{\sup})$ *for* $P_0 = \varphi_0 \in (\theta_{\inf}, \theta_{\sup})$, *then* P_t *converges to the smallest root* $P^* \in (\varphi_0, \theta_{\sup})$ *as* $t \to \infty$.

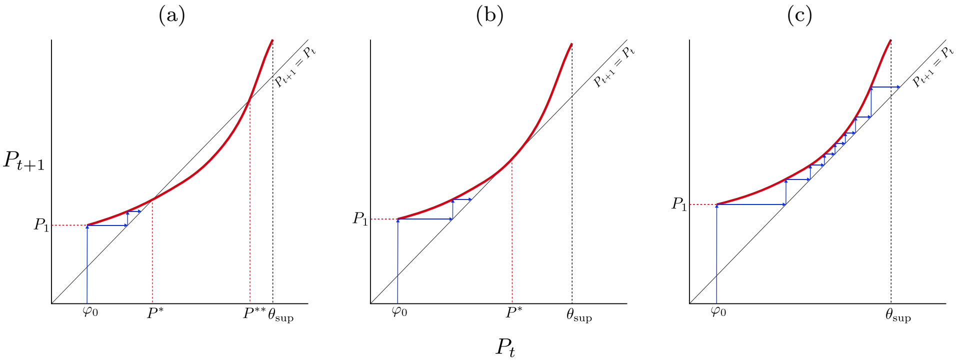

Fig. F.1. Illustrative figures of the cobwebbing method about the behavior of the sequence $\{P_t\}$ generated by (4.4) with $P_0 = \varphi_0 \in (\theta_{\inf}, \theta_{\sup})$. The equation $\mathscr{G}(P; \varphi_0) = 0$ in terms of P has (a) two different roots P^* and P^{**}; (b) a single root P^*; (c) no root in $(\theta_{\inf}, \theta_{\sup})$.

Proof. First we note that $P_t = P^*$ makes $P_{t+1} = P^*$ as easily found from (4.4). This means that P^* is an equilibrium value for the dynamical system with the recurrence relation (4.4) with $P_0 = \varphi_0 \in (\theta_{\inf}, \theta_{\sup})$. ext, since $\mathscr{G}(P; \varphi_0)$ is continuous in terms of $P \in [\varphi_0, \theta_{\sup}) \subset (\theta_{\inf}, \theta_{\sup})$ with $\mathscr{G}(P_0; \varphi_0) = \mathscr{G}(\varphi_0; \varphi_0) > 0$, we have $\mathscr{G}(P; \varphi_0) > 0$ for any $P \in [\varphi_0, P^*)$. Hence, from (4.4), we have $P_{t+1} = \Psi(P_t) > P_t$ for $P_t \in [\varphi_0, P^*)$.

Now, we apply the cobwebbing method[56–58] to find the behavior of the sequence $\{P_t\}$ in $[\varphi_0, \theta_{\sup})$, making use of the continuity and increasing monotonicity of $\Psi(P_t)$ which has been shown in Lemma 2.2. As already indicated in the first part of this appendix, the equation $\mathscr{G}(P; \varphi_0) = 0$ has at most two roots in terms of $P \in [\varphi_0, \theta_{\sup})$, and Lemma F.5 gives the condition for the existence of such a case. As shown in Fig. F.1, the cobwebbing method indicates that $P_t \to P^*$ as $t \to \infty$ when such a root P^* exists. This proves the lemma. □

As shown by Lemma F.1, the existence of the root $P = P^\dagger \in (\varphi_0, \theta_{\sup})$ of Eq. (F.1) with the condition (F.2) means the existence of a root P^* of the equation $\mathscr{G}(P; \varphi_0) = 0$ in $(\varphi_0, \theta_{\sup})$. Therefore, getting together the results and arguments by Lemmas 4.3, F.5, F.6, and their proofs, we can obtain the result given in Theorem 5.3.

