TOHOKU UNIVERSITY

Graduate School of Information Sciences

A Study on Concave Functions in Geometric Function Theory

Dissertation

written by: Rintaro Ohno
born 21. January 1984
in Tokyo (Japan)

Sendai 2014

Division of Mathematics
Prof. Toshiyuki Sugawa

Supervisor: Prof. Toshiyuki Sugawa
Examiners: Prof. Shigeru Sakaguchi
Prof. Reika Fukuizumi
Date of submission: 4. August 2014

Contents

Introduction

In the course of the last century, the field of geometric function theory presented many interesting and fascinating facts. Starting with the mapping theorem of Riemann, Bieberbach [5] gave a long-lasting conjecture in 1916, which attracted the attention of many mathematicians over the time. The conjecture concerned the class $\mathcal{S}$ of analytic and univalent functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, normalized at the origin to have $f(0) = f'(0) - 1 = 0$, and stated that the Taylor coefficients $a_n(f)$ of these functions $f(z) = z + \sum_{n=0}^{\infty} a_n(f) z^n \in \mathcal{S}$ satisfy the inequality $|a_n(f)| \leq n$. Although the proof for $n = 2$ was given in 1916 by Bieberbach himself, when he formulated the conjecture and the case $n = 3$ was proved shortly afterwards in 1923 by Loewner [17], it took another 32 years until the proof for the case $n = 4$ was given by Garabediean and Schiffer [12] in 1955.

During the time various approaches and many methods were developed in the attempt to reach this conclusion. One of the first thoughts was the analysis of subclasses of $\mathcal{S}$, which had additional geometric conditions and provided a different perspective on the problem. Similar like spirallike, starlike and close-to-convex functions, the class of convex functions was among these special subclasses, mapping the unit disk conformally onto a convex domain. A notable problem for this class was the Pólya-Schoenberg conjecture [27] stated in 1958, asking whether the convolution of two convex functions is again convex. Although disbelieved for some time, Ruscheweyh and Sheil-Small [30] succeeded in proving this conjecture 25 years later in 1973.

Another way to attack Bieberbach's conjecture was thought to be the class Σ, mapping the outside of the unit circle conformally onto a simply connected domain in the Riemann Sphere. This class was considered to be the counterpart to the class $\mathcal{S}$, which was concerning the interior of the unit circle, and therefore might lead to a new angle on the problem asserted for the original class. From Bieberbach's area theorem and its applications, it was easily shown, that the first two coefficients of a normalized function $g(z) = z + \sum_{n=0}^{\infty} b_n(g) z^{-n} \in \Sigma$ were bounded by $|b_1(g)| \leq 1$ and $|b_2(g)| \leq \frac{2}{3}$. Spencer [31] in 1947 then stated the conjecture that for all $n \in \mathbb{N}$ the inequality $|b_n(g)| \leq \frac{2}{n+1}$ should be valid for each function $g \in \Sigma$. However, in 1955 Garabedian and Schiffer [11] verified that for the third coefficient the inequality $|b_3(g)| \leq \frac{1}{2} + e^{-6}$ was sharp and therefore disproved the conjecture by Spencer.

In 1984 deBranges [8] was finally able to prove the original Bieberbach conjecture, but many of the problems which arouse during the time were still left open. Notably for the class Σ there is still no conjecture for the coefficients and the inequalities for $b_n(g)$ are unknown for $n \geq 4$.

As it was the case with the class $\mathcal{S}$, subclasses of Σ with additional geometric attributes were considered in an attempt to get closer to functions of the class. Among other types like meromorphically starlike functions, concave functions were considered. Originally these functions were defined to map the the outside of the unit circle conformally to the outside of a convex set, therefore giving the counterpart to the class of convex functions in the class Σ, fixing the point at infinity. However, in time it turned out to be more convenient to analyze meromorphic univalent functions in the unit disk, having a simple pole at some point $p \in \mathbb{D}$. In this case the two possible expansions, the Maclaurin expansion at the origin and the Laurent expansion at the pole, are of main interest.

First considerations were made by Goodman [13] in 1956 and Miller [18, 19] in 1970 and 1980. They considered the geometrical meaning of a function being "concave" and deduced several analytic characterizations. The analysis of concave functions was picked up again by Livingston [16] in 1994, where he considered a simple pole at $p \in (0,1)$ inside the unit disk for the first time. This point turned out to be important for the coefficient estimates of concave functions, giving additional information about the function.

Livingston's thoughts were continued and improved by Avkhadiev and Wirths [2, 3] in the years from 2002 to 2007. They mainly considered the Maclaurin series expansion of concave functions $f(z) = z \sum_{n=2} a_n(f) z^n$, $|z| < p$ having a simple pole at $p \in (0,1)$. Due to the pole, the range of the coefficients was also related to the value of p. In 2007 the finally succeeded in giving the range of the coefficients $a_n(f)$ of these concave functions for all $n \in \mathbb{N}$. The discussion about the Laurent series expansion $f(z) = \frac{c_{-1}(f)}{z-p} + \sum_{n=0}^{\infty} c_n(f)(z-p)^n$ about the pole p was started by Bhowmik, Ponnusamy and Wirths [6] in 2007, where they gave the range of the first coefficient $c_1(f)$ under special restrictions.

In the present thesis, we will provide a summary of the most important aspects concerning concave functions and give a detailed analysis for some of the coefficients as well as several new necessary and sufficient conditions for concave functions.

The first chapter deals with the basic properties of analytic univalent functions and the surrounding matter. We will introduce important tools as the Schwarz Lemma and the class of Carathéodory functions. The properties of convex functions, which can be considered to be a counterpart to the concave functions will also be a matter of this chapter.

Based on the discussion of the first chapter, we introduce basic properties and analytic characterizations of concave functions in the second chapter. They will be the basis for the following investigations. Some of the presented statements are actually quite well known. However, since the literature does not give the details and often uses them as definitions rather than theorems, we will show the proofs for these parts. During the course of this investigation, we will also see, that concave functions are closely related to convex functions. Integral representations as a tool to construct concave functions using simple holomorphic functions, satisfying certain conditions will also be the matter of this chapter. Parts of this were already discussed in the Diplom Thesis of the author as well as in [20].

Before we continue our investigation, we will take a closer look at know results concerning the coefficients of concave functions in the third chapter. Here we recall most of the resent work already mentioned above by Avkhadiev and Wirths. In the end, we will present an alternate proof for the range of the residue of Laurent series coefficients of concave functions using the integral representations of the second chapter. This discussion was also presented in the latter part of [20].

After dealing with the basics, we are going to improve and extend the theorems of the second chapter in order to obtain inequalities for the Laurent series coefficients of concave functions. The idea of the fourth chapter is based on the relation between two concave functions with the same image domain. This will provide additional information for the analysis and gives an estimate for the first and second coefficients, $c_1(f)$ and $c_2(f)$, in relation with the residue $c_{-1}(f)$. This part of the analysis will be published in [21].

The last chapter finally considers the coefficient body $\{a_1(f), c_{-1}(f), c_1(f)\}$ of concave functions, as also discussed in [22]. Historically, functions of $\mathcal{S}$ and Σ, and therefore also convex and concave functions, were usually normalized for the first coefficients as already stated. However, this normalization is not really necessary and interesting results are obtained if we do not assume them. For this matter, we will present an additional integral representation and finish with a conjecture considering the range of an arbitrary coefficient $c_n(f)$ of the Laurent expansion for concave functions.

Notations

This work uses the following notations.

Symbol	Explanation
$\mathbb{C}$	the complex plane
$\widehat{\mathbb{C}}$	the Riemann sphere
$\mathbb{D}$	$\{z \in \mathbb{C} : \|z\| < 1\}$; open unit disk in $\mathbb{C}$
$\mathbb{D}(c, r)$	$\{z \in \mathbb{C} : \|z - c\| < r\}$; open disk in $\mathbb{C}$ with radius r and center c
Δ	$\{z \in \widehat{\mathbb{C}} : \|z\| > 1\}$; the complement of the unit disk in $\widehat{\mathbb{C}}$
Ω	a simply connected domain in $\mathbb{C}$
$\mathcal{H}(\Omega)$	$\{f : \Omega \to \mathbb{C},\ f$ holomorphic in $\Omega\}$; the set of holomorphic (analytic) functions on Ω
$\mathcal{S}$	$\{f \in \mathcal{H}(\mathbb{D}) : f$ univalent, $f(0) = 0, f'(0) = 1\}$; the set of normalized univalent functions on $\mathbb{D}$
$\mathcal{P}$	$\{f \in \mathcal{H}(\mathbb{D})$: Re $f(z) > 0$ and $f(0) = 1\}$; Carathéodory class of functions, having positive real part
$\mathcal{C}$	$\{f \in \mathcal{S} : f(\mathbb{D})$ is convex$\}$; class of convex functions
$\mathcal{S}^*$	$\{f \in \mathcal{S} : f(\mathbb{D})$ is starlike$\}$; class of starlike functions
$\mathcal{C}o(\Omega)$	$\{f : \Omega \to \hat{\mathbb{C}} : \hat{\mathbb{C}} \backslash f(\Omega)$ convex$\}$; class of concave functions on Ω
$\mathcal{C}o_\Delta$	$\{f : \Delta \to \hat{\mathbb{C}} : \hat{\mathbb{C}} \backslash f(\Delta)$ convex$\}$; class of concave functions on Δ
$\mathcal{C}o_p$	$\{f : \mathbb{D} \to \hat{\mathbb{C}} : p := f^{-1}(\infty) \in \mathbb{D}$ simple pole, f univalent in $\mathbb{D} \backslash \{p\}$ and $\hat{\mathbb{C}} \backslash f(\mathbb{D})$ convex$\}$; class of concave functions in $\mathbb{D}$ with simple pole at $p \in \mathbb{D}$. For simplicity often $p \in (0, 1)$.
$a_n(f)$	n-th coefficient of the Maclaurin expansion of a function f
$c_n(f)$	n-th coefficient of the Laurent expansion of a functions f

1 Basic Properties and Preliminaries

In the first chapter, we will present basic properties of functions which are called *concave*. To understand the underlying cause, we start with the discussion of univalent functions in general and take a closer look at *convex* functions, as well as *starlike* functions.

1.1 Univalent Functions and Basic Principles

We begin with the introduction of basic notations and terms used throughout this work.

Let $\mathbb{C}$ be the complex plane, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk and $\Delta = \{z \in \widehat{\mathbb{C}} : |z| > 1\}$ the exterior of the unit circle.

In general, a function is called *univalent* in a domain, if it is meromorphic and injective, i.e. one-to-one. There are two classes of univalent functions notable.

A function f belongs to the class $\mathcal{S}$ of univalent functions, if f is injective, $f(0) = 0$ and $f'(0) = 1$.

$$\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} : f \text{ univalent}, \, f(0) = 0, \, f'(0) = 1\}$$

Functions in $\mathcal{S}$ can be expanded as

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n,$$

where $a_n(f)$ is the n-th coefficient of the Taylor series.

The other important class is Σ, containing all functions

$$f(z) = \frac{1}{z} \sum_{n=0}^{\infty} b_n z^{-n}$$

univalent for $z \in \Delta$, having a simple pole at ∞.

Univalent functions have a long history, going back to 1916 when the Bieberbach Conjecture was formulated.

Theorem 1.1 (Bieberbach Conjecture)**.** *The coefficients $a_n(f)$ of functions $f \in \mathcal{S}$ satisfy $|a_n(f)| \leq n$ for $n \in \mathbb{N}$. Equality is attained if and only if f is the Koebe function $f_k(z) = \frac{z}{(1-z)^{-2}}$ or one of its rotations.*

This statement was finally proved in 1984 by deBranges after almost seven decades. However, along the way, a lot of new problems involving univalent functions - originally with the aim to work towards the Bieberbach Conjecture - were formulated and discussed.

Before we look at some of these problems, we introduce two useful lemmas, which we will use later without further reference.

The first lemma considers holomorphic functions.

Lemma 1.2 (Schwarz Lemma, see e.g. [9]). *Let f be holomorphic with $f(0) = 0$ and $f(\mathbb{D}) \subset \mathbb{D}$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$ in $\mathbb{D}$. Equality is attained in both inequalities, if and only if f is a rotation of the disk, i.e. $f(z) = e^{i\vartheta} z$, $\vartheta \in \mathbb{R}$.*

The proof of this Lemma uses the maximum modulus principle and can be found in most textbooks.

Another useful tool is given in for the following restriction to holomorphic functions.

Let $\mathcal{P}$ be the class of normalized holomorphic functions with positive real part.

$$\mathcal{P} = \{f : \mathbb{D} \to \mathbb{C} : f \text{ holomorphic }, \mathrm{Re} f(z) > 0 \text{ and } f(0) = 1\}$$

This class is sometimes called the Carathéodory class and we know the following about functions belonging to $\mathcal{P}$.

Lemma 1.3 (Carathéodory, 1911, see e.g. [9]). *Let $P(z) = 1 + \sum_{n=0}^{\infty} a_n(P) z^n \in \mathcal{P}$. Then*

$$|a_n(p)| \leq 2$$

for all $n \in \mathbb{N}$.

Proof. First consider the function

$$q(z) = \frac{1}{k_0} \sum_{l=1}^{k_0} P\left(e^{\frac{2\pi i l}{k_0}} z\right)$$

with $k_0 \in \mathbb{N}$. Since the properties of P are preserved, $q \in \mathcal{P}$ and we have

$$\begin{aligned} q(z) &= \frac{1}{k_0} \sum_{l=1}^{k_0} \sum_{k=0}^{\infty} p_k \left(e^{\frac{2\pi i l}{k_0}} z\right)^k \\ &= \sum_{k=0}^{\infty} p_k \left(\sum_{l=1}^{k_0} \frac{1}{k_0} e^{\frac{2\pi i l k}{k_0}} \right) z^k. \end{aligned}$$

Now

$$\sum_{l=1}^{k_0} \frac{1}{k_0} e^{\frac{2\pi i l k}{k_0}} = \begin{cases} 0 & \text{for } k \nmid k_0 \\ 1 & \text{for } k \mid k_0 \end{cases}.$$

Therefore

$$q(z) = 1 + p_{k_0} z^{k_0} + p_{2k_0} z^{2k_0} + \ldots.$$

The function

$$z \mapsto \frac{1-z}{1+z}$$

maps the right half of the complex plane onto the unit disk, such that

$$\begin{aligned} Q(z) &= \frac{1-q(z)}{1+q(z)} \\ &= \frac{-p_{k_0} z^{k_0} - p_{2k_0} z^{2k_0} - \ldots}{2\left(1 + \frac{p_{k_0}}{2} z^{k_0} + \ldots\right)} \\ &= -\frac{p_{k_0}}{2} z^{k_0} - \ldots \end{aligned}$$

is a mapping $Q : \mathbb{D} \to \mathbb{D}$. With the well known Cauchy-Formula we obtain

$$\left| -\frac{p_{k_0}}{2} \right| \leq 1$$

for all $k_0 \in \mathbb{N}$, which leads to the statement. $\square$

1.2 Convex Functions

Using the previously introduced notations, we can define *convex* functions in the following way.

Definition 1.4. *A functions $f \in \mathcal{S}$ is convex, if and only if the domain $f(\mathbb{D})$ is convex. We denote this subclass of $\mathcal{S}$ by $\mathcal{C}$.*

These functions present a subclass of $\mathcal{S}$ with additional restrictions, providing a way to get to Bieberbachs Conjecture.

From this geometrical Definition, we get the following analytic characterization.

Theorem 1.5 (see e.g. [9])**.** *Let f be a holomorphic function with $f(0) = f'(0) - 1 = 0$. The function f belongs to $\mathcal{C}$ if and only if*

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{D},$$

i.e.

$$f \in \mathcal{C} \Leftrightarrow 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}.$$

The proof is straight forward and can be found in e.g. [9] or [29].

A similar characterization can be made for the subclass $\mathcal{S}^*$ of functions starlike in $\mathbb{D}$.

Corollary 1.6. *A holomorphic function f with $f(0) = f'(0) - 1 = 0$ belongs to $\mathcal{S}^*$ if and only if*

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{D},$$

i.e.

$$f \in \mathcal{S}^* \Leftrightarrow \frac{zf'(z)}{f(z)} \in \mathcal{P}.$$

The connection between convex and starlike functions is given by the following *Theorem of Alexander*.

Theorem 1.7 (See e.g. [9, 29])**.** *A function f is convex, if and only if $g(z) = z\,f'(z)$ is starlike.*

Proof. A simple calculation gives

$$\frac{zg'(z)}{g(z)} = \frac{z(zf'(z))'}{zf'(z)} = \frac{f'(z) + zf''(z)}{f'(z)} = 1 + z\frac{f''(z)}{f'(z)}, \tag{1.1}$$

which shows the relation between the two presented analytic characterizations. □

For coefficients of convex functions we have the following result.

Theorem 1.8. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n \in \mathcal{C}$. Then*

$$|a_n(f)| \leq 1$$

for all $n \in \mathbb{N}$. The function $f(z) = \frac{z}{1-z}$ provides equality.

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} a_k(f) z^k \in \mathcal{C}$. Using Theorem 1.7 we define

$$g(z) := zf'(z) = \sum_{k=1}^{\infty} k a_k(f) z^k \in \mathcal{S}^*.$$

Obviously the analytic characterizations for convex and starlike functions are valid, such that there exists a function $P(z) = \sum_{k=0}^{\infty} a_k(P) z^k \in \mathcal{P}$ for which

$$\operatorname{Re}\left(\sum_{k=0}^{\infty} a_k(P) z^k\right) = \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) > 0$$

holds. Setting $b_k := k a_k(f)$, we have

$$zg'(z) = \sum_{k=1}^{\infty} k b_k z^k = \left(\sum_{k=1}^{\infty} b_k z^k\right)\left(\sum_{k=0}^{\infty} a_k(P) z^k\right).$$

Equating the coefficients gives

$$n b_n = \sum_{k=0}^{n-1} p_k b_{n-k} = b_n + \sum_{k=1}^{n-1} p_k b_{n-k}$$

for all $n \in \mathbb{N}$. Using the Lemma of Carathéodory shows $|a_k(P)| \leq 2$ and inductively $|b_n| \leq n$. Due to the definition of b_n we obtain the statement. □

2 Concave Functions

Generally, a univalent function $f : \Omega \to \widehat{\mathbb{C}}$ is said to be concave, if the complement $\widehat{\mathbb{C}} \backslash f$ is convex, where Ω is some arbitrary domain. Since the class $\mathcal{S}$ dealt with the interior and Σ the exterior of the unit circle, concave functions (= functions mapping on the exterior of a convex curve) are considered to be a counterpart to the convex functions in Σ.

However, it is important to note, that so far there is no conjecture like Bieberbach's considering the coefficients of functions in Σ. Therefore the analysis of concave functions actually gives one of few footholds towards the more general class.

We also do not consider concave functions with Δ as the preimage, but rather take the unit disk and assume that the function has a simple pole inside.

Concerning the characteristics and properties, there are several types of concave functions:

1. A meromorphic, univalent function f is said to be in the class $\mathcal{C}o_0$, if it is concave, has a simple pole at the origin and the representation $f(z) = \frac{c_{-1}(f)}{z} + \sum_{n=0}^{\infty} c_n(f) z^n$.
2. A meromorphic, univalent function f is said to be in the class $\mathcal{C}o_p$ for $p \in (0,1)$, if it is concave and has a simple pole at p. The normalization for this class can be done by use of the Taylor series expansion at the origin with $f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n$.
3. An analytic, univalent function f is said to be in the class $\mathcal{C}o(\alpha)$, if it is concave, satisfies $f(1) = \infty$ with the representation $f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n$ around the origin and an opening angle of $f(\mathbb{D})$ at ∞ less than or equal to $\alpha\pi$ with $\alpha \in (1,2]$.

