ON THE REPRESENTATION AND THE RESIDUE OF CONCAVE FUNCTIONS

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ABSTRACT. In [2] we introduced several integral representation formulas for concave functions. Using those, we gave a general formula to describe the residue of concave functions with a pole at $p \in (0, 1)$. In the present article we will present alternate versions of the formulas, as well as a shortcut for the calculation to obtain the range of the residue.
Key words: concave univalent functions, integral representations

1. Introduction

Let $\mathbb{C}$ be the complex plane, $\widehat{\mathbb{C}}$ the Riemann sphere and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk. A univalent function $f : \mathbb{D} \to \widehat{\mathbb{C}}$ is said to be concave, if $f(\mathbb{D})$ is concave, i.e. $\mathbb{C}\backslash f(\mathbb{D})$ is convex. Commonly there are several types of concave functions, which map $\mathbb{D}$ conformally onto a simply connected, concave domain in $\widehat{\mathbb{C}}$:

(1) meromorphic, univalent functions f with a simple pole at the origin and the normalization $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$, said to belong to the class $\mathcal{C}o_0$,
(2) meromorphic, univalent functions f with a simple pole at the point $p \in (0, 1)$ and the normalization $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, said to belong to the class $\mathcal{C}o_p$ and
(3) analytic, univalent functions f satisfying $f(1) = \infty$ with the normalizations $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and an opening angle of $f(\mathbb{D})$ at ∞ less or equal to $\alpha\pi$ with $\alpha \in (1, 2]$, said to belong to the class $\mathcal{C}o(\alpha)$.

A detailed discussion of these classes has already been done in [2]. We therefore concentrate on the class $\mathcal{C}o_p$ for the present article.

2. Alternative formulas

In [2] we introduced the following integral representation formula for functions of $\mathcal{C}o_p$.

Theorem 1. [2] *Let $p \in (0, 1)$. For a meromorphic function $f : \mathbb{D} \to \widehat{\mathbb{C}}$ of class $\mathcal{C}o_p$, there exists a function $\varphi : \mathbb{D} \to \mathbb{D}$, holomorphic in $\mathbb{D}$ with $\varphi(p) = p$, such that the concave function can be represented as*

$$f'(z) = \frac{p^2}{(z-p)^2(1-zp)^2} \exp \int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta \tag{1}$$

for $z \in \mathbb{D}$. Conversely, for any holomorphic function φ mapping $\mathbb{D} \to \mathbb{D}$ with $\varphi(p) = p$, there exists a concave function of class $\mathcal{C}o_p$ described by (1).

However, a fixed point of the function φ at p is not very useful for further discussions. Using several transformations we obtain an alternate version of Theorem 1.

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Corollary 2. *Let $p \in (0,1)$. For a meromorphic function $f : \mathbb{D} \to \widehat{\mathbb{C}}$ of class $\mathcal{C}o_p$, there exists a function $\Psi : \mathbb{D} \to \mathbb{D}$, holomorphic in $\mathbb{D}$ with $\Psi(0) = 0$ such that the concave function can be represented as*

$$f'(z) = \frac{p^2}{(z-p)^2(1-zp)^2} \exp\left(2 \int_p^{\frac{p-z}{1-pz}} \frac{p}{1-p\zeta} - \frac{\Psi(\zeta)}{1-\zeta\Psi(\zeta)} d\zeta \right) \tag{2}$$

for $z \in \mathbb{D}$. Conversely, for any holomorphic function Ψ mapping $\mathbb{D} \to \mathbb{D}$ with $\Psi(0) = 0$, there exists a concave function of class $\mathcal{C}o_p$ described by (2).

Proof. Let $p \in (0,\,1)$ and $z \in \mathbb{D}$. Applying the transformation $\zeta = \frac{p-x}{1-px}$ and $\Phi(x) = \varphi(\zeta)$ we obtain

$$\begin{aligned} \int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta &= \int_p^{\frac{p-z}{1-pz}} \frac{-2\Phi(x)}{1-\frac{p-x}{1-px}\Phi(x)} \cdot \frac{p^2-1}{(1-px)^2} dx \\ &= \int_p^{\frac{p-z}{1-pz}} \frac{-2\Phi(x)(p^2-1)}{(1-px)^2-(p-x)\Phi(x)(1-px)} dx. \end{aligned}$$

Here the function Φ is holomorphic in $\mathbb{D}$ with $\Phi(0) = p$. Therefore there exists a function $\Psi : \mathbb{D} \to \mathbb{D}$ holomorphic in $\mathbb{D}$ with $\Psi(0) = 0$, such that $\Phi(x) = \frac{p-\Psi(x)}{1-p\Psi(x)}$. Then

$$\begin{aligned} \int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta &= \int_p^{\frac{p-z}{1-pz}} \frac{-2\frac{p-\Psi(x)}{1-p\Psi(x)}(p^2-1)}{(1-px)^2-(p-x)\frac{p-\Psi(x)}{1-p\Psi(x)}(1-px)} dx \\ &= \int_p^{\frac{p-z}{1-pz}} \frac{-2(p-\Psi(x))(p^2-1)}{(1-px)\left((1-p^2)-x\Psi(x)(1-p^2)\right)} dx \\ &= \int_p^{\frac{p-z}{1-pz}} \frac{-2(\Psi(x)-p)}{(1-px)(1-x\Psi(x)} dx \\ &= 2\int_p^{\frac{p-z}{1-pz}} \frac{p}{1-px} - \frac{\Psi(x)}{1-x\Psi(x)} dx. \end{aligned}$$

Changing the variable inside the integration and replacing the integral in (1) leads to the statement. □

The formula for the residue derived from the integral representation in [2] was given as follows.