Appendix G. Proof of Corollary 5.1

Since the equilibrium acceptor frequency P^* is uniquely determined as the smallest root $P = P^* \in (\varphi_0, \theta_{\sup})$ of the equation $\mathscr{G}(P; \varphi_0) = 0$ where $\mathscr{G}(P; \varphi_0)$ is a continuous function of P and φ_0 for $P \in [\varphi_0, 1]$ and $\varphi_0 \in (0, 1]$, it is clear that P^* is continuous in terms of $\varphi_0 \in (\theta_{\inf}, \varphi_0^c]$ when it exists under the condition (5.2) as shown by Theorem 5.3. It can be easily found that the equation $\mathscr{G}(P; \theta_{\inf}) = 0$ has the root $P = \theta_{\inf}$. Hence, from the continuity of P^* in terms of $\varphi_0 \in (\theta_{\inf}, \varphi_0^c]$, it is necessary that $P^* \to \varphi_0$ as $\varphi_0 \to \theta_{\inf} + 0$.

As shown in the proof of Lemma F.5 in Appendix F, we have $P^{\dagger} = P_c$ and $Q(P^{\dagger}) = Q(P_c) = 0$ for $\varphi_0 = \varphi_0^c$. Since the equation $Q(P_c) = 0$ is equivalent to Eq. (E.1) in Appendix E, we have $\mathscr{G}(P_c; \varphi_0^c) = 0$. Further, since $P^{\dagger}$ is defined as the value of P at which the slope of the right side of (E.1) is equal to $1/(1-\varphi_0)$, the equation $\mathscr{G}(P_c; \varphi_0^c) = 0$ with $P^{\dagger} = P_c$ means that the root of the equation $\mathscr{G}(P; \varphi_0^c) = 0$ in terms of P is unique and given by $P = P_c$. Then we have the case of Fig. F.1(b). Thus, we find that $P^* = P_c$ for $\varphi_0 = \varphi_0^c$. Consequently these arguments prove the corollary.

Appendix H. Proof of Theorem 5.4

Some parts in the proof of Theorem 5.4 follow the arguments corresponding to those in the proof of Theorem 5.3 (Appendix F), accompanied with some necessary modification according to the unimodal threshold distribution.

The equation $\mathscr{G}(P; \varphi_0) = 0$ is equivalent to (E.1) in Appendix E. With the unimodality of $f_+(\xi) > 0$ for $\xi \in (\xi_{\inf}, \xi_{\sup})$, the right side of (E.1) is continuous, monotonically increasing, and S-shaped in terms of $P \in [\varphi_0, \theta_{\sup})$, since its second derivative is positive for $P \in [\varphi_0, \theta_m)$ and negative for $P \in (\theta_m, \theta_{\sup})$ wherever $f_+(\alpha P)$ is differentiable.

First, the right side of (E.1) is continuous, monotonically increasing, and concave for $P \in [\varphi_0, \theta_{\sup})$ when $\varphi_0 \geq \theta_m$. Then, since it is greater than the left side for $P = \varphi_0 \geq \theta_m$ because of $\mathscr{G}(\varphi_0; \varphi_0) > 0$, and greater than or equal to the left side for $P = \theta_{\sup}$ (as shown in the last part of Appendix E), we have the following lemma.

Lemma H.1. *It holds that* $\mathscr{G}(P; \varphi_0) > 0$ *in* $(\varphi_0, \theta_{\sup})$ *for* $\varphi_0 \in [\theta_m, \theta_{\sup})$ *according to the unimodal threshold distribution.*

Hence if there exists a root of the equation $\mathscr{G}(P; \varphi_0) = 0$, it is only when $\varphi_0 \in (\theta_{\inf}, \theta_m)$. Besides this result of Lemma H.1 indicates the following result.

Lemma H.2. *According to the unimodal threshold distribution, the sequence* $\{P_t\}$ *monotonically increases and converges to* 1 *as* $t \to \infty$ *for the initial frequency of acceptors* $\varphi_0 \in [\theta_m, \theta_{\sup})$.