As discussed in the previous section

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}$$

characterizes *convex* functions, mapping the unit disk onto a convex domain. Due to the similarity, the inequality

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) < 0, \quad z \in \mathbb{D}$$

is used - sometimes also as a definition - for concave functions $f \in \mathcal{C}o_0$ (see e.g. [26] and others). Since a complete proof for this statement could not be found in the literature, we are going to present the details in Section 2.1. Adaptations for the other classes considered in this chapter were discussed e.g. by Miller in [19] and by Livingston in [16].

Using the given inequalities, several integral representations can be deduced for concave functions. This was first analyzed by Pfaltzgraff and Pinchuk [26], who stated the following for $\mathcal{C}o_0$.

Theorem 2.1 (see [26])**.** *Let $f : \mathbb{D} \to \widehat{\mathbb{C}}, f(z) = \frac{c_{-1}(f)}{z} + \sum_{n=0}^{\infty} c_n(f)z^n$ be a meromorphic function. Then $f \in \mathcal{C}o_0$, if and only if there exists a positive measure $\mu(t)$ with $\int_{-\pi}^{\pi} d\mu(t) = 1$ and $\int_{-\pi}^{\pi} e^{-it} d\mu(t) = 0$, such that for $z \in \mathbb{D}$*

$$f'(z) = -\frac{1}{z^2} \exp \int_{-\pi}^{\pi} 2 \log(1 - e^{-it}z) d\mu(t). \tag{2.1}$$

They used this expression in combination with a linear transformation T, to obtain a characterization for concave functions with pole at $p \in (0, 1)$.

Theorem 2.2 (see [26])**.** *For $p \in (0, 1)$, $f \in \mathcal{C}o_p$ if and only if there exists a positive measure $\mu(t)$ with $\int_{-\pi}^{\pi} d\mu(t) = 1$ and $\int_{-\pi}^{\pi} T(e^{it}) d\mu(t) = 0$, such that for $z \in \mathbb{D}$*

$$f'(z) = \frac{p^2}{(z - p)^2(1 - zp)^2} \exp \int_{-\pi}^{\pi} 2 \log(1 - e^{-it}z) d\mu(t). \tag{2.2}$$

We are going to show different representations for these classes in Section 2.2, which avoid the use of logarithms and measures.

An analysis for functions in $\mathcal{C}o(\alpha)$ was done by Avkhadiev and Wirths in [3]. The connection to an inequality was discussed by Cruz and Pommerenke [7]. This chapter will also deal with the remaining integral representation, using methods presented by Pfaltzgraff and Pinchuk.

As an application of the presented theorems, we will prove the following formula for residues of functions in $\mathcal{C}o_p$.

Theorem 2.3. *Let $f(z) \in \mathcal{C}o_p$ be a concave function with a simple pole at some point $p \in (0, 1)$. Then the residue of this function f can be described by some function*

$$\varphi : \mathbb{D} \to \mathbb{D} \text{ with } \varphi(p) = p$$

holomorphic in $\mathbb{D}$, such that

$$\operatorname{Res}_p f = -\frac{p^2}{(1 - p^2)^2} \exp \int_0^p \frac{-2\varphi(x)}{1 - x\varphi(x)} dx. \tag{2.3}$$

A proof of this theorem will be given in Section 3.3, as well as some further analysis.

The content of this chapter can also be found in [20].

2.1 Characterizations for Concave Functions

In this section, we are going to present a variety of characterizations for the different types of concave functions introduced previously.

At first we consider functions in the class $\mathcal{C}o_0$, where the pole lies at the origin.

Theorem 2.4. *Let $f : \mathbb{D} \to \widehat{\mathbb{C}}$, $f(z) = \frac{c_{-1}(f)}{z} + \sum_{n=0}^{\infty} c_n(f) z^n$ be a meromorphic function. The function f is of class $\mathcal{C}o_0$, if and only if the inequality*

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) < 0 \tag{2.4}$$

holds for every $z \in \mathbb{D}$.

A rough idea can be found in [29, p.47]. However, since the details are not carried out, we give a complete presentation of the proof.

First we need the following Lemma.

Lemma 2.5. *Let $\Delta := \{z \in \widehat{\mathbb{C}} : 1 < |z|\}$ be the exterior of the unit circle and $f : \Delta \to \widehat{\mathbb{C}}$ be a meromorphic univalent function, mapping Δ onto the outside of a bounded Jordan curve Γ and $\infty \mapsto \infty$. This curve Γ is analytic, if and only if f is analytic and univalent for $\{z \in \mathbb{C} : r < |z|\}$ with some $r < 1$.*

A similar statement can be found in [28, p.41] and the construction of the proof goes accordingly.

Proof. If f is analytic and univalent in $\{z \in \mathbb{C} : r < |z|\}$, the curve Γ is obviously analytic. Therefore, let Γ be analytic. Then there exists a univalent function $\varphi : \{z \in \mathbb{C} : \rho < |z| < \frac{1}{\rho}\} \to \mathbb{C}$ with $\rho < 1$, such that $\varphi(\partial\mathbb{D}) = \Gamma$. Furthermore, there exists an $r < 1$, so that $h := \varphi^{-1} \circ f$ is univalent in $\{z \in \mathbb{C} : 1 < |z| < \frac{1}{r}\}$ and $1 < |h(z)| < \frac{1}{\rho}$. Since $|h(z)| \to 1$ as $|z| \to 1$, we can apply the reflexion principle and it follows, that f can be extended to a holomorphic function on $r < |z| < \frac{1}{r}$, where $\rho < |h(z)| < \frac{1}{\rho}$ is satisfied. Thus $f = \varphi \circ h$ is holomorphic on $\{z \in \mathbb{C} : r < |z| < \frac{1}{r}\}$ and therefore analytic and univalent on $r < |z|$. □

Proof of Theorem 2.4. We may assume that the nonempty compact convex set $\mathbb{C}\backslash\tilde{f}(\Delta)$ is not a line segment, since otherwise the theorem is trivial. Then $\mathbb{C}\backslash\tilde{f}(\Delta)$ is a convex closed Jordan domain bounded by a simple closed curve.

Let $f(z) = \frac{c_{-1}(f)}{z} + \sum_{n=0}^{\infty} c_n(f) z^n \in \mathcal{C}o_0$ for $z \in \mathbb{D}$. Applying the transformation $u : \Delta \to \mathbb{D}$, $z \mapsto \frac{1}{z}$ and setting $\tilde{f} := f \circ u$, we get a concave function, which maps Δ conformally onto the concave domain $f(\mathbb{D})\backslash\{\infty\}$. Therefore there exists a convex domain $G = \mathbb{C}\backslash\overline{\tilde{f}(\Delta)}$, a curve $\Gamma = \partial G$ and a convex function $g : \mathbb{D} \to \operatorname{Int}(\Gamma)$ by use of the Riemann mapping theorem. The curves $\Gamma_k = \{g(z) : |z| = 1 - \frac{1}{k}\}$, $k = 2, 3, \ldots$ are analytic and convex because of the properties of g.

Now let $\tilde{f}_k$ be the functions, which map Δ onto $\operatorname{Ext}(\Gamma_k)$, such that $\tilde{f}_k(\infty) = \infty$ and $\tilde{f}_k'(\infty) > 0$. Due to the definition of Γ_k and Lemma 2.5, each curve can also be described by $\tilde{f}_k(e^{i\vartheta})$ with $\vartheta \in [0, 2\pi)$. Since the interior of the curve Γ_k is convex, $\arg\left(\tilde{f}_k(e^{it}) - \tilde{f}_k(e^{i\vartheta})\right)$ is non-decreasing for $t \in (\vartheta, \vartheta + 2\pi)$. Therefore

$$\partial_t \arg\left(\tilde{f}_k(e^{it}) - \tilde{f}_k(e^{i\vartheta})\right) \quad = \quad \partial_t \operatorname{Im}\log\left(\tilde{f}_k(e^{it}) - \tilde{f}_k(e^{i\vartheta})\right)$$

$$= \operatorname{Im}\frac{ie^{it}\tilde{f}_k'(e^{it})}{\tilde{f}_k(e^{it}) - \tilde{f}_k(e^{i\vartheta})}$$

$$= \operatorname{Re}\frac{z\tilde{f}_k'(z)}{\tilde{f}_k(z) - \tilde{f}_k(\zeta)} \quad \geq 0 \tag{2.5}$$

for $z = e^{it} \neq e^{i\vartheta} = \zeta$ and

$$\operatorname{Re}\frac{\zeta + z}{\zeta - z} = \operatorname{Re}\frac{e^{i\vartheta} + e^{it}}{e^{i\vartheta} - e^{it}} = \operatorname{Re}\frac{1 + e^{i(t-\vartheta)}}{1 - e^{i(t-\vartheta)}} = 0 \tag{2.6}$$

holds for the given z, ζ, t and ϑ.
Using the Taylor series expansion $\tilde{f}_k(\zeta) = \sum_{n=0}^{\infty} \frac{\tilde{f}_k^{(n)}(z)}{n!}(\zeta - z)^n$ for z, $\zeta \in \overline{\Delta}$, we obtain

$$\begin{aligned}
\frac{2z\tilde{f}_k'(z)}{\tilde{f}_k(z) - \tilde{f}_k(\zeta)} + \frac{\zeta + z}{\zeta - z} &= \frac{2z\tilde{f}_k'(z)}{\tilde{f}_k(z) - \tilde{f}_k(\zeta)} + \frac{2z}{\zeta - z} + 1 \\
&= 1 + z\frac{2\left(\tilde{f}_k'(z)(\zeta - z) + \tilde{f}_k(z) - \tilde{f}_k(\zeta)\right)}{(\tilde{f}_k(z) - \tilde{f}_k(\zeta))(\zeta - z)} \\
&= 1 + z\frac{-\tilde{f}_k''(z)(\zeta - z)^2 - 2\sum_{n=3}^{\infty}\frac{\tilde{f}_k^{(n)}(z)}{n!}(\zeta - z)^n}{-\tilde{f}_k'(z)(\zeta - z)^2 - \sum_{n=2}^{\infty}\frac{\tilde{f}_k^{(n)}(z)}{n!}(\zeta - z)^{n+1}} \\
&= 1 + z\frac{\tilde{f}_k''(z) + 2\sum_{n=3}^{\infty}\frac{\tilde{f}_k^{(n)}(z)}{n!}(\zeta - z)^{n-2}}{\tilde{f}_k'(z) + \sum_{n=2}^{\infty}\frac{\tilde{f}_k^{(n)}(z)}{n!}(\zeta - z)^{n-1}}.
\end{aligned}$$

Since

$$\lim_{\zeta \to z}\left(\frac{2z\tilde{f}_k'(z)}{\tilde{f}_k(z) - \tilde{f}_k(\zeta)} + \frac{\zeta + z}{\zeta - z}\right) = 1 + \frac{z\tilde{f}''(z)}{\tilde{f}'(z)}$$

$\zeta = z$ is a removable singularity.
From (2.5) and (2.6) we obtain

$$\operatorname{Re}\left(\frac{2z\tilde{f}_k'(z)}{\tilde{f}_k(z) - \tilde{f}_k(\zeta)} + \frac{\zeta + z}{\zeta - z}\right) \geq 0$$

for all $|z| = |\zeta| = 1$. Applying the maximum principle first for $|z| > 1$ and then for $|\zeta| > 1$ gives

$$\operatorname{Re}\left(1 + \frac{z\tilde{f}_k''(z)}{\tilde{f}_k'(z)}\right) > 0$$

for all $z \in \Delta$ and $k = 2, 3, \ldots$.

Since convex curves Γ_k converge to Γ for $k \to \infty$, $\tilde{f}_k$ converges locally uniformly to $\tilde{f}$ in Δ

due to the kernel theorem of Carathéodory. Therefore

$$\operatorname{Re}\left(1+\frac{z\tilde{f}''(z)}{\tilde{f}'(z)}\right)>0 \tag{2.7}$$

for all $z\in\Delta$.

Considering $\tilde{f}=f\circ u$ with $\tilde{f}'(z)=-\frac{1}{z^2}f'(u)$ and $\tilde{f}''(z)=\frac{1}{z^4}f''(u)+\frac{2}{z^3}f'(u)$, we obtain

$$\begin{aligned}1+\frac{z\tilde{f}''(z)}{\tilde{f}'(z)} &= 1+\frac{z\left(\frac{1}{z^4}f''(u)+\frac{2}{z^3}f'(u)\right)}{-\frac{1}{z^4}f'(u)}\\ &= 1-\frac{\frac{1}{z}f''(u)+2f'(u)}{f'(u)}\\ &= -1-\frac{uf''(u)}{f'(u)},\end{aligned} \tag{2.8}$$

hence (2.4).

The second implication is the same as for the convex case, when one considers $z\in\Delta$. This can be found in various textbooks, see e.g. [29]. Applying transformation (2.8) yields the statement. □

Remark 2.6. It is also

$$\begin{aligned}\frac{2z\tilde{f}'(z)}{\tilde{f}(z)-\tilde{f}(\zeta)}+\frac{\zeta+z}{\zeta-z} &= \frac{2z\frac{1}{z^2}\tilde{f}'(u(z))}{f(u(z))-f(u(\zeta))}+\frac{\frac{1}{u(\zeta)}+\frac{1}{u(z)}}{\frac{1}{u(\zeta)}-\frac{1}{u(z)}}\\ &= -\frac{2u(z)f'(u(z))}{f(u(z))-f(u(\zeta))}-\frac{u(\zeta)+u(z)}{u(\zeta)-u(z)}\end{aligned}$$

for $z,\,\zeta\in\Delta$. Therefore we obtain

$$\operatorname{Re}\left(\frac{2zf'(z)}{f(z)-f(\zeta)}+\frac{\zeta+z}{\zeta-z}\right)\le 0 \tag{2.9}$$

for $f\in\mathcal{C}o_0$ and $z,\,\zeta\in\mathbb{D}$ from the proof of Theorem 2.4.

Livingston [16] adapted the characterization for functions in $\mathcal{C}o_0$ for the class of concave functions with pole at $p\in(0,1)$, using the transformation $z\mapsto\frac{z+p}{1+pz}$.

Theorem 2.7 (see [16]). *Let $p\in(0,1)$ and f be a meromorphic function. It is $f\in\mathcal{C}o_p$, if and only if*

$$\operatorname{Re}\left(1+p^2-2pz+\frac{(z-p)(1-pz)f''(z)}{f'(z)}\right)<0 \tag{2.10}$$

for $z\in\mathbb{D}$.

From this theorem, we can obtain a statement similar to Remark 2.6 for functions in $\mathcal{C}o_p$. It should be mentioned, that Theorem 2.7 and the following statements are valid regardless of the normalization sometimes introduced to simplify the discussion. We will also take a closer look at the Theorem by Livingston in Chapter 2.

Corollary 2.8. *Let $p \in (0,1)$, $f \in \mathcal{C}o_p$ and $z, \zeta \in \mathbb{D}$. Then*

$$\operatorname{Re}\left(\frac{2zf'(z)}{f(z)-f(\zeta)} + \frac{\zeta+z}{\zeta-z} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz}\right) < 0. \tag{2.11}$$

Originally this was proved by Miller in [19]. Considering Livingston's analysis in [16], we can give an alternate proof.

Proof. Let

$$P(z) := -1-p^2+2pz-2(z-p)(1-pz)\frac{f'(z)(\zeta-z)+f(z)-f(\zeta)}{(f(z)-f(\zeta))(\zeta-z)}. \tag{2.12}$$

Using $f(z)=\sum_{n=-1}^{\infty} b_n(z-p)^n$ for $\zeta \neq p$ and

$$\begin{aligned}(z-p)\frac{f'(z)}{f(z)-f(\zeta)} &= \frac{\left[-b_{-1}(z-p)^{-1}+b_1(z-p)+\cdots\right]}{b_{-1}(z-p)^{-1}+b_0+\cdots-f(\zeta)}\\ &= \frac{\left[-b_{-1}+b_1(z-p)^2+\cdots\right]}{b_{-1}+b_0(z-p)+\cdots-f(\zeta)(z-p)}\end{aligned}$$

we have

$$P(p) = -1-p^2+2p^2-2(1-p^2)\frac{-b_{-1}(\zeta-p)}{b_{-1}(\zeta-p)} = 1-p^2.$$

Furthermore, observing that

$$\begin{aligned} zP(z)+pz^2-p &= 3pz^2-z-p^2z-p\\ &\qquad -2z(z-p)(1-pz)\frac{f'(z)(\zeta-z)+f(z)-f(\zeta)}{(f(z)-f(\zeta))(\zeta-z)}\\ \Leftrightarrow\qquad \frac{zP(z)+pz^2-p}{(z-p)(1-pz)} &= \frac{2pz^2-2p^2z+p^2z-z+pz^2-p}{(z-p)(1-pz)}\\ &\qquad -2z\frac{f'(z)(\zeta-z)+f(z)-f(\zeta)}{(f(z)-f(\zeta))(\zeta-z)}\\ &= \frac{2pz}{1-pz}-\frac{z+p}{z-p}-2z\frac{f'(z)(\zeta-z)+f(z)-f(\zeta)}{(f(z)-f(\zeta))(\zeta-z)}\\ &= \frac{1+pz}{1-pz}-\frac{z+p}{z-p}-1-2z\frac{f'(z)(\zeta-z)+f(z)-f(\zeta)}{(f(z)-f(\zeta))(\zeta-z)}\\ &= \frac{1+pz}{1-pz}-\frac{z+p}{z-p}-\frac{2zf'(z)}{f(z)-f(\zeta)}-\frac{\zeta+z}{\zeta-z} \end{aligned} \tag{2.13}$$

and defining

$$Q(z) := \frac{zP(z) + pz^2 - p}{(z-p)(1-pz)}, \tag{2.14}$$

we obtain $Q(p) = \frac{1+p^2}{1-p^2}$ and

$$\lim_{\zeta \to z} Q(z) = -1 - \frac{zf''(z)}{f'(z)} + \frac{1+pz}{1-pz} - \frac{z+p}{z-p}.$$

Therefore the function

$$F(z,\zeta) = \begin{cases} 1 + \dfrac{zf''(z)}{f'(z)} + \dfrac{z+p}{z-p} - \dfrac{1+pz}{1-pz}, & \text{for } z = \zeta \\ \dfrac{2zf'(z)}{f(z)-f(\zeta)} + \dfrac{\zeta+z}{\zeta-z} + \dfrac{z+p}{z-p} - \dfrac{1+pz}{1-pz}, & \text{for } z \neq \zeta \end{cases} \tag{2.15}$$

is holomorphic for $z, \zeta \in \mathbb{D}$.

Considering Theorem 2.4, Remark 2.6 and the fact that

$$\operatorname{Re}\left(\frac{z+p}{z-p} - \frac{1+pz}{1-pz}\right) = 0,$$

we obtain (2.11) by the maximum principle. □

The case $z = \zeta$ in (2.15) was deduced by different means by Pfaltzgraff and Pinchuk in [26]. $\operatorname{Re} F(z,z) < 0$ for $z \in \mathbb{D}$ also holds as a necessary and sufficient condition for a meromorphic function f to be in $\mathcal{C}o_p$.