Theorem 3. [2] *Let $f(z) \in \mathcal{C}o_p$ be a concave function with a simple pole at some point $p \in (0,1)$. Then the residue of this function f can be described by some function $\varphi : \mathbb{D} \to \mathbb{D}$, holomorphic in $\mathbb{D}$ and $\varphi(p) = p$, such that*

$$\operatorname{Res}_p f = -\frac{p^2}{(1-p^2)^2} \exp \int_0^p \frac{-2\varphi(z)}{1-x\varphi(z)} dz. \tag{3}$$

Applying the alternative representation from Corollary 2, we obtain

Corollary 4. *Let $f(z) \in \mathcal{C}o_p$ be a concave function with a simple pole at some point $p \in (0,1)$. Then the residue of this function f can be described by some function $\Psi : \mathbb{D} \to \mathbb{D}$, holomorphic in $\mathbb{D}$ and $\Psi(0) = 0$, such that*

$$\operatorname{Res}_p f = -\frac{p^2}{(1-p^2)^2} \exp 2 \int_0^p \frac{\Psi(x)}{1-x\Psi(x)} - \frac{p}{1-px} dx. \tag{4}$$

The advantage of Corollary 4 over the original presentation is the fixed point of Ψ at the origin. This provide much easier means for the construction, than a fixed point at p. Furthermore, the Schwarz Lemma can be applied directly without any complicated analysis, giving a way for the estimate of special values. We will show an application in the next section.

3. Range of the residue

Wirths proved the following statement in [3] using the inequality

$$\left| \frac{1}{f(z)} - \frac{1}{z} + \frac{1+p^2}{p} \right| \leq 1$$

provided by Miller in [1].

Theorem 5. [3] *Let $p \in (0, 1)$. For $a \in \mathbb{C}$ there exists a function $f \in \mathcal{C}o_p$ such that $a = \operatorname{Res}_p f$ if and only if*

$$\left| a + \frac{p^2}{1-p^4} \right| \leq \frac{p^4}{1-p^4}. \tag{5}$$

Let $\vartheta \in [0, 2\pi)$. A function $f \in \mathcal{C}o_p$ has the residue

$$a = -\frac{p^2}{1-p^4} + e^{i\vartheta} \frac{p^4}{1-p^4}$$

if and only if

$$f_\vartheta(z) = \frac{z - \frac{p}{1+p^2}(1+e^{i\vartheta})z^2}{\left(1-\frac{z}{p}\right)(1-pz)}. \tag{6}$$

The established representation formula for the residue can be used for a different approach of the same statement as described in [2]. For the present discussion we will use Corollary 4, which provides a shortcut for the proof. We also present some details, omitted in [2]

Proof. Let $p \in (0,1)$ and $\Psi : \mathbb{D} \to \mathbb{D}$ be holomorphic in $\mathbb{D}$ with fixed point at the origin. For $a = Res_p f$ with $f \in \mathcal{C}o_p$ we obtain with the use of Corollary 4

$$\left| a + \frac{p^2}{1-p^4} \right| \overset{(4)}{=} \frac{p^2}{1-p^4} \left| \frac{1+p^2}{1-p^2} \exp\left(2 \int_0^p \frac{\Psi(x)}{1-x\Psi(x)} - \frac{p}{1-px} dx \right) - 1 \right|.$$

Some basic calculations yield

$$\frac{1+p^2}{1-p^2} = \exp 2 \left(\frac{1}{2} \log \frac{1+p^2}{1-p^2} \right) = \exp \int_0^p \frac{2p}{1-p^2x^2} dx$$

and therefore

$$\begin{aligned} \left| a + \frac{p^2}{1-p^4} \right| &= \frac{p^2}{1-p^4} \left| \exp \int_0^p 2 \left(\frac{p}{1-p^2x^2} - \frac{p}{1-px} + \frac{\Psi(x)}{1-x\Psi(x)} \right) dx - 1 \right| \\ &= \frac{p^2}{1-p^4} \left| \exp \int_0^p 2 \frac{\Psi(x) - p^2 x}{(1-x\Psi(x))(1-p^2x^2)} dx - 1 \right|. \end{aligned}$$

From the triangle inequality, we know that

$$|e^w - 1| = \left| \sum_{n=1}^{\infty} \frac{w^n}{n!} \right| \leq \sum_{n=1}^{\infty} \frac{|w|^n}{n!} = e^{|w|} - 1.$$

Hence

$$\left|a+\frac{p^2}{1-p^4}\right| \leq \frac{p^2}{1-p^4}\left(\exp\int_0^p 2\left|\frac{\Psi(x)-p^2x}{(1-x\Psi(x))(1-p^2x^2)}\right|dx-1\right).$$

Due to the fixed point at the origin, we can apply the Schwarz Lemma and have $|\Psi(x)| \leq x$ for $0 < x < p$. Furthermore, since $\left|\frac{w-p^2x}{1-xw}\right| \leq \frac{(1-p^2)x}{1-x^2}$ for $|w| \leq x$, we have

$$\left|\frac{\Psi(x)-p^2x}{1-x\Psi(x)}\right| \leq \frac{(1-p^2)x}{1-x^2}. \tag{7}$$

Using the above, we finally obtain

$$\begin{aligned}\left|a+\frac{p^2}{1-p^4}\right| &\overset{(7)}{\leq} \frac{p^2}{1-p^4}\left(\exp\int_0^p 2\,\frac{(1-p^2)x}{(1-x^2)(1-p^2x^2)}dx-1\right)\\ &= \frac{p^2}{1-p^4}\left(\exp(\log(1+p^2))-1\right)\\ &= \frac{p^4}{1-p^4}.\end{aligned}$$

The rest of the proof goes according to the way described in [2].

□

References

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