Next, when $\varphi_0 \in (\theta_{\inf}, \theta_m)$, we have the following lemma on the root of the equation $\mathscr{G}(P; \varphi_0) = 0$:

Lemma H.3. *If the equation* $\mathscr{G}(P; \varphi_0) = 0$ *in terms of* P *for* $\varphi_0 \in (\theta_{\inf}, \theta_m)$ *has a root in* $(\varphi_0, \theta_{\sup})$ *according to the unimodal threshold distribution, the smallest root must be in* (φ_0, θ_m).

Proof. If there is no root of the equation $\mathscr{G}(P; \varphi_0) = 0$ in (φ_0, θ_m), we have $\mathscr{G}(\theta_m; \varphi_0) \geq 0$ because $\mathscr{G}(P; \varphi_0)$ is continuous in terms of $P \in [\varphi_0, \theta_{\sup})$ with

$\mathscr{G}(\varphi_0;\varphi_0) > 0$. In addition, when $\mathscr{G}(P;\varphi_0) > 0$ in (φ_0, θ_m) for $\varphi_0 \in (\theta_{\rm inf}, \theta_m)$, we find it impossible that $\mathscr{G}(\theta_m;\varphi_0) = 0$. Suppose that $\mathscr{G}(P;\varphi_0) > 0$ in (φ_0, θ_m) and $\mathscr{G}(\theta_m;\varphi_0) = 0$ for $\varphi_0 \in (\theta_{\rm inf}, \theta_m)$. Then, since the right side of (E.1) is continuous, monotonically increasing, convex for $P \in (\varphi_0, \theta_m)$, and concave for $P \in (\theta_m, \theta_{\rm sup})$, it necessarily holds that $\mathscr{G}(P;\varphi_0) < 0$ for $P \in (\theta_m, \theta_{\rm sup}]$. This is contradictory because the right side of (E.1) is greater than or equal to the left side for $P = \theta_{\rm sup}$ (as shown in the last part of Appendix E). Then this arguments indicates also that $P = \theta_m$ cannot be the smallest root for the equation $\mathscr{G}(P;\varphi_0) = 0$.

Now, we have $\mathscr{G}(\theta_m;\varphi_0) > 0$ when there is no root of the equation $\mathscr{G}(P;\varphi_0) = 0$ in (φ_0, θ_m). Since the right side of (E.1) is continuous, monotonically increasing, concave for $P \in (\theta_m, \theta_{\rm sup})$, and greater than or equal to the left side for $P = \theta_{\rm sup}$, this result subsequently indicates that $\mathscr{G}(P;\varphi_0) > 0$ in $(\theta_m, \theta_{\rm sup})$ for $\varphi_0 \in (\theta_{\rm inf}, \theta_m)$ when there is no root of the equation $\mathscr{G}(P;\varphi_0) = 0$ in (φ_0, θ_m).

As a result, when there is no root of the equation $\mathscr{G}(P;\varphi_0) = 0$ in (φ_0, θ_m), we have $\mathscr{G}(P;\varphi_0) > 0$ for $P \in (\theta_m, \theta_{\rm sup})$. This means that the equation $\mathscr{G}(P;\varphi_0) = 0$ has no root in $(\varphi_0, \theta_{\rm sup})$. Hence if the equation $\mathscr{G}(P;\varphi_0) = 0$ has a root in $(\varphi_0, \theta_{\rm sup})$, there must be a root in $(\varphi_0, \theta_{\rm inf})$. This arguments consequently prove the lemma. □