Theorem 2.9 (see [26]). *Let $f : \mathbb{D} \to \widehat{\mathbb{C}}$ be a meromorphic function. The function f is of class $\mathcal{C}o_p$, if and only if for $z \in \mathbb{D}$*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz}\right) < 0. \tag{2.16}$$

Remark 2.10. Equation (2.16) can also be formulated for an arbitrary pole $p \in \mathbb{D}$, in which case we have

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} + \frac{z+p}{z-p} - \frac{1+\bar{p}z}{1-\bar{p}z}\right) < 0. \tag{2.17}$$

as a necessary and sufficient condition for a function to be in $\mathcal{C}o_p$. We obtain this inequality by considering a rotation of a function with a pole at some point on $(0,1)$.

For the class $\mathcal{C}o(\alpha)$, the following inequality can be deduced.

Theorem 2.11 (see [3, 7]). *Let $\alpha\pi$, $\alpha \in (1,2]$. An analytic function f with $f(0) = f'(0) - 1 = 0$ is of class $\mathcal{C}o(\alpha)$, if and only if for $z \in \mathbb{D}$*

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)} - \frac{\alpha+1}{2} \cdot \frac{1+z}{1-z}\right) < 0. \tag{2.18}$$

Avkhadiev and Wirths considered this in [3] and Cruz and Pommerenke discussed a variation of the theorem in detail in [7]. A factor $\frac{2}{\alpha-1}$ has to be added to the characterization in case a normalization is required.

2.2 Integral Representations for Concave Functions

The inequalities from the previous section provide new representation formulas for concave functions. These are equivalent to the already presented characterizations.

Theorem 2.12. *Let $f : \mathbb{D} \to \widehat{\mathbb{C}}, f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ be a meromorphic function. If $f \in \mathcal{C}o_0$, then there exists a function*

$$\varphi : \mathbb{D} \to \mathbb{D} \text{ with } \varphi(0) = 0$$

holomorphic in $\mathbb{D}$, such that for $z \in \mathbb{D}$

$$f'(z) = -\frac{1}{z^2} \exp \int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta. \tag{2.19}$$

On the other hand, for any holomorphic function φ mapping $\mathbb{D} \to \mathbb{D}$ with $\varphi(0) = 0$, there exists a function $f \in \mathcal{C}o_0$ described by (2.19).

Proof. It is known, that a function which maps $\mathbb{D}$ into the right half plane and the origin to 1 can be expressed as $\frac{1+z\varphi(z)}{1-z\varphi(z)}$, where $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$ is a function holomorphic in $\mathbb{D}$. We combine this fact with Theorem 2.4. Therefore there exists a holomorphic function φ with the given properties such that

$$1 + z\frac{f''(z)}{f'(z)} = -\frac{1+z\varphi(z)}{1-z\varphi(z)}.$$

Hence it is also

$$\begin{aligned}
& & 2 + z\frac{f''(z)}{f'(z)} &= -\frac{2z\varphi(z)}{1-z\varphi(z)} \\
&\Leftrightarrow & \frac{2}{z} + \frac{f''(z)}{f'(z)} &= \frac{-2\varphi(z)}{1-z\varphi(z)} \\
&\Leftrightarrow & \frac{d}{dz}\log\left(-z^2 f'(z)\right) &= \frac{-2\varphi(z)}{1-z\varphi(z)} \\
&\Leftrightarrow & \log\left(-z^2 f'(z)\right) - \log\left(-\zeta^2 f'(\zeta)\right)\big|_{\zeta=0} &= \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta.
\end{aligned}$$

Using $\zeta^2 f'(\zeta) = \zeta^2 \cdot \left(\frac{-1}{\zeta^2} + \sum_{n=1}^{\infty} n a_n \zeta^{n-1}\right) = -1 + \mathcal{O}(\zeta^2)$, with $\mathcal{O}$ being the Landau symbol as in the proof of Corollary 2.8, we obtain

$$\begin{aligned}
& & \log\left(-z^2 f'(z)\right) &= \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta \\
&\Leftrightarrow & f'(z) &= -\frac{1}{z^2} \exp \int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta.
\end{aligned}$$

Since f' must not have a residue, it has to be

$$\left(z^2 f'(z)\right)'\Big|_{z=0} = 0.$$

Considering $\kappa(z) := \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta$ and $g(z) := z^2 f'(z) = -e^{\kappa(z)}$, we obtain

$$\frac{g'(z)}{g(z)} = \kappa'(z) = \frac{\varphi(z)}{1 - z\varphi(z)}.$$

Therefore it has to be $\varphi(0) = 0$.

Conversely, if $\varphi : \mathbb{D} \to \mathbb{D}$ is a holomorphic function with $\varphi(0) = 0$, a function f defined by

$$f'(z) = -\frac{1}{z^2} \exp \int_0^z \frac{-2\,\varphi(\zeta)}{1 - \zeta\varphi(\zeta)} d\zeta$$

does not have a residue of its own. Furthermore we obtain

$$1 + z\frac{f''(z)}{f'(z)} = -\frac{1 + z\varphi(z)}{1 - z\varphi(z)}.$$

By the use of Theorem 2.4, concavity follows immediately. □

Using the inequality obtained from Theorem 2.9, it is possible to prove a similar statement for the class $\mathcal{C}o_p$.

Theorem 2.13. *Let $p \in (0,1)$. If a meromorphic function $f : \mathbb{D} \to \widehat{\mathbb{C}}$ belongs to the class $\mathcal{C}o_p$, then there exists a function*

$$\varphi : \mathbb{D} \to \mathbb{D} \text{ with } \varphi(p) = p$$

holomorphic in $\mathbb{D}$, such that the concave function can be represented as

$$f'(z) = \frac{p^2}{(z-p)^2(1-zp)^2} \exp \int_0^z \frac{-2\,\varphi(\zeta)}{1 - \zeta\varphi(\zeta)} d\zeta \tag{2.20}$$

for $z \in \mathbb{D}$. Conversely, for any holomorphic function φ mapping $\mathbb{D} \to \mathbb{D}$ with $\varphi(p) = p$, there exists a concave function of class $\mathcal{C}o_p$ described by (2.20).

Proof. From Theorem 2.9 it is known, that $f \in \mathcal{C}o_p$ is equivalent to

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)} + \frac{z+p}{z-p} - \frac{1+zp}{1-zp}\right) < 0$$

for $p \in (0,1)$. Therefore there exists a function $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$, holomorphic in $\mathbb{D}$ such that

$$\begin{aligned} & 1 + z\frac{f''(z)}{f'(z)} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz} = -\frac{1+z\varphi(z)}{1-z\varphi(z)} \\ \Leftrightarrow \quad & 1 + z\frac{f''(z)}{f'(z)} + \left(\frac{2z}{z-p} - 1\right) - \left(\frac{2pz}{1-pz} + 1\right) = -1 - \frac{2z\varphi(z)}{1-z\varphi(z)} \end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \quad & z\frac{f''(z)}{f'(z)} + \frac{2z}{z-p} - \frac{2pz}{1-pz} = -\frac{2z\varphi(z)}{1-z\varphi(z)} \\
\Leftrightarrow \quad & \frac{f''(z)}{f'(z)} + \frac{2}{z-p} - \frac{2p}{1-pz} = -\frac{2\varphi(z)}{1-z\varphi(z)} \\
\Leftrightarrow \quad & \frac{d}{dz}\big(\log(f'(z)) + 2\log(z-p) + 2\log(1-pz)\big) = -\frac{2\varphi(z)}{1-z\varphi(z)}.
\end{aligned}$$

Integration yields

$$\begin{aligned}
& \log\left(f'(z)(z-p)^2(1-pz)^2\right) - \log p^2 = -\,2\int_0^z \frac{\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta \\
\Leftrightarrow \quad & f'(z) = \frac{p^2}{(z-p)^2(1-zp)^2}\exp\int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta.
\end{aligned}$$

Similar to the case of Theorem 2.12, the representation (2.20) must not have a residue of its own because of the properties of $f'(z)$. It has to be

$$\left((z-p)^2\, f'(z)\right)'\Big|_{z=p} = 0. \tag{2.21}$$

Setting $\kappa(z) := \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta$ and $g(z) := (z-p)^2\, f'(z) = \frac{p^2}{(1-zp)^2}e^{\kappa(z)}$, we obtain

$$\frac{g'(z)}{g(z)} = \kappa'(z) + \frac{2p}{1-zp}.$$

Therefore it is necessary to be

$$\begin{aligned}
& \frac{-2\varphi(p)}{1-p\varphi(p)} + \frac{2p}{1-p^2} = 0 \\
\Leftrightarrow \quad & \varphi(p) = p,
\end{aligned}$$

so that (2.21) is satisfied.

On the other hand, if $\varphi : \mathbb{D} \to \mathbb{D}$ is a holomorphic function with $\varphi(p) = p$, the function f defined by (2.20) does not have a residue of its own due to the consideration of the above. Furthermore it satisfies

$$1 + z\frac{f''(z)}{f'(z)} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz} = -\frac{1+z\varphi(z)}{1-z\varphi(z)}.$$

Using Theorem 2.9, we obtain $f \in \mathcal{C}o_p$. □

The fixed point at p of the function φ in Theorem 2.13 might however be complicated and not very useful, in case one wants to construct a concave function with pole at p. Using several transformations we obtain an alternate version of Theorem 2.13, where the fixed point of the involved function is at the origin.

Corollary 2.14. *Let $p \in (0,1)$. If a meromorphic function $f : \mathbb{D} \to \widehat{\mathbb{C}}$ belongs to the class $\mathcal{C}o_p$, then there exists a function*

$$\Psi : \mathbb{D} \to \mathbb{D} \text{ with } \Psi(0) = 0$$

holomorphic in $\mathbb{D}$, such that the concave function can be represented as

$$f'(z) = \frac{p^2}{(z-p)^2(1-zp)^2} \exp \int_p^{\frac{p-z}{1-pz}} \left(\frac{1}{1-p\zeta} - \frac{1}{1-\zeta\Psi(\zeta)} \right) \frac{2}{\zeta} d\zeta \tag{2.22}$$

for $z \in \mathbb{D}$. Conversely, for any holomorphic function Ψ mapping $\mathbb{D}$ to $\mathbb{D}$ with $\Psi(0) = 0$, there exists a concave function of class $\mathcal{C}o_p$ described by (2.22).

Proof. Let $p \in (0,1)$ and $z \in \mathbb{D}$. Applying the transformation $\zeta = \frac{p-x}{1-px}$ and $\Phi(x) = \varphi(\zeta)$ we obtain

$$\begin{aligned} \int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta &= \int_p^{\frac{p-z}{1-pz}} \frac{-2\Phi(x)}{1-\frac{p-x}{1-px}\Phi(x)} \cdot \frac{p^2-1}{(1-px)^2} dx \\ &= \int_p^{\frac{p-z}{1-pz}} \frac{-2\Phi(x)(p^2-1)}{(1-px)^2-(p-x)\Phi(x)(1-px)} dx. \end{aligned}$$

Here the function Φ is holomorphic in $\mathbb{D}$ with $\Phi(0) = p$. Therefore there exists a function $\Psi : \mathbb{D} \to \mathbb{D}$ holomorphic in $\mathbb{D}$ with $\Psi(0) = 0$, such that $\Phi(x) = \frac{p-\Psi(x)}{1-p\Psi(x)}$. Then

$$\begin{aligned} \int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta &= \int_p^{\frac{p-z}{1-pz}} \frac{-2\frac{p-\Psi(x)}{1-p\Psi(x)}(p^2-1)}{(1-px)^2-(p-x)\frac{p-\Psi(x)}{1-p\Psi(x)}(1-px)} dx \\ &= \int_p^{\frac{p-z}{1-pz}} \frac{-2(p-\Psi(x))(p^2-1)}{(1-px)\left((1-p^2)-x\Psi(x)(1-p^2)\right)} dx \\ &= \int_p^{\frac{p-z}{1-pz}} \frac{-2(\Psi(x)-p)}{(1-px)(1-x\Psi(x)} dx \\ &= \int_p^{\frac{p-z}{1-pz}} \left(\frac{1}{1-x\Psi(x)} - \frac{1}{1-px} \right) \frac{-2}{x} dx. \end{aligned}$$

Changing the variable inside the integration leads to the statement. □

Considering the class $\mathcal{C}o(\alpha)$ Avkhadiev and Wirths proved the following in [3].

Theorem 2.15 (see [3]). *Let $\alpha \in (1,2]$ and $f : \mathbb{D} \to \widehat{\mathbb{C}}$ be an analytic function with $f(0) = f'(0) - 1 = 0$. Then $f \in \mathcal{C}o(\alpha)$ if and only if there exists a function $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$, holomorphic in $\mathbb{D}$, such that for $z \in \mathbb{D}$*

$$f'(z) = \frac{1}{(1-z)^{\alpha+1}} \exp \int_0^z -(\alpha-1) \frac{\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta. \tag{2.23}$$

Using a positive measure $\mu(t)$ as in Theorem 2.1, we can obtain the next statement.

Theorem 2.16. *Let $\alpha \in (1,2]$ and f be an analytic function with $f(0) = f'(0) - 1 = 0$. Then $f \in \mathcal{C}o(\alpha)$ if and only if there exists a positive measure $\mu(t)$ with $\int_{-\pi}^{\pi} \mu(t)dt = 1$, such that*

$$f'(z) = \frac{1}{(1-z)^{\alpha+1}} \exp \int_{-\pi}^{\pi} (\alpha - 1) \log(1 - e^{-it}z) d\mu(t). \tag{2.24}$$

Proof. The normalized, analytic function f is of class $\mathcal{C}o(\alpha)$ if and only if (2.18) of Theorem 2.11 is valid. It is known, that every function $P(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ with $\operatorname{Re} P(z) > 0$ for $z \in \mathbb{D}$ can be represented as

$$P(z) = \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t),\ z \in \mathbb{D}$$

with some positive measure $\mu(t)$ due to the Herglotz representation formula.
From the normalized form of (2.18) we therefore obtain the existence of a positive measure $\mu(t)$, with $\int_{-\pi}^{\pi} d\mu(t) = 1$, such that

$$\begin{aligned}
& -\frac{2}{\alpha-1}\left(1 + z\frac{f''(z)}{f'(z)} - \frac{\alpha+1}{2} \cdot \frac{1+z}{1-z}\right) = \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) \\
\Leftrightarrow \quad & \frac{2}{\alpha-1}\left((\alpha+1)\frac{z}{1-z} + \frac{\alpha-1}{2} - z\frac{f''(z)}{f'(z)}\right) - 1 = \int_{-\pi}^{\pi} \frac{2z}{e^{it}-z} d\mu(t) \\
\Leftrightarrow \quad & \frac{2z(\alpha+1)}{(\alpha-1)(1-z)} - \frac{2}{\alpha-1} \cdot \frac{zf''(z)}{f'(z)} = \int_{-\pi}^{\pi} \frac{2z}{e^{it}-z} d\mu(t) \\
\Leftrightarrow \quad & \frac{\alpha+1}{1-z} z - z\frac{f''(z)}{f'(z)} = (\alpha-1) \int_{-\pi}^{\pi} \frac{z}{e^{it}-z} d\mu(t).
\end{aligned}$$

Considering the derivative leads to

$$\begin{aligned}
& \frac{d}{dz}\left(-(\alpha+1)\log(1-z) - \log f'(z)\right) = -(\alpha-1)\int_{-\pi}^{\pi} \frac{d}{dz} \log(1 - e^{it}z) d\mu(t) \\
\Leftrightarrow \quad & \log(1-z)^{\alpha+1} f'(z) = (\alpha-1)\int_{-\pi}^{\pi} \log(1-e^{it}z) d\mu(t),
\end{aligned}$$

which is obviously equivalent to the desired representation formula. □

Since we do not have to deal with any complications concerning the logarithm during the proof of Theorem 2.16, there are no additional conditions for the measure, as it was the case in the previous theorems.

Remark 2.17. As it can easily be observed, there is a similarity between the representation formula using a function φ (see e.g. Theorem 2.12) and the version considering a positive measure $\mu(t)$ (see e.g. Theorem 2.1).

Since the expression $z \mapsto \frac{1+z\varphi(z)}{1-z\varphi(z)}$, with $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$, φ holomorphic in $\mathbb{D}$, maps the unit disk onto the right half of the complex plane and is normalized by $0 \mapsto 1$, it can also be described by means of the Herglotz representation formula. Therefore

$$\begin{aligned}
&& \frac{1+z\varphi(z)}{1-z\varphi(z)} &= \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) \\
\Leftrightarrow && \frac{2z\varphi(z)}{1-z\varphi(z)} + 1 &= \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) \\
\Leftrightarrow && \frac{2\varphi(z)}{1-z\varphi(z)} &= \int_{-\pi}^{\pi} \frac{2z}{e^{it}-z} d\mu(t) \\
\Leftrightarrow && \int_0^z \frac{\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta &= \int_{-\pi}^{\pi} \log(1-e^{it}z) d\mu(t) \qquad (2.25)
\end{aligned}$$

for $z \in \mathbb{D}$. The existence of a certain function φ is hereby equivalent to the existence of a positive measure $\mu(t)$ so that (2.25) holds.

However, since the representation using the measure involves logarithms, we have to be careful with additional conditions, to ensure that the results are well-defined. On the other hand, additional conditions for φ are provided by the fact, that $f'(z)$ must not have a residue of its own, as shown in the previous proofs.

3 Coefficients of Concave Functions - Known Results

In this chapter we will introduce the previously known results for the coefficients of concave functions. As it is the case for the Bieberbach Conjecture, the variability of the coefficients is of great interest and there are two different ways for an approach.

Since we have a pole at some point in the unit disk (and we assume this is not the origin), we have the Taylor Series expansion at the origin, valid up to the pole, and the Laurent Series expansion, valid up to the closest boundary point.

We begin with the discussion about the domain of the Taylor series coefficients.

3.1 Coefficients of the Taylor Series

First, we need some additional tools describing concave functions. In 1980 Miller [19] showed that the following theorem is valid for functions in $f \in \mathcal{C}o_p$.

Theorem 3.1 (see [19]). *Let $p \in (0,1)$ and $f \in \mathcal{C}o_p$. For $z \in \mathbb{D}\backslash\{0\}$*

$$\left| \frac{1}{f(z)} - \frac{1}{z} + \frac{1+p^2}{p} \right| \leq 1. \tag{3.1}$$

Proof. Let $z, \zeta \in \mathbb{D}$. We know that a function p defined by

$$-P(z) = \frac{2zf'(z)}{f(z)-f(\zeta)} + \frac{\zeta+z}{\zeta-z} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz}$$

has positive real part and $P(0) = 1$. Further

$$\begin{aligned} P'(z) = &-\frac{2f'(z)}{f(z)-f(\zeta)} - \frac{2zf''(z)}{f(z)-f(\zeta)} + \frac{2zf'(z)}{(f(z)-f(\zeta))^2} \\ &-\frac{1}{\zeta-z} - \frac{\zeta+z}{(\zeta-z)^2} - \frac{1}{z-p} + \frac{z+p}{(z-p)^2} + \frac{p}{1-pz} + \frac{p+p^2z}{(1-pz)^2} \end{aligned}$$

For $\zeta \neq 0$, we obtain

$$P'(0) = \frac{2}{f(\zeta)} - 2\frac{1}{\zeta} + 2\frac{1}{p} + 2p$$

which leads to a Maclaurin Series expansion of the form

$$P(z) = 1 + \left(\frac{1}{f(\zeta)} - \frac{1}{\zeta} + \frac{1}{p} + p\right) 2z + \cdots .$$

Since $P \in \mathcal{P}$, we can use Lemma 1.3, which leads to the statement. $\square$

Using the above, we have the following lemma, which will be helpful for the following discussion.

Lemma 3.2 (see [4]). *Let $f \in Co_p$, $p \in (0,1)$. There exists a holomorphic function $\omega : \mathbb{D} \to \mathbb{D}$, such that*

$$f(z) = \frac{z - \frac{p}{1+p^2}(1+\omega(z))z^2}{\left(1 - \frac{z}{p}\right)(1 - zp)} \tag{3.2}$$

for $z \in \mathbb{D}$.

Proof. Setting

$$\omega(z) = \frac{1}{f(z)} - \frac{1}{z} + \frac{1+p^2}{p}, \tag{3.3}$$

we have a holomorphic function ω due to the properties of f and the fact, that there is a holomorphic continuation at both the origin and p, where $\omega(p) = -\frac{1}{p} + \frac{1+p^2}{p} = p$. From the previous Theorem, we know that $\omega(\mathbb{D}) \subset \bar{\mathbb{D}}$. Therefore, there exists a holomorphic function v with $v \in \bar{\mathbb{D}}$, such that

$$\omega(z) = \frac{p + \frac{z-p}{1-zp}v(z)}{1 + p\frac{z-p}{1-zp}v(z)}.$$

Using (3.3) we obtain

$$f(z) = \frac{zp\left(1 + p\frac{z-p}{1-zp}v(z)\right)}{zp^2 + zp\frac{z-p}{1-zp}v(z) + p + p^2\frac{z-p}{1-zp}v(z) - (1+p^2)z\left(1 + p\frac{z-p}{1-zp}v(z)\right)}$$

and the denominator can be written as

$$\begin{aligned}
& zp^2 + zp\frac{z-p}{1-zp}v(z) + p + p^2\frac{z-p}{1-zp}v(z) - (1+p^2)z\left(1 + p\frac{z-p}{1-zp}v(z)\right) \\
= \;& zp^2(1-zp) + zp(z-p)v(z) + p(1-zp) + p^2(z-p)v(z) \\
& \qquad -(1+p^2)z(1-zp) - (1+p^2)zp(z-p)v(z) \\
= \;& (1-zp)(zp^2 + p - z - zp^2) + v(z)(z-p)(zp + p^2 - zp - zp^3) \\
= \;& p(1-zp)(1-\frac{z}{p}) - v(z)p^3(1-\frac{z}{p})(1-zp) \\
= \;& p(1-zp)(1-\frac{z}{p})(1-p^2v(z)).
\end{aligned}$$

Therefore

$$f(z) = z\frac{1 - zp + p(z-p)v(z)}{\left(1 - \frac{z}{p}\right)(1 - zp)(1 - p^2 v(z))}. \tag{3.4}$$

Rewriting v in the form

$$v(z) = \frac{p^2 - \omega(z)}{1 - p^2\omega(z)}$$

and inserting into (3.4) leads to

$$\begin{aligned} f(z) &= \frac{z - zp^2\omega(z) - z^2p(1 - p^2\omega(z)) + zp(z-p)(p^2 - \omega(z))}{\left(1 - \frac{z}{p}\right)(1 - zp)(1 - p^2\omega(z) - p^2(p^2 - \omega(z)))} \\ &= \frac{z(1-p^4) + z^2(p^3 - p) + \omega(z)(z^2p^3 - zp^2 - z^2p + p^2z)}{\left(1 - \frac{z}{p}\right)(1 - zp)(1 - p^2)(1 + p^2)} \\ &= \frac{z + z^2p\left(\frac{p^2-1}{(1-p^2)(1+p^2)} + \omega(z)\frac{p^2-1}{(1-p^2)(1+p^2)}\right)}{\left(1 - \frac{z}{p}\right)(1 - zp)}, \end{aligned}$$

which is (3.2). □

Using these statements, we are able to prove a general estimate for Taylor Series coefficients of functions in the class $\mathcal{C}o_p$.

Theorem 3.3 (see [4]). *Let $p \in (0,1)$, $f \in \mathcal{C}o_p$ and $n \geq 2$. Then*

$$\left| a_n(f) - \frac{1 - p^{2n+2}}{p^{n-1}(1 - p^4)} \right| \leq \frac{p^2(1 - p^{2n-2})}{p^{n-1}(1 - p^4)}. \tag{3.5}$$

Proof. We consider the function

$$g(z) = \frac{z - \frac{pz^2}{1+p^2}}{\left(1 - \frac{z}{p}\right)(1 - zp)}.$$

By polynomial division we calculate

$$\begin{aligned} g'(0) &= 1 \\ \frac{g''(0)}{2} &= \frac{p^2+1}{p} + \frac{-p}{1+p^2} = \frac{p^4 + p^2 + 1}{p(1+p^2)} = \frac{1-p^6}{p(1-p^4)} \\ \frac{g^{(3)}(0)}{3!} &= \frac{1-p^6}{p(1-p^4)}\frac{p^2+1}{p} - 1 = \frac{1-p^8}{p^2(1-p^4)} \end{aligned}$$

and obtain by induction

$$\frac{g^{(n)}(0)}{n!} = \frac{g^{(n-1)}(0)}{(n-1)!}\frac{p^2+1}{p} - \frac{g^{(n-2)}(0)}{(n-2)!}.$$

in general.

Therefore

$$\begin{aligned}
\frac{g^{(n)}(0)}{n!} &= \frac{1-p^{2(n-1)+2}}{p^{n-2}(1-p^4)}\cdot\frac{p^2+1}{p}-\frac{1-p^{2(n-2)+2}}{p^{n-3}(1-p^4)}\\
&= \frac{1}{(1-p^4)p^{n-1}}\left((1-p^{2n})(p^2+1)-p^2(1-p^{2n-2})\right)\\
&= \frac{1}{(1-p^4)p^{n-1}}\left(1-p^{2n}+p^2-p^{2n+2}-p^2+p^{2n}\right)\\
&= \frac{1-p^{2n+2}}{p^{n-1}(1-p^4)}.
\end{aligned}$$

which leads to

$$g(z)=\frac{z-\frac{pz^2}{1+p^2}}{\left(1-\frac{z}{p}\right)(1-zp)}=\sum_{n=1}^{\infty}\frac{1-p^{2n+2}}{p^{n-1}(1-p^4)}z^n,\ |z|<p.$$

Furthermore we define a function h by

$$h(z)=g(z)-f(z)=\frac{\frac{pz^2}{1+p^2}\omega(z)}{\left(1-\frac{z}{p}\right)(1-zp)}=\sum_{n=1}^{\infty}b_n(\omega)z^n,\ |z|<p,$$

where $f(z)$ is the function from (3.2) and ω a holomorphic function with $\omega(\mathbb{D})\subset\bar{\mathbb{D}}$. For the statement, we need to show

$$|b_n(\omega)|\le\frac{p^2(1-p^{2n-2})}{p^{n-1}(1-p^4)}. \tag{3.6}$$

Using $\omega(z)=\sum_{k=0}^{\infty}c_kz^k$ we obtain

$$h(z)=\frac{pc_0}{1+p^2}z^2+\left(c_0+\frac{pc_1}{1+p^2}\right)z^3+\left(\frac{(1-p^6)c_0}{(1-p^4)p}+c_1+\frac{pc_2}{1+p^2}\right)z^4+...$$

meaning

$$b_n(\omega)=\sum_{k=0}^{n-2}c_k\frac{p^2}{p^{n-k-1}}\frac{1-p^{2(n-k)-2}}{1-p^4}.$$

Setting $m=n-2$ and rescaling, we have the equivalent formulation

$$\begin{aligned}
&\left|\sum_{k=0}^{n-2}c_k\frac{p^2}{p^{n-k-1}}\frac{1-p^{2(n-k)-2}}{1-p^4}\right|\le\frac{p^2(1-p^{2n-2})}{p^{n-1}(1-p^4)}\\
\Leftrightarrow\quad&\left|\sum_{k=0}^{n-2}c_k\frac{1-p^{2(n-k)-2}}{p^{n-k-1}}\right|\le\frac{(1-p^{2n-2})}{p^{n-1}}\\
\Leftrightarrow\quad&\left|\sum_{k=0}^{m}c_k\frac{1-p^{2(m-k)+2}}{p^{m-k}}\right|\le\frac{1-p^{2m+2}}{p^m}.
\end{aligned} \tag{3.7}$$

To prove the validity of (3.7), we regard this as a problem of linear functionals in H^p-Spaces. A detailed presentation of this theory can be found e.g. in [10].

We consider the linear functional

$$\Phi_m(\omega) = \sum_{k=0}^{m} c_k \frac{1-p^{2(m-k)+2}}{p^{m-k}}, \tag{3.8}$$

on H^∞ and therefore need to show

$$|\Phi_m(\omega)| \leq \frac{1-p^{2m+2}}{p^m}. \tag{3.9}$$

Choosing

$$\kappa_m(z) = \sum_{k=0}^{m} \frac{1-p^{2(m-k)+2}}{p^{m-k}} z^{-k-1}, \tag{3.10}$$

as the kernel, we obtain

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \omega(z)\kappa_m(z)dz = \sum_{k=0}^{m} c_k \frac{1-p^{2(m-k)+2}}{p^{m-k}} = \Phi_m(\omega).$$

using the Residue Theorem.

Shifting the indices in (3.10) by $k = m-1$ and setting

$$K_m(z) = z^{-m-1}P_m(z) \tag{3.11}$$

with

$$P_m(z) = \sum_{l=0}^{m-1} \frac{1-p^{2l+2}}{p^l}\left(z^l + z^{2m-l}\right) + \frac{1-p^{2m+2}}{p^m} z^m,\ m \geq 0, \tag{3.12}$$

we obtain an alternate kernel K_m to the original κ_m, which produces the same functional.

Now we consider the trigonometrical polynomials Q_m with $m \geq 0$ and $\vartheta \in [0, 2\pi]$ of the form

$$\begin{aligned} Q_m(\vartheta) &= e^{-im\vartheta}P_m(e^{i\vartheta}) \\ &= \sum_{l=0}^{m-1} \frac{1-p^{2l+2}}{p^l}\left(e^{-i\vartheta(m-l)} + e^{i\vartheta(m-l)}\right) + \frac{1-p^{2m+2}}{p^m} \\ &= \sum_{l=0}^{m-1} \frac{1-p^{2l+2}}{p^l} 2\cos((m-l)\vartheta) + \frac{1-p^{2m+2}}{p^m}. \end{aligned} \tag{3.13}$$

Let

$$\Lambda(z) = \frac{1}{2}\frac{1-p^2}{\left(1-\frac{z}{p}\right)(1-zp)}\left(\frac{1+e^{i\vartheta}z}{1-e^{i\vartheta}z} + \frac{1+e^{-i\vartheta}z}{1-e^{-i\vartheta}z}\right).$$

Then

$$\frac{1+e^{i\vartheta}z}{1-e^{i\vartheta}z} + \frac{1+e^{-i\vartheta}z}{1-e^{-i\vartheta}z} = \frac{2-2z^2}{1-z\left(e^{i\vartheta}+e^{-i\vartheta}\right)+z^2}$$

$$= \frac{1-z^2}{\frac{1}{2} - z\cos\vartheta + \frac{1}{2}z^2}$$

and we obtain

$$\begin{aligned}
\Lambda(z) &= \frac{1}{2}\frac{(1-p^2)(1-z^2)}{\left(1-\frac{z}{p}\right)(1-zp)(1-e^{i\vartheta}z)(1-e^{-i\vartheta}z)} \\
&= \frac{1}{2}\frac{1-p^2}{1-\left(\frac{p^2+1}{p}z\right)+z^2}\cdot\frac{1-z^2}{\frac{1}{2}-z\cos\vartheta+\frac{1}{2}z^2} \\
&= \frac{1-p^2-(1-p^2)z^2}{1-\left(\frac{p^2+1}{p}+2\cos\vartheta\right)z+2\left(1+\cos\vartheta\frac{p^2+1}{p}\right)z^2-\left(\frac{p^2+1}{p}+2\cos\vartheta\right)+z^4}.
\end{aligned}$$

Polynomial division leads to

$$\Lambda(z) = \sum_{m=0}^{\infty} Q_m(\vartheta)z^m.$$

Decomposition at the poles $z_1 = p$, $z_2 = 1/p$, $z_3 = e^{i\vartheta}$ and $z_4 = e^{-i\vartheta}$ gives

$$Q_m(\vartheta) = \frac{(1-p^2)(1+p^{2m+2}-2p^{m+1}\cos((m+1)\vartheta))}{p^m(1+p^2-2p\cos(\vartheta))}.$$

This expression is always positive for $m \geq 0$. Therefore we have

$$\begin{aligned}
e^{i\vartheta}K_m(e^{i\vartheta}) &\overset{(3.11)}{=} e^{-im\vartheta}P_m(e^{i\vartheta}) \\
&\overset{(3.13)}{=} Q_m(\vartheta) \geq 0. \qquad (3.14)
\end{aligned}$$

Remembering $\|\omega\|_\infty \leq 1$, we have

$$\begin{aligned}
\left|\sum_{k=0}^{m} c_k \frac{1-p^{2(m-k)+2}}{p^{m-k}}\right| &= \left|\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \omega(z)\kappa_m(z)dz\right| \\
&= \left|\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \omega(z)K_m(z)dz\right| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \left|e^{i\vartheta}K_m(e^{i\vartheta})\right| d\vartheta\|\omega\|_\infty \\
&= \frac{1}{2\pi}\int_0^{2\pi} Q_m(\vartheta)d\vartheta\|\omega\|_\infty \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} Q_m(\vartheta)d\vartheta \\
&= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} K_m(z)dz \overset{(3.12)}{=} \frac{1-p^{2m+2}}{p^m}. \qquad (3.15)
\end{aligned}$$

which is the statement of the theorem. □

Remark 3.4. Considering the function $\omega \equiv 1$, we obtain equality at every step in (3.15).

Applying the theory of extremal problems for linear functionals on Φ_m, there exists a unique normalized extremal function ω_e such that

$$\max\{|\Phi_m(\omega)||\omega \in H^\infty,\ \|\omega\|_\infty \leq 1\} = \Phi_m(\omega_e),$$

where this function is $\omega_e \equiv 1$ due to the above. As a general extremal function, we have therefore $\omega \equiv e^{i\vartheta}$ with $\vartheta \in [0, 2\pi)$, which leads to

$$f_\vartheta(z) = \frac{z - \frac{p}{1+p^2}(1+e^{i\vartheta})z^2}{\left(1 - \frac{z}{p}\right)(1 - zp)}. \tag{3.16}$$

using the lemma above.

3.2 Coefficients of the Laurent Series

Instead of the Maclaurin Series expansion we can also look at the Laurent Series expansion of functions in $\mathcal{C}o_p$. We consider concave functions in $\mathcal{C}o_p$ of the form

$$f(z) = \sum_{n=-1}^{\infty} c_n(f)\,(z-p)^n \tag{3.17}$$

in $|z - p| < 1 - p$. Again we are concerned with the variability of the coefficients.

First we take a closer look at the residue.

Theorem 3.5 (see [33]). *Let $p \in (0,1)$ and $f(z) = \sum_{n=-1}^{\infty} c_n\,(z-p)^n \ \in \mathcal{C}o_p$. Then for the residue $c_{-1}(f)$ we have*

$$\left| c_{-1}(f) + \frac{p^2}{1-p^4} \right| \leq \frac{p^4}{1-p^4}. \tag{3.18}$$

Equality occurs if and only if

$$f_\vartheta(z) = \frac{z - \frac{p}{1+p^2}(1+e^{i\vartheta})z^2}{\left(1 - \frac{z}{p}\right)(1 - zp)} \tag{3.19}$$

for $\vartheta \in [0;\ 2\pi]$.

Proof. According to Lemma 3.1 we have

$$\left| \frac{1}{f(z)} - \frac{1}{z} + \frac{1+p^2}{p} \right| \leq 1.$$

for a concave function $f \in \mathcal{C}o_p$ and $z \in \mathbb{D}\backslash\{0\}$.

Setting

$$\omega(z) = \frac{1}{f(z)} - \frac{1}{z} + \frac{1+p^2}{p}, \tag{3.20}$$

as in the proof of Lemma 3.2, we have $|\omega(z)| \le 1$ for $z \in \mathbb{D}$ and $\omega(p) = p$. Due to the Schwarz Lemma,

$$|\omega'(p)| \le \frac{1 - |\omega(p)|^2}{1 - p^2} = 1. \tag{3.21}$$

Equality is attained if and only if ω is an automorphism of the unit disk with fixed point p. A short calculation gives

$$\begin{aligned}
\omega'(z) &= \frac{-f'(z)}{f^2(z)} + \frac{1}{z^2} \\
&= \frac{-\left(\frac{-c_{-1}(f)}{(z-p)^2} + a_1 + \ldots\right)}{\left(\frac{c_{-1}(f)}{z-p} + c_0(f) + \ldots\right)^2} + \frac{1}{z^2} \\
&= \frac{c_{-1}(f) - c_1(f)(z-p)^2 + \ldots}{(c_{-1}(f) + c_0(f)(z-p) + \ldots)^2} + \frac{1}{z^2} \\
\omega'(p) &= \frac{1}{c_{-1}(f)} + \frac{1}{p^2}. \qquad (3.22)
\end{aligned}$$

Combining with (3.21) leads to

$$\left| \frac{1}{c_{-1}(f)} + \frac{1}{p^2} \right| \le 1.$$

This gives the first part of the Theorem.

Considering

$$\begin{aligned}
c_{-1}(f) &= \lim_{z \to p} (z-p) f_\vartheta(z) = \frac{p^2 - \frac{p^4}{1+p^2}(1 + e^{i\vartheta})}{-(1-p^2)} \\
&= -\frac{p^2}{1-p^4} + \frac{p^4}{1-p^4} e^{i\vartheta}, \qquad (3.23)
\end{aligned}$$

we obtain the second part of the Theorem. □

After dealing with the residue, the analysis of the coefficient $c_0(f)$ will be the second step. For this we need another Lemma provided by Jenkins in 1962 [15].

Lemma 3.6. *(see [15]) Let $f(z) = z + \sum_{n=0}^{\infty} b_n z^n$ be a univalent function with simple pole at $p \in (0, 1)$. Then*

$$|b_n| \le \frac{1 + p^2 + \ldots + p^{2n-2}}{p^{n-1}} \tag{3.24}$$

for all $n \in \mathbb{N}$.

Using this lemma, we are now able to obtain an estimate concerning $c_0(f)$ with respect to the residue.