From Lemmas H.1 and H.3, we find it sufficient to consider the condition to have a root of the equation $\mathscr{G}(P;\varphi_0) = 0$ in (φ_0, θ_m) for $\varphi_0 \in (\theta_{\rm inf}, \theta_m)$. Since $\mathscr{G}(P;\varphi_0)$ is continuous and convex in (φ_0, θ_m) with $\mathscr{G}(\varphi_0;\varphi_0) > 0$, the equation $\mathscr{G}(P;\varphi_0) = 0$ has a root in $(\varphi_0, \theta_m]$ if $\mathscr{G}(\theta_m;\varphi_0) \le 0$, that is, if the right side of (E.1) is less than or equal to the left side for $P = \theta_m$, which leads to the condition $\varphi_0 \le \varphi_0^s$ where φ_0^s is defined by

$$\varphi_0^s := \frac{\theta_m - \int_{\xi_{\rm inf}}^{\xi_m} f_+(\xi)\, d\xi}{1 - \int_{\xi_{\rm inf}}^{\xi_m} f_+(\xi)\, d\xi} \quad \in (\theta_{\rm inf}, \theta_m). \tag{H.1}$$

It can be easily proved that φ_0^s is necessarily less than θ_m. However the critical value φ_0^s is valid if and only if it is greater than $\theta_{\rm inf}$, because the condition $\varphi_0 \le \varphi_0^s$ is meaningful only for $\varphi_0 > \theta_{\rm inf}$. Therefore we can take the condition $\varphi_0 \le \varphi_0^s$ only if it is satisfied that $\varphi_0^s > \theta_{\rm inf}$, that is,

$$\frac{1}{\theta_m - \theta_{\rm inf}} \int_{\xi_{\rm inf}}^{\xi_m} f_+(\xi)\, d\xi < \frac{1}{1 - \theta_{\rm inf}}. \tag{H.2}$$

Unless the condition (H.2) is satisfied, then $\varphi_0^s \le \theta_{\rm inf}$, and the condition $\varphi_0 \le \varphi_0^s$ does not hold for any $\varphi_0 \in (\theta_{\rm inf}, \theta_m)$, so that $\mathscr{G}(\theta_m;\varphi_0) \le 0$ cannot hold for any $\varphi_0 \in (\theta_{\rm inf}, \theta_m)$. In such a case, we have $\mathscr{G}(\theta_m;\varphi_0) > 0$ for any $\varphi_0 \in (\theta_{\rm inf}, \theta_m)$. From these arguments and Lemma H.3, we obtain the following result.

Lemma H.4. *When the condition* (H.2) *is satisfied, the equation* $\mathscr{G}(P;\varphi_0) = 0$ *has a root in* (φ_0, θ_m) *for any* $\varphi_0 \in (\theta_{\rm inf}, \varphi_0^s]$ *where* $\varphi_0^s \in (\theta_{\rm inf}, \theta_m)$ *is defined by* (H.1).

Note that, even when the condition (H.2) is satisfied, we have $\mathscr{G}(\theta_m;\varphi_0) > 0$ for $\varphi_0 \geq \varphi_0^s$ with $\varphi_0 < \theta_m$. The existence of a root for the equation $\mathscr{G}(P;\varphi_0) = 0$ is still nontrivial when $\mathscr{G}(\theta_m;\varphi_0) > 0$. Let us consider it next.

Since the right side of (E.1) is continuous, monotonically increasing, and covex in terms of $P \in (\varphi_0, \theta_m)$, being greater than the left side for $P = \varphi_0$ because of $\mathscr{G}(\varphi_0;\varphi_0) > 0$, the existence of a root for the equation $\mathscr{G}(P;\varphi_0) = 0$ with $\mathscr{G}(\theta_m;\varphi_0) > 0$ can be argued out along the same line as the proof of Theorem 5.3 in Appendix F about the monotonically increasing threshold distribution.

When $\mathscr{G}(\theta_m;\varphi_0) > 0$, the equation $\mathscr{G}(P;\varphi_0) = 0$ has a root in (φ_0, θ_m) for $\varphi_0 \in (\theta_{\inf}, \theta_m)$, if and only if the right side of (E.1) has the slope $1/(1-\varphi_0)$ in the convex part in (φ_0, θ_m) as a function of P at a unique value $P = P^\dagger \in (\varphi_0, \theta_m)$, and becomes less than or equal to the left side for $P = P^\dagger$. This is from the same arguments for the proof of Theorem 5.3 in Appendix F. Then, correspondingly to Lemma F.1, we have the following lemma.