Theorem 3.7 (see [16]). *Let $f(z) = \sum_{n=-1}^{\infty} c_n(f) (z-p)^n \in Co_p$ with $p \in (0, 1)$. Then*

$$\left| p + \frac{c_0(f)(1-p^2)}{c_{-1}(f)} \right| \le \frac{1+p^2}{p}. \tag{3.25}$$

This inequality is sharp

Proof. We define

$$h(z) = \frac{-c_{-1}(f)}{(1-p^2)f\left(\frac{p-z}{1-pz}\right)}.$$

Due to the properties of f we know that h is univalent in $\mathbb{D}$ and has a simple pole at the origin. Furthermore

$$\begin{aligned}
h(0) &= \left.\frac{-c_{-1}(f)}{(1-p^2)f(\zeta)}\right|_{\zeta=p} = 0 \\
h'(z) &= \frac{-c_{-1}(f)}{(1-pz)^2}\frac{f'\left(\frac{p-z}{1-pz}\right)}{f^2\left(\frac{p-z}{1-pz}\right)} \\
h'(0) &= \left.\frac{-c_{-1}(f)\left(-c_{-1}(f)\frac{1}{(\zeta-p)^2}+c_1(f)+...\right)}{\left(c_{-1}(f)\frac{1}{\zeta-p}+a_0+...\right)^2}\right|_{\zeta=p} \\
&= \left.\frac{c_{-1}(f)^2+c_{-1}(f)c_1(f)(\zeta-p)^2+...}{(c_{-1}(f)+c_0(f)(\zeta-p)+...)^2}\right|_{\zeta=p} = 1 \\
h''(0) &= 2p+\frac{2c_0(f)(1-p^2)}{c_{-1}(f)}.
\end{aligned}$$

For $|z-p|<1-p$ we therefore have

$$h(z) = z + \left(p+\frac{(1-p^2)c_0(f)}{c_{-1}(f)}\right)z^2+...$$

and with the previous Lemma for $n=2$

$$\left|p+\frac{(1-p^2)c_0}{c_{-1}(f)}\right| \leq \frac{1+p^2}{p}$$

which gives the statement.

Equality is attained for the extremal function

$$f_e(z) = \frac{z}{(1-\frac{z}{p})(1-pz)}. \tag{3.26}$$

□

Combining this with the result about the residue, we have the following theorem.

Theorem 3.8. *(see [6]) Let $p\in(0,1)$ and $f(z)=\sum_{n=-1}^{\infty}c_n(f)\,(z-p)^n\in\mathcal{C}o_p$. Then*

$$\operatorname{Re} c_0(f) \geq -\frac{p}{(1-p^2)^2}. \tag{3.27}$$

Proof. Due to the previous proof, for each $f \in \mathcal{C}o_p$ there exists a number $\zeta \in \bar{\mathbb{D}}$ such that

$$c_0(f) = \frac{c_{-1}(f)}{1-p^2}\left(-p + \zeta \frac{1+p^2}{p}\right). \tag{3.28}$$

Since we want to find the smallest real part, it is sufficient to look at $\zeta = e^{i\vartheta}$, $\vartheta \in [0, 2\pi]$, and $c_{-1}(f)$ as in (3.18). We therefore have to find the minimum of the expression

$$\frac{-p}{(1-p^4)(1-p^2)}\left((1+p^2)\cos\vartheta - p^2\right) - \frac{p^3}{(1-p^4)(1-p^2)}\left((1+p^2)^2\sin^2\vartheta + \left((1+p^2)\cos\vartheta - p^2\right)^2\right)^{\frac{1}{2}}.$$

Setting $x = \cos\vartheta \in [-1; 1]$ and calculating the derivative for x, there is no local extremum in the interval $(-1, 1)$. Therefore we have a minimum for $\zeta = 1$ and $c_{-1}(f) = \frac{-p^2}{1-p^2}$, which gives the statement. □

If the pole is close enough to the origin, we have a more refined statement.

Theorem 3.9 (see [1])**.** *Let $p \in (0, \sqrt{3}-1]$ and $f(z) = \sum_{n=-1}^{\infty} c_n(f)\,(z-p)^n \in \mathcal{C}o_p$. Then*

$$\left| c_0 + \frac{1-p^2+p^4}{1-p^4} \right| \leq \frac{p^2(2-p^2)}{1-p^4}. \tag{3.29}$$

Equality occurs for the previously mentioned f_ϑ.

Proof. Considering the function $f_\omega(z) = \frac{z - \frac{p}{1+p^2}(1+\omega(z))z^2}{\left(1-\frac{z}{p}\right)(1-zp)}$ from Lemma 3.2, multiplying with the denominator and equating the coefficients on both sides using the expansion $\omega(z) = \sum_{n=0}^{\infty} c_n(\omega)\,(z-p)^n$ in $z \in \mathbb{D}$ around p leads to

$$\begin{aligned} -\frac{1-p^2}{p}c_{-1}(f) &= p - \frac{p^3}{1+p^2} - \frac{p^3}{1+p^2}c_0(\omega) \\ \Leftrightarrow \qquad c_{-1}(f) &= -\frac{p^2}{1-p^4} + \frac{p^4}{1-p^4}c_0(\omega) \end{aligned} \tag{3.30}$$

and

$$c_{-1}(f) + \frac{1-p^2}{p}c_0(f) = \frac{1-p^2}{1+p^2} - \frac{2p^2}{1+p^2}c_0(\omega) - \frac{p^3}{1+p^2}c_1(\omega). \tag{3.31}$$

Combining (3.30) and (3.31) gives

$$\frac{1-p^2}{p}c_0(f) + \frac{1-p^2+p^4}{1-p^4} = \frac{2p^2-p^4}{1-p^4}c_0(\omega) + \frac{p^3}{1+p^2}c_1(\omega). \tag{3.32}$$

Since $|c_0(\omega)| \leq 1$ and $|c_1(\omega)| \leq \frac{1-|c_0(\omega)|^2}{1-p^2}$

$$\left| \frac{1-p^2}{p}c_0(f) + \frac{1-p^2+p^4}{1-p^4} \right| \leq \frac{p^2}{1-p^4}\left((2-p^2)|c_0(\omega)| + p(1-|c_0(\omega)|^2)\right).$$

If we now look at the function

$$g(x) = (2-p^2)x + p(1-x^2),$$

this has a local maximum at $x_p = \frac{2-p^2}{2p}$. Since $x_p \geq 1$ for $p \in]0, \sqrt{3}-1]$ we obtain

$$\max\{g(x) \mid x \in [0;1]\} = g(1) = 2 - p^2.$$

For equality it has to be $|\omega(z)| = |c_0| = 1$, which is only the case for a function of the form f_ϑ. □

Remark 3.10. With $|c_0| \leq 1$ we obtain the result of Theorem 2.3 directly from (3.30).

Similarly to the previous discussion, we can analyze further Laurent series coefficients of concave functions. To do so, we need some further Lemmas.

Lemma 3.11 (see [16]). *Let $P(z)$ be holomorphic in $\mathbb{D}$ with $\mathrm{Re}P(z) > 0$, $P(p) = 1-p^2$ and $P'(p) = 0$ for $p \in (0,1]$. If p has the expansion*

$$P(z) = (1-p^2) + d_2\,(z-p)^2 + d_3\,(z-p)^3 + \ldots \tag{3.33}$$

for $|z-p| < 1-p$, then

$$|d_2| \leq \frac{2}{1-p^2}, \tag{3.34}$$

$$\left|\frac{p}{1-p^2}d_2 + d_3\right| \leq \frac{6p}{(1-p^2)^2}, \qquad \frac{2}{3} \leq p < 1, \qquad \text{and} \tag{3.35}$$

$$\left|\frac{p}{1-p^2}d_2 + d_3\right| \leq \frac{2(1+\frac{9}{4}p^2)}{1-p^2}, \qquad 0 < p \leq \frac{2}{3}. \tag{3.36}$$

All inequalities are sharp.

Proof. Let

$$g(z) = \frac{P(z) - (1-p^2)}{P(z) + 1 - p^2}. \tag{3.37}$$

Then $g(p) = 0$ and $|g(z)| \leq 1$ for $z \in \mathbb{D}$, as well as

$$g'(z) = \frac{2(1-p^2)P'(z)}{(P(z)+1-p^2)^2},$$

with $g'(p) = 0$. Multiplying (3.37) with the denominator and considering the expansion at p, we have

$$\left(2(1-p^2) + \sum_{n=2}^{\infty} d_n (z-p)^n\right) \sum_{k=2}^{\infty} \frac{g^{(k)}(p)}{k!}(z-p)^k = \sum_{n=2}^{\infty} d_n (z-p)^n. \tag{3.38}$$

Therefore

$$d_2 = (1-p^2)g''(p) \tag{3.39}$$

and

$$\frac{p}{1-p^2}d_2 + d_3 = pg''(p) + (1-p^2)\frac{g'''(p)}{3}. \tag{3.40}$$

Rewriting g as

$$g(z) = \phi\left(\frac{z-p}{1-pz}\right),$$

where ϕ is holomorphic with $\phi(0) = \phi'(0) = 0$ and $|\phi(z)| \leq 1$ for $z \in \mathbb{D}$. Furthermore

$$g''(p) = \frac{\phi''(0)}{(1-p^2)^2}.$$

From $|\frac{\phi''(0)}{2}| \leq 1$ we have $|g''(p)| \leq \frac{2}{(1-p^2)^2}$. In combination with (3.39) this leads to

$$|d_2| = (1-p^2)|g''(p)| \leq \frac{2}{1-p^2},$$

which is (3.34). Considering (3.40) with

$$g'''(p) = \frac{6p}{(1-p^2)^3}\phi''(0) + \frac{\phi'''(0)}{(1-p^2)^3},$$

we obtain

$$\frac{p}{1-p^2}d_2 + d_3 = \frac{1}{(1-p^2)^2}\left(3p\phi''(0) + \frac{\phi'''(0)}{3}\right).$$

Setting $\phi(z) = a_2 z^2 + a_3 z^3 + ...$ for $z \in \mathbb{D}$ leads to

$$\frac{p}{1-p^2}d_2 + d_3 = \frac{2}{(1-p^2)^2}(3pa_2 + a_3).$$

Since ϕ is bounded, we have

$$|3pa_2 + a_3| \leq 3p|a_2| + |a_3| \leq 1 + 3p|a_2| - |a_2|^2$$

and therefore

$$\left|\frac{p}{1-p^2}d_2 + d_3\right| \leq \frac{2}{(1-p^2)^2}(1 + 3p|a_2| - |a_2|^2).$$

With $x = |a_2|$ and $h(x) = 1 + 3px - x^2$ we calculate $h'(x) = 3p - 2x$.

In case $p \geq \frac{2}{3}$, it has to be $h'(x) \geq 0$ for $x \in [0,1]$. The maximum is at $x = |a_2| = 1$ for $h(x) \leq h(1) = 3p$, which leads to (3.35). If $0 < p \leq \frac{2}{3}$ the function h attains its maximum at $x = |a_2| = \frac{3p}{2}$. Since $h(x) \leq 1 + \frac{9}{4}p^2$ and therefore (3.36).

Setting $g(z) = \left(\frac{z-p}{1-pz}\right)^2$ and therefore

$$P(z) = \frac{1 + p^2 - 4pz + (1+p^2)z^2}{1-z^2},$$

we obtain equality in (3.34) and (3.35). In case $0 < p \leq \frac{2}{3}$, we choose

$$\phi(z) = \frac{z^2(z + \frac{3}{2}p)}{1 + \frac{3}{2}pz},$$

for which we can construct a function P satisfying (3.36). □

Using this result, we are able to give a range for $c_1(f)$ and $c_2(f)$.

Theorem 3.12 (see [16]). *Let $p \in (0,1)$ and $f(z) = \sum_{n=-1}^{\infty} c_n(f)\,(z-p)^n \in \mathcal{C}o_p$. Then*

$$\begin{aligned} |c_1(f)| &\leq \frac{p^2}{(1-p^2)^3}, && && (3.41)\\ |c_2(f)| &\leq \frac{(4+9p^2)|c_{-1}(f)|}{12(1-p^2)^3}, && 0 < p \leq \frac{2}{3}, && (3.42)\\ \text{and} \quad |c_2(f)| &\leq \frac{p\,|c_{-1}(f)|}{(1-p^2)^3} \leq \frac{p^3}{(1-p^2)^4}, && \frac{2}{3} \leq p < 1. && (3.43) \end{aligned}$$

As in the previous Lemma all inequalities are sharp.

Proof. Let

$$P(z) = 2pz - 1 - p^2 - \frac{(z-p)(1-pz)f''(z)}{f'(z)}. \tag{3.44}$$

Then P satisfies the conditions for the previous Lemma and we can use the expression

$$\left(2p(z-p) - (1-p^2)\right) f'(z) - \left((z-p)(1-p^2) - p(z-p)^2\right) f''(z) = P(z)f'(z)$$

to equate the coefficients. This leads to

$$\begin{aligned} 2c_1(f)(1-p^2) &= c_{-1}(f)d_2 && (3.45)\\ \text{and} \qquad 6(1-p^2)c_2(f) &= 2pc_1(f) + c_{-1}(f)d_3. && (3.46) \end{aligned}$$

Using (3.34) and (3.45), we obtain

$$|c_1(f)| \leq \frac{|c_{-1}(f)|}{(1-p^2)^2}.$$

With $|c_{-1}(f)| \leq \frac{p^2}{1-p^2}$ from Theorem 2.3 this leads to (3.41).

From (3.45) and (3.46) we obtain

$$c_2(f) = \frac{1}{6(1-p^2)} \left(\frac{p}{1-p^2} d_2 + d_3 \right) c_{-1}(f). \tag{3.47}$$

For $0 < p \leq \frac{2}{3}$ (3.35) and (3.47) lead to the desired statement. In case $\frac{2}{3} \leq p < 1$ we combine (3.36) and (3.47) to obtain (3.43).

Equality in (3.41) and (3.43) is attained again for the function $f_e(z) = \frac{z}{(1-\frac{z}{p})(1-pz)}$ from equation (3.26).

In case $0 < p \leq \frac{2}{3}$ we have equality if f has the properties of P of (3.44). □

If we assume as in Theorem 3.9, that the poles are closer to the origin, we have the following statement for the coefficient $c_1(f)$.

Theorem 3.13. *(see [6]) Let $p \in (0, 1-\frac{\sqrt{2}}{2}]$ and $f(z) = \sum_{n=-1}^{\infty} c_n(f)\,(z-p)^n \in Co_p$. Then*

$$\left| c_1(f) \left(\frac{1-p^2}{p} \right)^2 + \frac{p^2}{1-p^4} \right| \leq \frac{1}{1-p^4}. \tag{3.48}$$

We have again equality for f_ϑ from (3.19).

Proof. Additionally to (3.30) and (3.31) we can equate the coefficients for $(z-p)^2$ and obtain

$$c_0(f) - \frac{1-p^2}{p} c_1(f) = -\frac{p}{1+p^2}(1 + c_0(\omega) + 2pc_1(\omega) + p^2 c_2(\omega)). \tag{3.49}$$

Inserting this into (3.32) gives

$$c_1(f) \left(\frac{1-p^2}{p} \right)^2 + \frac{p^2}{1-p^4} = \frac{c_0(\omega)}{1-p^4} + \frac{2p-p^3}{1+p^2} c_1(\omega) + \frac{p^2-p^4}{1+p^2} c_2(\omega) =: \Phi_p(\omega). \tag{3.50}$$

We have to show that

$$|\Phi_p(\omega)| \leq \frac{1}{1-p^4} \tag{3.51}$$

holds, where ω is chosen as in the proof for Theorem 3.9.

Considering Φ_p as a linear functional in H^∞ similar to the proof of Theorem 3.3, we can present the function $\Phi_p(\omega)$ in the form

$$\Phi_p(\omega) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \kappa_p(z)\omega(z)dz \tag{3.52}$$

where

$$\kappa_p(z) = \frac{1}{(1-p^4)(z-p)} + \frac{2p-p^3}{(1+p^2)(z-p)^2} + \frac{p^2-p^4}{(1+p^2)(z-p)^3}.$$

The functional Φ_p does not change, if we consider an equivalent kernel K_p holomorphic in $\bar{\mathbb{D}}$ except for the same singularity p instead of κ_p.

Therefore let

$$\begin{aligned} K_p(z) =& \frac{1}{1-p^4} \left(\frac{1}{z-p} + \frac{p}{1-pz} \right) + \frac{2p-p^3}{1+p^2} \left(\frac{1}{(z-p)^2} + \frac{1}{(1-pz)^2} \right) \\ &+ \frac{p^2-p^4}{1+p^2} \left(\frac{1}{(z-p)^3} + \frac{z}{(1-pz)^3} \right), \end{aligned}$$

which leads to

$$\begin{aligned}
e^{i\vartheta}K_p(e^{i\vartheta})(1+p^2)|1-pe^{i\vartheta}|^6 &= (1-2p\cos\vartheta+p^2)^2 \\
&\quad + (2p-p^3)(-4p+2(1+p^2)\cos\vartheta)(1-2p\cos\vartheta+p^2) \\
&\quad + (p^2-p^4)(4\cos^2\vartheta-(2p^3+6p)\cos\vartheta-2+6p^2) \\
&= 4p^4(-2+p^2)\cos^2\vartheta \\
&\quad + 4p^3(3-p^2)\cos\vartheta+1-8p^2+5p^4-2p^6 \\
&=: Q_p(\cos(\vartheta))
\end{aligned}$$

with $\vartheta \in [0, 2\pi]$.

Considering $x = \cos\vartheta$, the function Q_p has a local minimum at $x_p = \frac{3-p^2}{2p(2-p^2)}$. Since $x_p > 1$ for $p \in (0,1)$ we have

$$Q_p(\cos(\vartheta)) \geq Q_p(-1) = 1-8p^2-12p^3-3p^4+4p^5+2p^6 =: S(p).$$

From $S'(p) < 0$ for $p \in (0,1)$ and $S\left(1-\frac{\sqrt{2}}{2}\right) = 0$ we obtain

$$e^{i\vartheta}K_p(e^{i\vartheta}) \geq 0, \quad \vartheta \in [0,2\pi] \quad \text{and} \quad p \in \left(0, 1-\frac{\sqrt{2}}{2}\right]. \tag{3.53}$$

Therefore

$$\begin{aligned}
|\Phi_p(\omega)| &= \left|\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \kappa_p(z)\omega(z)dz\right| \\
&= \left|\frac{1}{2\pi i}\int_{\partial\mathbb{D}} K_p(z)\omega(z)dz\right| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} |e^{i\vartheta}K_p(e^{i\vartheta})|d\vartheta\|\omega\|_\infty \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i\vartheta}K_p(e^{i\vartheta})d\vartheta\|\omega\|_\infty \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} e^{i\vartheta}K_p(e^{i\vartheta})d\vartheta \\
&= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} K_p(z)dz \\
&= \frac{1}{1-p^4}
\end{aligned} \tag{3.54}$$

leads to (3.51), which means (3.48).

In (3.54) we obtain equality at each step, if $\omega \equiv 1$ (also see Remark 3.4). Furthermore with (3.53) the condition extremal problems in H^∞ for Φ_p is satisfied.

We therefore have a unique normalized function ω_e, such that

$$\max\{|\Phi_p(\omega)| \mid \omega \in H^\infty,\ \|\omega\|_\infty \leq 1\} = \Phi_p(\omega_e),$$

which has to be $\omega_e \equiv 1$ from the above consideration. Equality in (3.51) therefore occurs for $\omega(z) = e^{i\vartheta}$ with a certain $\vartheta \in [0, 2\pi)$, which means, that f has to be of the form f_ϑ. This proves the statement.

□

3.3 Alternative Proof for the Residue

Using the integral representation formula of the previous section for the class $\mathcal{C}o_p$, we can obtain Theorem 2.3 for the residue of concave functions.

To recall, the statement of the theorem was as follows.

Theorem. *Let $f(z) \in \mathcal{C}o_p$ be a concave function with a simple pole at some point $p \in (0,1)$. Then the residue of this function f can be described by some function*

$$\varphi : \mathbb{D} \to \mathbb{D} \textit{ with } \varphi(p) = p$$

holomorphic in $\mathbb{D}$, such that

$$\operatorname{Res}_p f = -\frac{p^2}{(1-p^2)^2} \exp \int_0^p \frac{-2\varphi(x)}{1 - x\varphi(x)} dx. \tag{3.55}$$

Proof of Theorem 2.3. Since a function $f \in \mathcal{C}o_p$ is represented by

$$f(z) = \frac{b_{-1}}{z-p} + b_0 + \sum_{n=1}^{\infty} b_n (z-p)^n$$

for $|z-p| < 1-p$, we obtain

$$f'(z) = -\frac{b_{-1}}{(z-p)^2} + b_1 + \sum_{n=2}^{\infty} n\, b_n (z-p)^{n-1}.$$

Applying (2.20) from Theorem 2.13, the following equality is valid.

$$\frac{-b_{-1}}{(z-p)^2} + b_1 + \sum_{n=2}^{\infty} n\, b_n (z-p)^{n-1} = \frac{p^2}{(z-p)^2(1-zp)^2} \exp \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta.$$

Multiplying both sides with $-(z-p)^2$ we have

$$b_{-1} - b_1(z-p)^2 - \sum_{n=2}^{\infty} n\, b_n (z-p)^{n+1} = \frac{-p^2}{(1-zp)^2} \exp \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta.$$

Considering $z = p$ leads to the theorem. □

Similarly to Corollary 2.14 we can describe the residue in ways of a holomorphic function Ψ, which has a fixed point at the origin.

We omit the detailed proof, since the result can be obtained by direct calculation using Corollary 2.14 instead of Theorem 2.13 in the proof above.

Corollary 3.14. *Let $f(z) \in \mathcal{C}o_p$ be a concave function with a simple pole at some point $p \in (0,1)$. Then the residue of this function f can be described by some function*

$$\Psi : \mathbb{D} \to \mathbb{D} \text{ with } \Psi(0) = 0$$

holomorphic in $\mathbb{D}$, such that

$$\operatorname{Res}_p f = -\frac{p^2}{(1-p^2)^2} \exp \int_0^p \left(\frac{1}{1 - x\Psi(x)} - \frac{1}{1-px} \right) \frac{2}{x} dx. \tag{3.56}$$

As we already discussed in Theorem 3.5, Wirths determined this range of the residue in [33], using the inequality

$$\left| \frac{1}{f(z)} - \frac{1}{z} + \frac{1+p^2}{p} \right| \leq 1$$

for $f \in \mathcal{C}o_p$ provided by Miller in [19].

4 Extension of Necessary and Sufficient Conditions for Concave Functions

In the previous chapter we discussed the basic properties of concave functions and introduced necessary and sufficient conditions. These conditions can be extended to have a more complicated, yet useful form. As introduced before, we know that a function f_0 belongs to $\mathcal{C}o_0$ if and only if

$$\mathrm{Re}\left(1+\frac{zf_0''(z)}{f_0'(z)}\right)<0$$

for all $z\in\mathbb{D}$. For the class $\mathcal{C}o_q$ with $q\in\mathbb{D}$ the inequality

$$\mathrm{Re}\left(1+\frac{zf_q''(z)}{f_q'(z)}+\frac{z+q}{z-q}-\frac{1+\bar{q}z}{1-\bar{q}z}\right)<0 \tag{4.1}$$

is a necessary and sufficient condition provided by Pfaltzgraff and Pinchuk in [26].

For simplicity, in this chapter we will only consider real q, meaning $q\in(-1,1)$.

Again, a concave functions of class $\mathcal{C}o_0$ can be expanded as

$$f_0(z)=\frac{c_{-1}(f)}{z}+c_0(f_0)+c_1(f_0)\,z+\cdots,\quad |z|<1, \tag{4.2}$$

and we have

$$f_q(z)=\frac{\mathrm{Res}_q\, f_q}{z-q}+c_0(f_q)+c_1(f_q)\,(z-q)+\cdots,|z-q|<1-|q|, \tag{4.3}$$

as a typical expression for functions $f_q\in\mathcal{C}o_q$, $q\in(-1,1)$, since the normalization usually considers the Maclaurin expansion for this class (see e.g. [1, 4]). Here $\mathrm{Res}_q\, f_q=c_{-1}(f_q)$ is the residue of f_q at the point $z=q$.

In the present chapter we shall prove the following:

Theorem 4.1. *Let $p,q\in(-1,1)$. A meromorphic function f_q with simple pole at q belongs to the class $\mathcal{C}o_q$ if and only if for all $z\in\mathbb{D}$*

$$\begin{aligned}\mathrm{Re}\Bigg(1-q^2&+\frac{2p(1-q^2)}{1+p^2}\cdot\frac{1-qz}{z-q}\\&-\left(\frac{z-q}{1-qz}+p\right)\left(1+p\frac{z-q}{1-qz}\right)\left(\frac{2q}{1+p^2}+\frac{1-qz}{1+p^2}\frac{f_q''(z)}{f_q'(z)}\right)\Bigg)<0.\end{aligned} \tag{4.4}$$

For the case $q = 0$ we actually have

Corollary 4.2. *Let $p \in (-1, 1)$. A meromorphic function f_0 with a simple pole at the origin belongs to the class $\mathcal{C}o_0$ if and only if for all $z \in \mathbb{D}$*

$$\operatorname{Re}\left(1 + \frac{2p}{1+p^2} \cdot \frac{1}{z} + \frac{1}{1+p^2}(z+p)(1+pz)\frac{f_0''(z)}{f_0'(z)}\right) < 0. \tag{4.5}$$

Remark 4.3. For $q = p$ in (4.4) we obtain the original inequality (4.1) after normalization. If we put $p = 0$ in (4.4), we have

$$\operatorname{Re}\left(1 + q^2 - 2qz + \frac{(z-q)(1-qz)f_q''(z)}{f_q'(z)}\right) < 0.$$

This is the same result as Livingston obtained in [16].

Additionally, functions in the class $\mathcal{C}o_q$ and $\mathcal{C}o_0$ have the following integral representations.

Theorem 4.4. *A meromorphic function f_q with a simple pole at the point $q \in (-1, 1)$ belongs to the class $\mathcal{C}o_q$ if and only if there exists a holomorphic function $\varphi : \mathbb{D} \to \mathbb{D}$ with $\varphi(p) = p$ such that f_q can be expressed as*

$$f_q'(z) = -\frac{(1 - qz + p(z-q))^2}{(z-q)^2(1-qz)^2}\operatorname{Res}_q f_q \exp \int_p^{T(z)} \frac{-2\varphi(\zeta)}{1 - \zeta\varphi(\zeta)} d\zeta \tag{4.6}$$

for $z \in \mathbb{D}$, where T is an automorphism of the unit disk, mapping $T(q) = p$ with $T'(q) = 1 - p^2 > 0$. In particular

$$T(z) = \frac{(1-pq)z + p - q}{1 - pq + (p-q)z}.$$

Furthermore in case f_0, meaning for $q = 0$ and $\operatorname{Res}_0 f_0 = 1$, we obtain the next representation.

Corollary 4.5. *A function f_0 belongs to the class $\mathcal{C}o_0$ if and only if there exists a holomorphic function $\varphi : \mathbb{D} \to \mathbb{D}$ with $\varphi(p) = p$ such that f_0 can be expressed as*

$$f_0'(z) = -\left(\frac{1}{z} + p\right)^2 \exp \int_p^{\frac{z+p}{1+pz}} \frac{-2\varphi(\zeta)}{1 - \zeta\varphi(\zeta)} d\zeta \tag{4.7}$$

for $z \in \mathbb{D}$ and $p \in (-1, 1)$.

The second section will provide the proofs for the Theorems and in the last section, we take a look at an application of Theorem 4.1. This chapter presents the contents of [21].

4.1 Proofs of the Extended Formluas

We shall begin with Corollary 4.2 and use the steps of the proof for the proof of Theorem 4.1.

Proof of Corollary 4.2. Let $f_0 \in \mathcal{C}o_0$ and $p \in (0,1)$. Then there exists a function $f_p \in \mathcal{C}o_p$ such that $C_0 \cdot f_0(\mathbb{D}) = f_p(\mathbb{D})$ with a constant $C_0 \in \mathbb{C}\backslash\{0\}$ and f_p can be written as

$$f_p(z) = C_0 \cdot f_0\left(\frac{z-p}{1-pz}\right). \tag{4.8}$$

For any function of $\mathcal{C}o_p$ we also know that (4.1) is valid.

Setting

$$Q_1(z) = 1 + \frac{zf_p''(z)}{f_p'(z)} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz}$$

we obtain

$$Q_1(z) = 1 + \frac{2p}{z-p} + \frac{(1-p^2)zf_0''\left(\frac{z-p}{1-pz}\right)}{(1-pz)^2 f_0'\left(\frac{z-p}{1-pz}\right)}$$

in relation to f_0.

Since $\operatorname{Re} Q_1(z) < 0$ for all $z \in \mathbb{D}$ is only valid if and only if $\operatorname{Re} Q_1\left(\frac{z+p}{1+pz}\right) < 0$ for all $z \in \mathbb{D}$, we obtain by a short calculation that

$$Q_1\left(\frac{z+p}{1+pz}\right) = \frac{1+p^2}{1-p^2} + \frac{2p}{(1-p^2)z} + \frac{(z+p)(1+pz)f_0''(z)}{(1-p^2)f_0'(z)}. \tag{4.9}$$

Normalizing (4.9) for $z=0$ by multiplication with $\frac{1-p^2}{1+p^2}$ leads to

$$\frac{1-p^2}{1+p^2} \cdot Q_1\left(\frac{z+p}{1+pz}\right) = 1 + \frac{2p}{1+p^2} \cdot \frac{1}{z} + \frac{1}{1+p^2}(z+p)(1+pz)\frac{f_0''(z)}{f_0'(z)}, \tag{4.10}$$

which has also negative real part for all $z \in \mathbb{D}$ since $\frac{1-p^2}{1+p^2} > 0$.

This proves Corollary 4.2. □

For Theorem 4.1 we continue the above proof at (4.10).

Proof of Theorem 4.1. Let $p, q \in (0,1)$ and $f_q \in \mathcal{C}o_q$. We set

$$Q_2(z) = 1 + \frac{2p}{1+p^2} \cdot \frac{1}{z} + \frac{1}{1+p^2}(z+p)(1+pz)\frac{f''(z)}{f'(z)}$$

where

$$C_0\, f(z) = C_q\, f_q\left(\frac{z+q}{1+qz}\right)$$

with constants $C_0, C_q \in \mathbb{C}\backslash\{0\}$. Therefore

$$\begin{aligned} Q_2(z) = 1 + \frac{2p}{1+p^2} \cdot \frac{1}{z} - \frac{2q}{1+q^2}\,\frac{(z+p)(1+pz)}{1+qz} \\ + \frac{1-q^2}{1+p^2}\,\frac{(z+p)(1+pz)}{(1+qz)^2}\,\frac{f_q''\left(\frac{z+q}{1+qz}\right)}{f_q'\left(\frac{z+q}{1+qz}\right)} \end{aligned}$$

Again, we have $\operatorname{Re} Q_2(z) < 0$ for all $z \in \mathbb{D}$ if and only if $\operatorname{Re} Q_2\left(\frac{z-q}{1-qz}\right) < 0$ for all $z \in \mathbb{D}$. Therefore we know that

$$Q_2\left(\frac{z-q}{1-qz}\right) = 1+\frac{2p}{1+p^2}\cdot\frac{1-qz}{z-q} - \left(\frac{z-q}{1-qz}+p\right)\left(1+p\frac{z-q}{1-qz}\right) \\ \times\left(\frac{2q}{(1+p^2)(1-q^2)} - \frac{1-qz}{(1+p^2)(1-q^2)}\frac{f_q''(z)}{f_q'(z)}\right)$$

has negative real part for all $z \in \mathbb{D}$. Multiplying with $1-q^2 > 0$ results in (4.4).

This completes the proof for Theorem 4.1. □

As in the previous proof, we shall start with the proof for functions $f_0 \in \mathcal{C}o_0$.

Proof of Corollary 4.5. From [20] we know for functions $f_p \in \mathcal{C}o_p$, that the integral representation is given as

$$f_p'(z) = \frac{p^2}{(z-p)^2(1-zp)^2}\exp\int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta$$

with a holomorphic function $\varphi : \mathbb{D} \to \mathbb{D}$, $\varphi(p) = p$.

Using (4.8), we obtain

$$\frac{1-p^2}{(1-pz)^2}\,C_0\cdot f_0'\left(\frac{z-p}{1-pz}\right) = \frac{p^2}{(z-p)^2(1-zp)^2}\exp\int_0^z \frac{-2\,\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta.$$

Applying the transformation $z \mapsto \frac{z+p}{1+zp}$ yields

$$f_0'(z) = \frac{p^2\,(1+pz)^2}{C_0\,(1-p^2)^3\,z^2}\exp\int_0^{\frac{z+p}{1+pz}} \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta. \tag{4.11}$$

From [20] we also know, that this residue of functions in $\mathcal{C}o_p$ can be represented as

$$\operatorname{Res}_p f_p = -\frac{p^2}{(1-p^2)^2}\exp\int_0^p \frac{-2\varphi(x)}{1-x\varphi(x)}dx \tag{4.12}$$

with some holomorphic function $\varphi : \mathbb{D} \to \mathbb{D}$, $\varphi(p) = p$.

Combining (4.11) with (4.12) we have

$$\begin{aligned} -z^2\,f_0'(z)\big|_{z=0} &= -\frac{p^2\,(1+pz)^2}{C_0\,(1-p^2)^3}\exp\int_0^{\frac{z+p}{1+pz}} \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta\bigg|_{z=0} \\ &= \frac{\operatorname{Res}_p f_p}{(1-p^2)\,C_0}. \end{aligned}$$

Using the expansion (4.2) for functions in $\mathcal{C}o_0$, this leads to

$$C_0 = \frac{\operatorname{Res}_p f_p}{1-p^2} = \frac{p^2}{(1-p)^3}\exp\int_0^p \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta.$$

Inserting this result in (4.11) leads to the statement for functions of class $\mathcal{C}o_0$. □

For functions in $\mathcal{C}o_q$ we can continue in (4.11).

Proof of Theorem 4.4. By replacing $C_0 \cdot f_0(z) = C_q f_q\left(\frac{z+q}{1+qz}\right)$ we obtain

$$f_q'\left(\frac{z+q}{1+qz}\right) = \frac{p^2(1+pz)^2(1+qz)^2}{C_q(1-p^2)^3(1-q^2)z^2} \exp \int_0^{\frac{z+p}{1+pz}} \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta.$$

Applying the transformation $z \mapsto \frac{z-q}{1-qz}$ leads to

$$f_q'(z) = \frac{p^2(1-q^2)}{C_q(1-p^2)^3} \frac{(1-qz+p(z-q))^2}{(z-q)^2(1-qz)^2} \exp \int_0^{\frac{(1-pq)z+p-q}{1-pq+(p-q)z}} \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta. \tag{4.13}$$

Again with the use of (4.12) we have

$$\begin{aligned} -(z-q)^2 f_q'(z)\big|_{z=q} &= \frac{-p^2(1-q^2)\,(1-qz+p(z-q))^2}{C_q(1-p^2)^3(1-qz)^2} \exp \int_0^{\frac{(1-pq)z+p-q}{1-pq+(p-q)z}} \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta \Bigg|_{z=q} \\ &= -\frac{(1-q^2)}{C_q} \cdot \frac{p^2}{(1-p^2)^3} \exp \int_0^p \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta \\ &= \frac{1-q^2}{1-p^2} \cdot \frac{\operatorname{Res}_p f_p}{C_q}. \end{aligned}$$

Therefore by (4.3) we have

$$\begin{aligned} C_q &= \frac{1-q^2}{1-p^2} \cdot \frac{\operatorname{Res}_p f_p}{\operatorname{Res}_q f_q} \\ &= \frac{-p^2(1-q^2)}{(1-p^2)^3 \cdot \operatorname{Res}_q f_q} \exp \int_0^p \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta. \end{aligned}$$

Using this fact with (4.13) leads to the representation of Theorem 4.4.

This completes the proof.

□

4.2 Application of the Extended Condition

As discussed in Theorem 3.13, Bhowmik, Ponnusamy and Wirths gave the range of the coefficient for $c_1(f)$ in [6]. For non-normalized concave functions, this is equivalent to the following theorem.

Theorem (see [6])**.** *Let $q \in (0, 1-\frac{\sqrt{2}}{2})$ and $f_q \in \mathcal{C}o_q$. Then the variability region of $c_1(f_q)$ is given by*

$$\left| \frac{c_1(f_q)}{a_1(f_q)} + \frac{q^4}{(1+q^2)(1-q^2)^3} \right| \leq \frac{q^2}{(1+q^2)(1-q^2)^3}$$

where equality holds if and only if f_q is some specific function.

Here the value $a_1(f_q)$ is the first coefficient of the non-normalized Maclaurin expansion of f_q.

As an application of the Theorems, we will now take a closer look at $\{c_{-1}(f_q), c_1(f_q)\}$ and $\{c_{-1}(f_q), c_2(f_q)\}$.

First we set for $p, q \in (-1,1)$ and $z \in \mathbb{D}$

$$\begin{aligned} P(z) = &-(1-q^2) - \frac{2p(1-q^2)}{1+p^2} \cdot \frac{1-qz}{z-q} \\ &+ \left(\frac{z-q}{1-qz} + p\right)\left(1 + p\frac{z-q}{1-qz}\right)\left(\frac{2q}{1+p^2} + \frac{1-qz}{1+p^2}\frac{f_q''(z)}{f_q'(z)}\right). \end{aligned}$$

Let P have the expansion of the form

$$P(z) = d_0 + d_1(z-q) + d_2(z-q)^2 + \cdots .$$

We calculate

$$\begin{aligned} P(q) &= 1 - q^2 = d_0, \\ P'(q) &= \frac{2p}{1+p^2}\left(1 + (1-q^2)^2\frac{c_1(f_q)}{c_{-1}(f_q)}\right) = d_1 \end{aligned}$$

and

$$\frac{P''(q)}{2} = \frac{2}{(1+p^2)(1-q^2)}\left(-pq - (1-2pq+p^2)(1-q^2)\frac{c_1(f_q)}{c_{-1}(f_q)} + 3p(1-q^2)^3\frac{c_2(f_q)}{c_{-1}(f_q)}\right) = d_2.$$

Then the function $\tilde{P}(z)$ defined by

$$\begin{aligned} P\left(\frac{z+q}{1+qz}\right) &= (1-q^2)\big(1 + d_1 z + \left((1-q^2)d_2 - qd_1\right) z^2 + \cdots\big) \\ &= (1-q^2)\tilde{P}(z) \end{aligned}$$

has positive real part for all $z \in \mathbb{D}$ with $\tilde{P}(0) = 1$ and we can write

$$\tilde{P}(z) = 1 + a_1 z + a_2 z^2 + \cdots .$$

Since $\tilde{P}$ belongs to the Carathéodory class of functions, $|a_n| \leq 2$ for all $n \in \mathbb{N}$ and

$$|a_2 + \lambda a_1| \leq 2(1 + |\lambda|) \tag{4.14}$$

for $\lambda \in \mathbb{C}$. Furthermore, we have $a_1 = d_1$ and $a_2 = (1-q^2)d_2 - qd_1$ by equating the coefficients.