Lemma H.5. *When $\mathscr{G}(\theta_m;\varphi_0) > 0$, the equation $\mathscr{G}(P;\varphi_0) = 0$ has a root in (φ_0, θ_m) for $\varphi_0 \in (\theta_{\inf}, \theta_m)$ if and only if Eq.* (F.1) *has a root $P = P^\dagger \in (\varphi_0, \theta_m)$ such that it satisfies* (F.2).

Such a root $P^\dagger \in (\varphi_0, \theta_m)$ is uniquely determined by (F.1) for $\varphi_0 \in (\theta_{\inf}, \theta_m)$ if it exists, and it is monotonically increasing in terms of φ_0. This is because the right side of (F.1) is monotonically increasing in terms of $\varphi_0 \in (\theta_{\inf}, \theta_m)$ while the left side is monotonically increasing in terms of $P \in (\theta_{\inf}, \theta_m)$. To find the condition for the existence of such a root $P^\dagger$ of (F.1) satisfying (F.2), we first show the following lemma corresponding to Lemma F.2.

Lemma H.6. *When $\mathscr{G}(\theta_m;\varphi_0) > 0$, suppose that there exists a root $P = P^\dagger \in (\varphi_0, \theta_m)$ for Eq.* (F.1) *with $\varphi_0 \in (\theta_{\inf}, \theta_m)$, satisfying the condition* (F.2)*. Then the condition* (5.2) *is necessarily satisfied.*

Proof. When the root $P = P^\dagger$ for (F.1) exists in (φ_0, θ_m), we have $\varphi_0 = 1 - 1/\{\alpha f_+(\alpha P^\dagger)\}$, and then the condition (F.2) can be expressed as $Q(P^\dagger) \geq 0$ with $Q(P)$ defined by (5.5). Since $f_+(\alpha P)$ is continuous, monotonically increasing, and almost everywhere differentiable for $P \in (\theta_{\inf}, \theta_m)$ from the definition, we find that $Q'(P) = -(1-P)\alpha^2 f'_+(\alpha P) < 0$ for almost every $P \in (\theta_{\inf}, \theta_m)$. and $Q'(P) > 0$ for almost every $P \in (\theta_m, \theta_{\sup})$. Since $Q(P)$ is continuous in terms of $P \in (\theta_{\inf}, \theta_m)$, we can hence find that $Q(P)$ is decreasing for $P \in (\theta_{\inf}, \theta_m)$ and increasing for $P \in (\theta_m, \theta_{\sup})$. As shown in the proof of Lemma F.2, we have $\lim_{P\to\theta_{\sup}-0} Q(P) \leq 0$ where the equality holds when and only when $\theta_{\sup} = 1$. Thus, since $Q(P)$ is monotonically increasing for $P \in (\theta_m, \theta_{\sup})$, we have $Q(P) < 0$ for $P \in [\theta_m, \theta_{\sup})$. Hence, if $\lim_{P\to\theta_{\inf}+0} Q(P) \leq 0$, there is no value of $P \in (\theta_{\inf}, \theta_m)$ such that $Q(P) > 0$. Therefore, to have a value of $P \in (\theta_{\inf}, \theta_m)$ such that $Q(P) > 0$, it is necessary that $\displaystyle\lim_{P\to\theta_{\inf}+0} Q(P) > 0$. Then with the same arguments as in the proof of Lemma F.2, we can prove that the condition (5.2) is necessarily satisfied when $P^\dagger$ satisfying the condition (F.2) exists. □

Next, correspondingly to Lemma F.3, we have the following feature about the unimodal $f_+(\xi)$.

Lemma H.7. *For the density distribution function* (4.2) *with the unimodal* $f_+(\xi)$, *it is satisfied that*

$$\alpha f_+^{\max} > \frac{1}{1-\theta_{\inf}}. \tag{H.3}$$

Proof. Since $f_+(\xi)$ is unimodal, we have

$$\begin{aligned}\int_{\xi_{\inf}}^{\xi_{\sup}} f_+(\xi)\,d\xi &< \int_{\xi_{\inf}}^{\xi_{\sup}} f_+(\xi_m)\,d\xi \\ &= (\xi_{\sup}-\xi_{\inf})f_+^{\max} = (\theta_{\sup}-\theta_{\inf})\alpha f_+^{\max} \le (1-\theta_{\inf})\alpha f_+^{\max}.\end{aligned}$$

The left side becomes 1 from the definition of the density distribution function $f(\xi)$ by (4.2). This inequality proves the lemma. □

Then from Lemmas H.6 and H.7, we can obtain the following result on the existence of $P^\dagger \in (\varphi_0, \theta_m)$, correspondingly to Lemma F.4.

Lemma H.8. *When* $\mathscr{G}(\theta_m; \varphi_0) > 0$, *if and only if the condition* (F.4) *is satisfied, Eq.* (F.1) *has a unique root* $P = P^\dagger \in (\varphi_0, \theta_m)$ *for every* $P_0 = \varphi_0 \in (\theta_{\inf}, \varphi_0^{**}) \subset (\theta_{\inf}, \theta_m)$ *with*

$$\varphi_0^{**} := \min\left\{\theta_m, 1 - \frac{1}{\alpha f_+^{\max}}\right\}. \tag{H.4}$$

For any $P_0 = \varphi_0 \ge \varphi_0^{**}$, *no such root* $P^\dagger$ *exists.*

Proof. For $\varphi_0 \in (\theta_{\inf}, \theta_m)$, we have

$$\frac{1}{1-\varphi_0} \in \left(\frac{1}{1-\theta_{\inf}}, \frac{1}{1-\theta_m}\right).$$

On the other hand, if the condition (F.4) is satisfied, we have

$$\alpha f_+^{\inf} \le \frac{1}{1-\theta_{\inf}} < \alpha f_+^{\max},$$

taking account of Lemma H.7. Then we have a unique root $P^\dagger \in (\theta_{\inf}, \theta_m)$ of Eq. (F.1) for every

$$\frac{1}{1-\varphi_0} \in \left(\frac{1}{1-\theta_{\inf}}, \min\left\{\frac{1}{1-\theta_m}, \alpha f_+^{\max}\right\}\right).$$

This result means that we have a unique root $P^\dagger \in (\theta_{\inf}, \theta_m)$ of Eq. (F.1) for every $\varphi_0 \in (\theta_{\inf}, \varphi_0^{**})$ with φ_0^{**} given by (H.4) if the condition (F.4) is satisfied.

Inversely if there is a unique root $P^\dagger \in (\theta_{\inf}, \theta_m)$ of Eq. (F.1) for *every* $\varphi_0 \in (\theta_{\inf}, \varphi_0^{**})$ with φ_0^{**} given by (H.4), the condition (F.4) must be satisfied. Unless the condition (F.4) is satisfied, we have

$$\alpha f_+^{\inf} > \frac{1}{1-\theta_{\inf}},$$

which means that there could be no root $P^\dagger \in (\theta_{\inf}, \theta_m)$ of Eq. (F.1) for φ_0 sufficiently near and greater than $\theta_{\inf}$. This is contradictory. Hence if there is a unique root $P^\dagger \in (\theta_{\inf}, \theta_m)$ of Eq. (F.1) for *every* $\varphi_0 \in (\theta_{\inf}, \varphi_0^{**})$ with φ_0^{**} given by (H.4), the condition (F.4) must be satisfied. □

The result given by Lemma H.8 subsequently gives us the following result corresponding to Lemma F.5.

Lemma H.9. *When* $\mathscr{G}(\theta_m; \varphi_0) > 0$, *if and only if the condition* (5.2) *is satisfied, there exists a certain value* $\varphi_0^c \in (\theta_{\inf}, \varphi_0^{**}) \subset (\theta_{\inf}, \theta_m)$ *such that Eq.* (F.1) *has a unique root* $P = P^\dagger \in (\varphi_0, \theta_m)$ *which satisfies the condition* (F.2) *for every* $\varphi_0 \in (\theta_{\inf}, \varphi_0^c]$, *while no such root* $P^\dagger$ *exists for any* $\varphi_0 > \varphi_0^c$.