This immediately leads us to

$$\left|1 + (1-q^2)^2\frac{c_1(f_q)}{c_{-1}(f_q)}\right| \leq \frac{1+p^2}{|p|} \tag{4.15}$$

for $\{c_{-1}(f_q), c_1(f_q)\}$.

Since (4.15) is valid for all $p \in (-1, 1)$, we can minimize the right hand side by taking $p \to 1$. This yields

$$\left|1 + (1-q^2)^2 \frac{c_1(f_q)}{c_{-1}(f_q)}\right| \le 2,$$

what is similar to a known result from [22, Theorem 1.1]. We will discuss detailes of this fact in the next chapter.

In case $q = 0$ and $c_{-1}(f_0) = 1$ we have

$$|1 + c_1(f_0)| \le 2,$$

which is the same result as we would have obtained, if we used the term of Corollary 4.2 for the definition of $P(z)$ instead of the term from Theorem 4.1.

For $\{c_{-1}(f_q), c_2(f_q)\}$ we calculate

$$d_2 + \frac{1+p^2-2pq}{(1-q^2)p} d_1 = \frac{2}{(1+p^2)(1-q^2)} \left(1 + p^2 - 3pq + 3p(1-q^2)^3 \frac{c_2(f_q)}{c_{-1}(f_q)}\right).$$

In terms of a_1 and a_2 we obtain

$$a_2 + \frac{1-pq+p^2}{p} a_1 = \frac{2}{(1+p^2)} \left(1 + p^2 - 3pq + 3p(1-q^2)^3 \frac{c_2(f_q)}{c_{-1}(f_q)}\right).$$

Therefore using (4.14) we obtain for all $p \in (-1, 1)$

$$\left|\frac{1+p^2}{3p} - q + (1-q^2)^3 \frac{c_2(f_q)}{c_{-1}(f_q)}\right| \le \frac{1+p^2}{3p^2}(1 + |p| - pq + p^2),$$

which in case $q = 0$ becomes

$$\left|\frac{1+p^2}{3p} + c_2(f_0)\right| \le \frac{1+p^2}{3p^2}(1 + |p| + p^2).$$

5 On a Coefficient Body of Concave Functions

As introduced in the first chapter and discussed in the previous, a meromorphic function f with a simple pole at p in $\mathbb{D}$, belongs to $\mathcal{C}o_p$ if and only if

$$\mathrm{Re}\left(1+p^2-2pz+\frac{(z-p)(1-pz)f''(z)}{f'(z)}\right)<0, \quad z\in\mathbb{D}. \tag{5.1}$$

.

A function $f\in\mathcal{C}o_p$ in general can also be expanded as

$$f(z)=a_0(f)+a_1(f)z+a_2(f)z^2+\cdots, \quad |z|<p$$

and

$$f(z)=\frac{c_{-1}(f)}{z-p}+c_0(f)+c_1(f)(z-p)+\cdots, \quad |z-p|<1-p.$$

The univalence of f forces $a_1(f)\neq 0$ and $c_{-1}(f)\neq 0$. Usually normalizations like $a_0=a_1(f)-1=0$ or $c_0(f)=c_{-1}(f)-1=0$ are assumed in the definitions of the class $\mathcal{C}o_p$. However we omit them for the following discussion.

The contents of this chapter can also be found in [22].

As already stated in 3.3, the variability region of $c_{-1}(f)$, when f ranges over $\mathcal{C}o_p$, was determined by Wirths. For the non-normalized case, we therefore have

Theorem A (Wirths [33])**.** *For $0<p<1$*

$$\left\{\frac{c_{-1}(f)}{a_1(f)}:f\in\mathcal{C}o_p\right\}=\overline{\mathbb{D}}\left(-\frac{p^2}{1-p^4},\frac{p^4}{1-p^4}\right). \tag{5.2}$$

Furthermore

$$\frac{c_{-1}(f)}{a_1(f)}+\frac{p^2}{1-p^4}=\frac{p^4}{1-p^4}e^{i\theta}$$

holds for some real θ if and only if

$$f(z)=a_1(f)\frac{z-\frac{p^2}{1+p^2}(1+e^{i\theta})z^2}{(1-\frac{z}{p})(1-pz)}+C \tag{5.3}$$

for some constant $C\in\mathbb{C}$ in $\mathbb{D}$.

The variability region of $c_1(f)/a_1(f)$, when f ranges over $\mathcal{C}o_p$ with $0<p\leq 1-2^{-1/2}$, was given by Bhowmik, Ponnusamy and Wirths, as discussed twice before.

Theorem B (Bhowmik, Ponnusamy and Wirths [6]). *For* $0 < p < 1 - \frac{\sqrt{2}}{2}$

$$\left\{\frac{c_1(f)}{a_1(f)} : f \in \mathcal{C}o_p\right\} = \overline{\mathbb{D}}\left(-\frac{p^4}{(1+p^2)(1-p^2)^3}, \frac{p^2}{(1+p^2)(1-p^2)^3}\right).$$

Furthermore

$$\frac{c_1(f)}{a_1(f)} = -\frac{p^4}{(1+p^2)(1-p^2)^3} + \frac{p^2}{(1+p^2)(1-p^2)^3}e^{i\theta}$$

holds for some real θ *in* $\mathbb{D}$ *if and only if* (5.3) *holds for some constant* $C \in \mathbb{C}$ *in* $\mathbb{D}$.

By making use of the same argument as in the proof of Theorem A we shall determine the coefficient body $\{(a_1(f), c_{-1}(f), c_1(f)) \in \mathbb{C}^3 : f \in \mathcal{C}o_p\}$. Let $\mathcal{P}$ be the class of analytic functions g in $\mathbb{D}$ such that $\operatorname{Re} g(z) > 0$ in $\mathbb{D}$ and $g(0) = 1$. Let

$$P(z) = \frac{1+z}{1-z}, \quad z \in \mathbb{D}. \tag{5.4}$$

Then P is a conformal mapping of $\mathbb{D}$ onto the right half plane $\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$ with $P(0) = 1$ and particularly $P \in \mathcal{P}$. For $\mu \in \mathbb{D}$ define τ_μ by

$$\tau_\mu(z) = \frac{z+\mu}{1+\overline{\mu}z}, \quad z \in \mathbb{D} \tag{5.5}$$

and $\tau_\mu(z) = \mu$, when $\mu \in \partial\mathbb{D}$. Now we define for $z \in \mathbb{D}$ and $w \in \overline{\mathbb{D}}$

$$Q_1(z,w) = \int_0^z \frac{P\left(w\zeta^2\right) - 1}{\zeta}\, d\zeta = -\log(1 - wz^2), \tag{5.6}$$

$$Q_2(z,w,\mu) = \int_0^z \frac{P\left(\zeta^2\tau_\mu(w\zeta)\right) - 1}{\zeta}\, d\zeta. \tag{5.7}$$

Notice that if $\mu \in \partial\mathbb{D}$, $Q_2(z,w,\mu) = Q_1(z,\mu)$.

Theorem 5.1. *Let* $0 < p < 1$. *For any fixed* $z_0 \in \mathbb{D}\backslash\{p\}$ *and* $\mu \in \overline{\mathbb{D}}$ *the function* $w \mapsto Q_2(z_0, w, \mu)$ *is convex univalent on* $\overline{\mathbb{D}}$ *and the function* $w \mapsto \exp\left(-Q_2(z_0,w,\mu)\right)$ *is univalent on* $\overline{\mathbb{D}}$. *Further*

$$\begin{aligned}&\{(a_1(f), c_{-1}(f), c_1(f)) \in \mathbb{C}^3 : f \in \mathcal{C}o_p\}\\ &= \Bigg\{(\alpha_1, \gamma_{-1}, \gamma_1) \in \mathbb{C}^3 : \gamma_{-1} \neq 0,\ \mu = (1-p^2)^2\frac{\gamma_1}{\gamma_{-1}} \in \overline{\mathbb{D}}\\ &\qquad\qquad \text{and } \alpha_1 \in -\frac{\gamma_{-1}}{p^2}\exp\left(-Q_2(p,\mathbb{D},\mu)\right)\Bigg\}.\end{aligned}$$

(i) *In case that* $\gamma_{-1} \neq 0$ *and* $\mu = (1-p^2)^2\frac{\gamma_1}{\gamma_{-1}} \in \partial\mathbb{D}$, $c_{-1}(f) = \gamma_{-1}$ *and* $c_1(f) = \gamma_1$ *holds for some* $f \in \mathcal{C}o_p$, *if and only if*

$$f(z) = C + \frac{\gamma_{-1}}{z-p} + (1-p^2)\frac{\gamma_1(z-p)}{1-pz}$$

for some constant $C \in \mathbb{C}$. *In this case specially* $Q_2(p, \overline{\mathbb{D}}, \mu)$ *reduces to a singleton* $\{-\log(1-\mu p^2)\}$ *and*

$$a_1(f) = -\frac{\gamma_{-1}}{p^2} + (1-p^2)^2\gamma_1.$$

(ii) *In case that* $\gamma_{-1} \neq 0$ *and* $\mu = (1-p^2)^2 \frac{\gamma_1}{\gamma_{-1}} \in \mathbb{D}$, *equalities* $c_{-1}(f) = \gamma_{-1}$, $c_1(f) = \gamma_1$ *and* $a_1(f) = -p^{-2}\gamma_{-1}\exp\left(-Q_2(p, w_0, \mu)\right)$ *hold for some* $f \in \mathcal{C}o_p$ *and* $w_0 \in \partial\mathbb{D}$ *if and only if*

$$f(z) = C - \int_p^z \frac{\gamma_{-1}}{(\zeta-p)^2} \exp\left(-Q_2\left(\frac{p-\zeta}{1-p\zeta}, w_0, \mu\right)\right) d\zeta \tag{5.8}$$

for some constant $C \in \mathbb{C}$.

Remark 5.2. We understand the integral in (5.8) as formal integration, i.e.

$$\begin{aligned}\int_p^z \frac{-\gamma_1}{(\zeta-p)^2}\exp(-Q_2)d\zeta &= \int_p^z \frac{-\gamma_1}{(\zeta-p)^2} + \gamma + \cdots d\zeta \\ &= \frac{\gamma_1}{z-p} + \gamma\,(z-p) + \cdots,\end{aligned}$$

giving a single-valued meromorphic function in $\mathbb{D}$, since the integrand has no residue.

5.1 Representation Formula for $\mathcal{C}o_p$ and Lemmas

For $f \in \mathcal{C}o_p$ with $0 < p < 1$ let

$$h_f(z) = -\left(1 + p^2 - 2pz + \frac{(z-p)(1-pz)f''(z)}{f'(z)}\right), \tag{5.9}$$

$$g_f(z) = \frac{1}{1-p^2} h_f\left(\frac{p-z}{1-pz}\right). \tag{5.10}$$

Then $\operatorname{Re} h_f(z) > 0$ and $\operatorname{Re} g_f(z) > 0$ in $\mathbb{D}$. Since $f(z) = c_{-1}(f)(z-p)^{-1} + c_0(f) + c_1(f)(z-p) + \cdots$ in $\mathbb{D}(p, 1-p)$, we have

$$\begin{aligned}&\frac{(z-p)(1-pz)f''(z)}{f'(z)} \\ =&\frac{(z-p)\{1-p^2-p(z-p)\}(2c_{-1}(f)(z-p)^{-3} + 2c_2(f) + \cdots)}{-c_{-1}(f)(z-p)^{-2} + c_1(f) + \cdots} \\ =& -2\left(1 - p^2 - p(z-p) + (1-p^2)\frac{c_1(f)}{c_{-1}(f)}(z-p)^2 + \cdots\right)\end{aligned}$$

and hence

$$h_f(z) = 1 - p^2 + 2(1-p^2)\frac{c_1(f)}{c_{-1}(f)}(z-p)^2 + \cdots.$$

From this, it follows that

$$h_f(p) = 1 - p^2, \quad h_f'(p) = 0, \quad h_f''(p) = 4(1-p^2)\frac{c_1(f)}{c_{-1}(f)} \tag{5.11}$$

and that

$$\begin{aligned} g_f(0) &= \frac{1}{1-p^2} h_f(p) = 1, \quad g_f'(0) = -h_f'(p) = 0, \\ g_f''(0) &= (1-p^2)h_f''(p) = 4(1-p^2)^2 \frac{c_1(f)}{c_{-1}(f)}. \end{aligned} \tag{5.12}$$

By (5.9) we have

$$\frac{d}{dz} \log \left\{ \frac{f'(z)(z-p)^2}{-c_{-1}(f)} \right\} = \frac{f''(z)}{f'(z)} + \frac{2}{z-p} = -\frac{h_f(z) - (1-p^2)}{(z-p)(1-pz)},$$

where the branch of logarithm is taken such that $\log \left\{ \frac{f'(z)(z-p)^2}{-c_{-1}(f)} \right\}\Big|_{z=p} = 0$. By integration

$$\begin{aligned} \log \left\{ \frac{f'(z)(z-p)^2}{-c_{-1}(f)} \right\} &= -\int_p^z \frac{h_f(\zeta) - (1-p^2)}{(\zeta-p)(1-p\zeta)} \, d\zeta \\ &= -\int_0^{\frac{p-z}{1-pz}} \frac{g_f(\zeta) - 1}{\zeta} \, d\zeta = \frac{g_f''(0)}{4(1-p^2)^2}(z-p)^2 + \cdots \end{aligned}$$

Thus

$$f(z) = c_0(f) - \int_p^z \frac{c_{-1}(f)}{(\zeta - p)^2} \exp\left(-\int_0^{\frac{p-\zeta}{1-p\zeta}} \frac{g_f(t) - 1}{t} \, dt \right) d\zeta. \tag{5.13}$$

From (5.1) it is easy to see that the following holds.

Proposition 5.3. *For $f \in \mathcal{C}o_p$ let h_f and g_f be the functions defined by (5.9) and (5.10), respectively. Then both h_f and g_f have positive real parts in $\mathbb{D}$ and satisfy (5.11) and (5.12). Particularly $g_f \in \mathcal{P}$ and (5.13) holds. Conversely for any $g \in \mathcal{P}$ with $g'(0) = 0$ and $\gamma_0, \gamma_{-1} \in \mathbb{C}$ with $\gamma_{-1} \neq 0$, the function f defined by*

$$f(z) = \gamma_0 - \int_p^z \frac{\gamma_{-1}}{(\zeta - p)^2} \exp\left(-\int_0^{\frac{p-\zeta}{1-p\zeta}} \frac{g(t) - 1}{t} \, dt \right) d\zeta. \tag{5.14}$$

belongs to $\mathcal{C}o_p$, and satisfies $c_{-1}(f) = \gamma_{-1}$, $c_0(f) = \gamma_0$ and $c_1(f) = \dfrac{\gamma_{-1} g''(0)}{4(1-p^2)^2}$.

Proposition 5.4. *For any fixed $z_0 \in \mathbb{D} \backslash \{0\}$, $Q_1(z_0, w)$ is convex univalent on $\overline{\mathbb{D}}$ and $\exp(-Q_1(z_0, w))$ is univalent on $\overline{\mathbb{D}}$.*

$$\left\{ \int_0^{z_0} \frac{g(\zeta) - 1}{\zeta} \, d\zeta : g \in \mathcal{P} \text{ with } g'(0) = 0 \right\} = Q_1(z_0, \overline{\mathbb{D}}) \tag{5.15}$$

Furthermore

$$\int_0^{z_0} \frac{g(\zeta)-1}{\zeta}\, d\zeta = Q_1(z_0, w_0)(= -\log(1-w_0 z_0^2))$$

holds for some $g \in \mathcal{P}$ *with* $g'(0) = 0$ *and* $w_0 \in \partial\mathbb{D}$ *if and only if* $g(z) = P(w_0 z^2)$ *in* $\mathbb{D}$.

Proposition 5.5. *For any fixed* $z_0 \in \mathbb{D}\backslash\{0\}$ *and* $\mu \in \mathbb{D}$, $Q_2(z_0, w, \mu)$ *is a convex univalent function of* $w \in \overline{\mathbb{D}}$ *and* $\exp(-Q_2(z_0, w, \mu))$ *is a univalent function of* $w \in \overline{\mathbb{D}}$, *and*

$$\begin{aligned}&\left\{\int_0^{z_0} \frac{g(\zeta)-1}{\zeta}\, d\zeta : g \in \mathcal{P} \text{ with } g'(0) = 0 \text{ and } g''(0) = 4\mu\right\} \\ =&Q_2(z_0, \overline{\mathbb{D}}, \mu).\end{aligned} \tag{5.16}$$

Furthermore

$$\int_0^{z_0} \frac{g(\zeta)-1}{\zeta}\, d\zeta = Q_2(z_0, w_0, \mu)$$

holds for some $g \in \mathcal{P}$ *with* $g'(0) = 0$, $g''(0) = 4\mu$ *and* $w_0 \in \partial\mathbb{D}$ *if and only if* $g(z) = P(z^2 \tau_\mu(w_0 z))$ *in* $\mathbb{D}$.

Remark 5.6. Notice that if $\mu \in \partial\mathbb{D}$, then $g(z) = P(z^2\tau_\mu(wz)) = P(\mu z^2)$ is the unique function satisfying $g \in \mathcal{P}$ with $g'(0) = 0$ and $g''(0) = 4\mu$.

We shall only prove Proposition 5.5. Because our proofs of Propositions 5.4 and 5.5 are quite similar and the former is much easier than the latter. First we provide some lemmas.

Lemma 5.7. *Let* G *be an analytic functions in* $\mathbb{D}$ *with* $G(z) = z^n + \cdots$ *for some positive integer* n *satisfying*

$$\mathrm{Re}\left(1 + z\frac{G''(z)}{G'(z)}\right) > 0$$

in $\mathbb{D}$. *Then there exists a starlike univalent analytic function* G_0 *in* $\mathbb{D}$ *satisfying* $G = G_0^n$.

For a proof see e.g. [34].