Proof. When such a unique root $P = P^\dagger$ exists, we have $\lim_{P\to\theta_{\inf}+0} Q(P) > 0$ and $Q(P) < 0$ for $P \in (\theta_m, \theta_{\sup})$ as shown in the proof of Lemmas F.2 or H.6 making use of the function $Q(P)$ defined by (5.5). Since the function $Q(P)$ is continuous in terms of $P \in (\theta_{\inf}, \theta_{\sup})$, there exists a specific value $P_c \in (\theta_{\inf}, \theta_m)$ such that $Q(P_c) = 0$, $Q(P) > 0$ for $P < P_c$, and $Q(P) < 0$ for $P > P_c$.

When $1 - 1/\alpha f_+^{\max} \leq \theta_m$, that is, when $1/(1-\varphi_0^{**}) = \alpha f_+^{\max}$, we note that $P^\dagger \to \theta_m - 0$ as $\varphi_0 \to \varphi_0^{**} - 0$ because of Eq. (F.1) and the definition of $\alpha f_+^{\max}$ given in (H.3). Then $Q(P^\dagger)$ becomes negative as $P^\dagger \to \theta_m - 0$ with $\varphi_0 \to \varphi_0^{**} - 0$, since $Q(P) < 0$ for $P \in [\theta_m, \theta_{\sup})$.

When $\varphi_0^{**} = \theta_m$ and $1/(1-\theta_m) < \alpha f_+^{\max}$, we have $P^\dagger < \theta_m$ for Eq. (F.1) since this is the case where

$$\begin{aligned}\alpha f_+(\alpha P^\dagger) &= \frac{1}{1-\varphi_0} \leq \frac{1}{1-\theta_m} < \alpha f_+^{\max} \\ &= \lim_{\xi\to\xi_m-0} \alpha f_+(\xi) = \lim_{P\to\theta_m-0} \alpha f_+(\alpha P).\end{aligned}$$

Then the left side of (F.2) becomes negative as $\varphi_0 \to \varphi_0^{**} - 0 = \theta_m - 0$ because $P^\dagger - \varphi_0 \to P^\dagger - \theta_m < 0$, while the right side is positive. This means that $Q(P^\dagger)$ becomes negative as $\varphi_0 \to \varphi_0^{**} - 0$.

On the other hand, from Lemma H.8, when the condition (5.2) is satisfied, we have a unique root of Eq. (F.1), $P^\dagger \in (\theta_{\inf}, \theta_m)$ for every $\varphi_0 \in (\theta_{\inf}, \varphi_0^{**})$ with φ_0^{**} given by (H.4). Then $P^\dagger$ approaches a value $P_{\inf}^\dagger$ in $(\theta_{\inf}, \theta_m)$ as $\varphi_0 \to \theta_{\inf} + 0$. With the same arguments as in the proof of Lemma F.5, we can subsequently show that the condition (F.2) comes to hold as $\varphi_0 \to \theta_{\inf} + 0$. Hence $Q(P^\dagger)$ necessarily becomes positive with $P^\dagger \to P_{\inf}^\dagger \in (\theta_{\inf}, \theta_m)$ as $\varphi_0 \to \theta_{\inf} + 0$.

As a result, we find that there exists a unique $P_c \in (P^\dagger_{\rm inf}, \theta_m) \subset (\theta_{\rm inf}, \theta_m)$ such that $Q(P_c) = 0$, $Q(P) > 0$ for $P < P_c$, and $Q(P) < 0$ for $P > P_c$. Since $P^\dagger$ is monotonically increasing in terms of $\varphi_0 \in (\theta_{\rm inf}, \varphi_0^{**}) \subset (\theta_{\rm inf}, \theta_m)$, a unique value of $\varphi_0 \in (\theta_{\rm inf}, \varphi_0^{**})$ is determined by (F.1) for $P = P_c \in (P^\dagger_{\rm inf}, \theta_m)$, which is denoted by φ_0^c and given by (5.4).

Now, we have $P^\dagger < P_c$ for $\varphi_0 < \varphi_0^c$, and $P^\dagger > P_c$ for $\varphi_0 > \varphi_0^c$. Thus, $Q(P^\dagger) > 0$ for $\varphi_0 \in (\theta_{\rm inf}, \varphi_0^c)$, $Q(P^\dagger) < 0$ for $\varphi_0 \in (\varphi_0^c, \varphi_0^{**})$, and $Q(P^\dagger) = Q(P_c) = 0$ for $\varphi_0 = \varphi_0^c$. Therefore the condition (F.2) holds for $\varphi_0 \in (\theta_{\rm inf}, \varphi_0^c]$ and does not for $\varphi_0 \in (\varphi_0^c, \varphi_0^{**})$.

Inversely, when such a value φ_0^c exists with some values $P^\dagger$ satisfying the condition (F.2) as the root for Eq. (F.1), Lemma H.6 indicates that the condition (5.2) holds. Consequently these arguments prove the lemma. □

We have now two critical values φ_0^s and φ_0^c for the initial frequency of acceptors $P_0 = \varphi_0$ according to the unimodal threshold distribution. As for their relation, we can find the following result:

Lemma H.10. *When the condition* (5.2) *is satisfied, it holds that* $\varphi_0^c > \varphi_0^s$.

Proof. Since $f_+(\xi)$ is monotonically increasing for $\xi \in (\xi_{\rm inf}, \xi_m)$, we have

$$\frac{1}{\theta_m - \theta_{\rm inf}} \int_{\xi_{\rm inf}}^{\xi_m} f_+(\xi)\, d\xi > \frac{1}{\theta_m - \theta_{\rm inf}} \int_{\xi_{\rm inf}}^{\xi_m} f_+^{\rm inf}\, d\xi$$
$$= \frac{\alpha}{\xi_m - \xi_{\rm inf}} f_+^{\rm inf}(\xi_m - \xi_{\rm inf}) = \alpha f_+^{\rm inf}$$

from the definition of $f_+^{\rm inf}$ in (5.2). This inequality shows that the condition (5.2) holds if the condition (H.2) is satisfied. Therefore, the condition (5.2) is rigorously wider than (H.2). When the condition (H.2) is satisfied, we can define two critical values φ_0^s and φ_0^c in $(\theta_{\rm inf}, \theta_m)$. As a result, for $\varphi_0 > \varphi_0^s$ when the condition (H.2) is unsatisfied, there is a range of φ_0 which satisfies the condition (5.2). This means that φ_0^c must be greater than φ_0^s.

On the other hand, even when the condition (H.2) is not satisfied, φ_0^s can be formally defined and then it becomes less than or equal to $\theta_{\rm inf}$, so that we have $\mathscr{G}(\theta_m; \varphi_0) > 0$. Hence from Lemma H.9, we have the critical value $\varphi_0^c \in (\theta_{\rm inf}, \theta_m)$ if the condition (5.2) is satisfied. In such a case, formally $\varphi_0^s < \varphi_0^c$. Lastly these arguments prove the lemma. □

Consequently, independently of the sign of $\mathscr{G}(\theta_m; \varphi_0)$, that is, independently of the condition (H.2), we may conclude that, if the condition (5.2) is satisfied, we have the critical value for the initial frequency of acceptors φ_0 as $\varphi_0^c \in (\theta_{\rm inf}, \theta_m)$. This result corresponds to that for the monotonically increasing threshold distribution. Unless the condition (5.2) is satisfied, there is no such critical value, because the condition (H.2) is not satisfied at the same time.

Finally getting together the results and arguments by Lemmas 4.3, F.6, H.9, H.10, and the arguments in their proofs, we can get the result in Theorem 5.4.