Proof of Proposition 5.5. Let $g \in \mathcal{P}$ with $g'(0) = 0$ and $g''(0) = 4\mu$. Then $\omega = P^{-1} \circ g$ satisfies $\omega(\mathbb{D}) \subset \overline{\mathbb{D}}$ with $\omega(z) = \mu z^2 + \cdots$ and P from (5.4). With τ_μ from (5.5), we can apply the Schwarz lemma and obtain for any fixed $\zeta \in \mathbb{D}$

$$\left|\tau_\mu^{-1}\left(\frac{\omega(\zeta)}{\zeta^2}\right)\right| = \left|\frac{\frac{\omega(\zeta)}{\zeta^2} - \mu}{1 - \overline{\mu}\frac{\omega(\zeta)}{\zeta^2}}\right| \leq |\zeta|.$$

Thus successively we have

$$\begin{aligned}&\tau_\mu^{-1}\left(\frac{\omega(\zeta)}{\zeta^2}\right) \in \overline{\mathbb{D}}(0, |\zeta|), \\ &\omega(\zeta) \in \zeta^2\tau_\mu(\overline{\mathbb{D}}(0, |\zeta|))\end{aligned}$$

and

$$g(\zeta) = P(\omega(\zeta)) \in P\left(\zeta^2 \tau_\mu(\overline{\mathbb{D}}(0, |\zeta|))\right).$$

Since P is a convex, univalent function in $\mathbb{D}$, P maps the closed disk $\zeta^2 \tau_\mu(\overline{\mathbb{D}}(0, |\zeta|))$ conformally onto the convex closed domain $P(\zeta^2 \tau_\mu(\overline{\mathbb{D}}(0, |\zeta|)))$ bounded by the curve $\partial\mathbb{D} \ni w \mapsto P(\zeta^2 \tau_\mu(w\zeta))$. Since $g(\zeta)$ belongs to the half plane left of the tangential line at $P(\zeta^2 \tau_\mu(w\zeta))$ of the boundary curve, the inequality

$$\mathrm{Re}\left(\frac{P(\zeta^2 \tau_\mu(w\zeta)) - g(\zeta)}{w\zeta^3 P'(\zeta^2 \tau_\mu(w\zeta)) \tau'_\mu(w\zeta)}\right) \geq 0$$

holds for all $\zeta \in \mathbb{D}\backslash\{0\}$ and $w \in \partial\mathbb{D}$. Furthermore equality holds at some ζ and $w \in \partial\mathbb{D}$ if and only if $g(z) = P(z^2 \tau_\mu(wz))$ in $\mathbb{D}$. Put

$$G(z) = \int_0^z w\zeta^2 P'(\zeta^2 \tau_\mu(w\zeta)) \tau'_\mu(w\zeta)\, d\zeta, \;\; z \in \mathbb{D}.$$

Then

$$\begin{aligned} G'(z) &= wz^2 P'(z^2 \tau_\mu(wz)) \tau'_\mu(wz) \\ &= \frac{2wz^2}{\left(1 - z^2 \frac{wz+\mu}{1+\overline{\mu}wz}\right)^2} \frac{1 - |\mu|^2}{(1 + \overline{\mu}wz)^2} \end{aligned}$$

and

$$\mathrm{Re}\left(\frac{P(z^2 \tau_\mu(wz)) - g(z)}{zG'(z)}\right) \geq 0, \quad z \in \mathbb{D}\backslash\{0\}. \tag{5.17}$$

Notice that

$$z \mapsto 1 - z^2 \frac{wz + \mu}{1 + \overline{\mu}wz}$$

from the denominator of $G'(z)$ is a rational function of z and has three zeros in $\partial\mathbb{D}$ counted with multiplicity. Let $\zeta_j \in \partial\mathbb{D}$, $j = 1$, 2, 3 be the zeros. Thus G' can be written of the form

$$G'(z) = \frac{2\,(1 - |\mu|^2) wz^2}{\prod_{j=1}^3 (1 - \zeta_j^{-1} z)^2}.$$

Hence we have

$$\mathrm{Re}\left(1 + z\frac{G''(z)}{G'(z)}\right) = \sum_{j=1}^3 \mathrm{Re}\left(\frac{1 + \zeta_j^{-1} z}{1 - \zeta_j^{-1} z}\right) > 0$$

in $\mathbb{D}$. Applying Lemma 5.7 there exists a starlike univalent function G_0 in $\mathbb{D}$ with $G = G_0^3$.

For any $z_1 \in \mathbb{D}\backslash\{0\}$ let γ be a path in $\mathbb{D}$ defined by

$$z(t) = G_0^{-1}(t^{1/3} G_0(z_1)), \quad 0 \leq t \leq 1.$$

Then since $G(z(t)) = G_0(z(t))^3 = tG_0(z_1)^3 = tG(z_1)$,

$$G'(z(t))z'(t) = G(z_1), \quad 0 < t \le 1.$$

Combining this and (5.17) we have

$$\begin{aligned}
&\mathrm{Re}\left[\frac{1}{G(z_1)}\left(Q_2(z_1,w,\mu) - \int_0^{z_1}\frac{g(\zeta)-1}{\zeta}\,d\zeta\right)\right] \\
&= \mathrm{Re}\left[\frac{1}{G(z_1)}\left(\int_0^{z_1}\frac{P(\zeta^2\tau_\mu(w\zeta))-1}{\zeta}\,d\zeta - \int_0^{z_1}\frac{g(\zeta)-1}{\zeta}\,d\zeta\right)\right] \\
&= \mathrm{Re}\left(\int_\gamma \frac{P(\zeta^2\tau_\mu(w\zeta))-g(\zeta)}{\zeta G(z_1)}\,d\zeta\right) \\
&= \mathrm{Re}\left[\int_0^1 \frac{\{P(z(t)^2\tau_\mu(wz(t)))-g(z(t))\}z'(t)}{z(t)G'(z(t))z'(t)}\,dt\right] \\
&= \int_0^1 \mathrm{Re}\left[\frac{\{P(z(t)^2\tau_\mu(wz(t)))-g(z(t))\}}{z(t)G'(z(t))}\right]dt \ge 0.
\end{aligned}$$

Let $V(z_1,\mu) = \{\int_0^{z_1}\zeta^{-1}(g(\zeta)-1)\,d\zeta : g \in \mathcal{P}\ g'(0) = 0 \text{ and } g''(0) = 4\mu\}$. Then the above inequality implies $V(z_1,\mu)$ is contained in a half plane. Precisely

$$V(z_1,\mu) \subset \mathbb{H}(z_1,w,\mu) = \left\{\chi \in \mathbb{C} : \mathrm{Re}\left(\frac{Q_2(z_1,w,\mu)-\chi}{G(z_1)}\right) \ge 0\right\}.$$

Since for any $w \in \overline{\mathbb{D}}$ the function $g_0(z) = P(z^2\tau_\mu(wz))$ belongs to $\mathcal{P}$ and satisfies $g_0'(0) = 0$ and $g_0''(0) = 4\mu$, we have $Q_2(z_1,\overline{\mathbb{D}},\mu) \subset V(z_1,\mu)$ and particularly when $w \in \partial\mathbb{D}$, $Q_2(z_1,w,\mu) \in V(z_1,\mu)\cap\partial\mathbb{H}(z_1,w,\mu))$. Therefore $Q_2(z_1,w,\mu) \in \partial V(z_1,\mu)$ for $w \in \partial\mathbb{D}$. Furthermore $\int_0^{z_1}\zeta^{-1}(g(\zeta)-1)\,d\zeta = Q_2(z_1,w,\mu)$ holds for some $g \in \mathcal{P}$ with $g'(0) = 0$ and $g''(0) = 4\mu$ and $w \in \partial\mathbb{D}$ if and only if $g(z) = Q_2(z,w,\mu)$ in $\mathbb{D}$.

Notice that $\mathcal{P}$ is a compact convex subset of the class of analytic functions in $\mathbb{D}$ with respect to the topology of locally uniform convergence and so is the closed subset $\{g \in \mathcal{P} : g'(0) = 0 \text{ and } g''(0) = 4\mu\}$. Hence $V(z_1,\mu)$ is a compact convex subset of $\mathbb{C}$. Now we show that the analytic function $w \mapsto Q_2(z_1,w,\mu)$ is a non constant open map. Indeed using (5.7) and $\tau_\mu(0) = \mu$ we obtain

$$\begin{aligned}
k(z) &= \frac{d}{dw}Q_2(z,w,\mu)\Big|_{w=0} \\
&= (1-|\mu|^2)\int_0^z P'(\mu\zeta^2)\zeta^2\,d\zeta,
\end{aligned}$$

which satisfies

$$\mathrm{Re}\left(1 + z\frac{k''(z)}{k'(z)}\right) = 1 + 2\,\mathrm{Re}\left(1 + \mu z^2\frac{P''(\mu z^2)}{P'(\mu z^2)}\right) > 0$$

in $\mathbb{D}$. By Lemma 5.7 there exists a starlike univalent function k_0 with $k(z) = k_0(z)^3$. In particular $k(z_1) = k_0(z_1)^3 \ne 0$ and $w \mapsto Q_2(z_1,w,\mu)$ is an open map. Thus $Q_2(z_1,0,\mu)$ is an interior point of $Q_2(z_1, \overline{\mathbb{D}}, \mu) \subset V(z_1, \mu)$.

Since $V(z_1, \mu)$ is a compact convex subset of $\mathbb{C}$ with nonempty interior, the boundary $\partial V(z_1, \mu)$ is a simple closed curve and $V(z_1, \mu)$ is a Jordan domain bounded by $\partial V(z_1, \mu)$. We have therefore shown that $Q_2(z_1, w, \mu) \in \partial V(z_1, \mu)$ for $w \in \partial\mathbb{D}$. Furthermore the mapping $\partial\mathbb{D} \ni w \mapsto Q_2(z_1, w, \mu)$ is simple. Indeed by uniqueness if $Q_2(z_1, w_1, \mu) = Q_2(z_1, w_2, \mu)$ for $w_1, w_2 \in \partial\mathbb{D}$, then $Q_2(z, w_1, \mu) = Q_2(z, w_2, \mu)$ holds for $w_1, w_2 \in \mathbb{D}$ and hence successively we have by (5.7)

$$\frac{P(z^2\tau_\mu(w_1 z)) - 1}{z} = \frac{P(z^2\tau_\mu(w_2 z)) - 1}{z}.$$

$$\tau_\mu(w_1 z) = \tau_\mu(w_2 z),$$

$$w_1 = w_2.$$

Thus the mapping $\partial\mathbb{D} \ni w \mapsto Q_2(z_1, w, \mu)$ gives a simple closed curve contained in $\partial V(z_1, \mu)$. This implies the mapping is a parameterization of $V(z_1, \mu)$.

Since the analytic function of $w \mapsto Q_2(z_1, w, \mu)$ maps $\partial\mathbb{D}$ univalently onto the convex Jordan curve $\partial V(z_1, \mu)$, it follows from Darboux's theorem that $Q_2(z_1, w, \mu)$ is convex univalent on $\overline{\mathbb{D}}$.

Since P is a conformal mapping of $\mathbb{D}$ onto the right half plane, it is starlike univalent with respect to 1 and for any $g \in \mathcal{P}$, $g - 1 \prec P - 1$, i.e. $g - 1$ is subordinate to $P - 1$. Thus by Suffridge's theorem (see [32]) $\int_0^z \zeta^{-1}(g(\zeta) - 1)\, d\zeta \prec \int_0^z \zeta^{-1}(P(\zeta) - 1)\, d\zeta = -2\log(1 - z)$. This implies $\int_0^z \zeta^{-1}(g(\zeta) - 1)\, d\zeta \in \{\chi \in \mathbb{C} : |\mathrm{Im}\,\chi| < \pi\}$ for all $g \in \mathcal{P}$ and $z \in \mathbb{D}$. In particular $Q_2(z_1, w, \mu) \subset \{\chi \in \mathbb{C} : |\mathrm{Im}\,\chi| < \pi\}$ and $\exp(-Q_2(z_1, w, \mu))$ is also a univalent function of $w \in \overline{\mathbb{D}}$. $\square$

5.2 Proof of the Theorems

Now we shall determine the variability region $\{\log \frac{f'(z_0)(z_0-p)^2}{-c_{-1}(f)} : f \in \mathcal{C}o_p\}$ for fixed $z_0 \in \mathbb{D}\backslash\{p\}$. Particularly by putting $z_0 = 0$ we can show that the variability region $\{\frac{a_1(f)}{c_{-1}(f)} : f \in \mathcal{C}o_p\}$ coincides with the closed disk $\overline{\mathbb{D}}(-p^{-2}, 1)$. This gives another formulation for Theorem A.

Theorem 5.8. *Let $0 < p < 1$. Then for any fixed $z_0 \in \mathbb{D}\backslash\{p\}$*

$$\left\{\frac{f'(z_0)(z_0 - p)^2}{-c_{-1}(f)} : f \in \mathcal{C}o_p\right\} = \left\{1 - w\left(\frac{z - p}{1 - pz}\right)^2 : w \in \overline{\mathbb{D}}\right\} \tag{5.18}$$

with

$$\frac{f'(z_0)(z_0 - p)^2}{-c_{-1}(f)} = 1 - w_0\left(\frac{z - p}{1 - pz}\right)^2 \tag{5.19}$$

for some $w_0 \in \partial\mathbb{D}$ if and only if

$$\frac{f(z)}{c_{-1}(f)} = C + \frac{1}{z - p} + \frac{w_0}{1 - p^2} \cdot \frac{z - p}{1 - pz} \tag{5.20}$$

in $\mathbb{D}$ for some constant C.

Proof. By Proposition 5.3 and 5.4 for any fixed $z_0 \in \mathbb{D}\backslash\{p\}$ the variability region of $\log\left\{\frac{f'(z_0)(z_0-p)^2}{-c_{-1}(f)}\right\}$ for $\mathcal{C}o_p$ can be expressed as

$$\begin{aligned}
&\left\{\log\left(\frac{f'(z_0)(z_0-p)^2}{-c_{-1}(f)}\right) : f \in \mathcal{C}o_p\right\} \qquad (5.21)\\
&= \left\{-\int_0^{\frac{p-z_0}{1-pz_0}} \frac{g(\zeta)-1}{\zeta}\, d\zeta : g \in \mathcal{P} \text{ with } g'(0)=0\right\}.\\
&= \left\{\log\left(1-w\left(\frac{p-z_0}{1-pz_0}\right)^2\right) : w \in \overline{\mathbb{D}}\right\}
\end{aligned}$$

This implies (5.18).

By the second part of Proposition 5.4, the expression (5.19) holds for some $w_0 \in \partial\mathbb{D}$ if and only if $g_f(z) = P(w_0 z^2)$ in $\mathbb{D}$. Using Proposition 5.3, this is equivalent to (5.20). □

We notice that (5.18) is equivalent to the known estimate

$$\left|\frac{f'(z_0)}{c_{-1}(f)} + \frac{1}{(z_0-p)^2}\right| \leq \frac{1}{|1-pz_0|^2}.$$

See also [26, Cor.5.2].

Corollary 5.9. *For* $0<p<1$

$$\left\{\frac{a_1(f)}{c_{-1}(f)} : f \in \mathcal{C}o_p\right\} = \overline{\mathbb{D}}\left(-\frac{1}{p^2}, 1\right) \qquad (5.22)$$

with

$$\frac{a_1(f)}{c_{-1}(f)} = -\frac{1}{p^2} + w_0$$

for some $w_0 \in \partial\mathbb{D}$ *if and only if* (5.20) *holds for some constant* C *in* $\mathbb{D}$.

We shall prove the above Corollary implies Theorem A and vice versa.

Proof. First since $w \in \overline{\mathbb{D}}\left(-p^{-2}, 1\right)$ if and only if $w^{-1} \in \overline{\mathbb{D}}\left(-p^2/(1-p^4), p^4/(1-p^4)\right)$, (5.2) is equivalent to (5.22).

Replace $c_{-1}(f)$ in (5.20) by $\tilde{a}_1$ and C by

$$\frac{C}{\tilde{a}_1} + \frac{1}{p} + \frac{pw_0}{1-p^2}.$$

Then we have

$$\begin{aligned}
f(z) &= \tilde{a}_1\left(\frac{1}{z-p} + \frac{1}{p} + \frac{w_0(z-p)}{(1-p^2)(1-pz)}\right) + C\\
&= \tilde{a}_1\left(\frac{z}{p(z-p)} + \frac{w_0 z}{1-pz}\right) + C
\end{aligned}$$

$$= \tilde{a}_1 \frac{z(1 - p^2 w_0 - p(1 - w_0)z)}{p(z - p)(1 - pz)} + C.$$

Replacing $\tilde{a}_1$ by $-p^2 a_1(f)/(1 - p^2 w_0)$ we have

$$f(z) = \frac{z(1 + \frac{p(1-w_0)}{1-p^2 w_0} z)}{(1 - z/p)(1 - pz)}.$$

Since the function $p(1 - w)/(1 - p^2 w)$ maps $\partial\mathbb{D}$ bijectively onto $\partial\mathbb{D}(p/(1 + p^2), p/(1 + p^2))$, for any $w_0 \in \partial\mathbb{D}$ there exists a real θ satisfying

$$\frac{p(1 - w_0)}{1 - p^2 w_0} = \frac{p(1 + e^{i\theta})}{1 + p^2}.$$

Thus extremal functions in (5.20) can be expressed of the form (5.3) and vice versa. □

Using the previous results, we can move to the proof of Theorem 5.1.

Proof of Theorem 5.1. Let $f \in \mathcal{C}o_p$ and $\mu = (1 - p^2)^2 \frac{c_1(f)}{c_{-1}(f)}$. Then by Proposition 5.3 we have $\mu = 4^{-1} g_f''(0)$. Since g_f satisfies $g_f'(0) = 0$, $\omega_f = P^{-1} \circ g_f$ satisfies $\omega_f(\mathbb{D}) \subset \overline{\mathbb{D}}$ and $\omega_f(0) = \omega_f'(0) = 0$. Hence by applying the Schwarz lemma to $z^{-1}\omega_f(z)$ we have $|\mu| = 4^{-1}|g_f''(0)| = 2^{-1}|\omega_f''(0)| \leq 1$.

Assume $|\mu| < 1$. Then by Propositions 5.3 and 5.5

$$\begin{aligned} \log\left(\frac{f'(z)(z - p)^2}{-c_{-1}(f)}\right) &= -\int_0^{\frac{p-z}{1-pz}} \frac{g_f(\zeta) - 1}{\zeta} \\ &\in -\left\{\int_0^{\frac{p-z}{1-pz}} \frac{g(\zeta) - 1}{\zeta} : g \in \mathcal{P},\ g'(0) = 0 \text{ and } g''(0) = 4\mu\right\} \\ &= -Q_2\left(\frac{p - z}{1 - pz}, \overline{\mathbb{D}}, \mu\right) \end{aligned}$$

for $z \in \mathbb{D}\backslash\{p\}$. Letting $z = 0$ we have

$$-\frac{a_1(f)}{c_{-1}(f)} p^2 \in \exp\left(-Q_2(p, \overline{\mathbb{D}}, \mu)\right).$$

Next assume $\mu \in \partial\mathbb{D}$. Then by the uniqueness part of the Schwarz lemma we successively have $\omega_f(z) = \mu z^2$, $g_f(z) = \frac{1+\mu z^2}{1-\mu z^2}$ and

$$\frac{f'(z)(z - p)^2}{-c_{-1}(f)} = 1 - \mu\left(\frac{p - z}{1 - pz}\right)^2.$$

From this it follow that

$$\frac{a_1(f)}{-c_{-1}(f)} p^2 = 1 - \mu p^2$$

and by integration

$$f(z) = c_0(f) + \frac{c_{-1}(f)}{z-p} + \frac{\mu c_{-1}(f)}{1-p^2}\frac{z-p}{1-pz} = c_0(f) + \frac{c_{-1}(f)}{z-p} + (1-p^2)c_1(f)\frac{z-p}{1-pz}.$$

Due to Remark 5.6 for $\mu \in \partial\mathbb{D}$ the set $Q_2(p,\overline{\mathbb{D}},\mu)$ is reduced to a singleton and $\{Q_1(p,\mu)\} = \{-\log(1-\mu p^2)\}$, $a_1(f) \in -\frac{c_{-1}(f)}{p^2}\exp(-Q_2(p,\overline{\mathbb{D}},\mu))$ holds.

Thus we have shown $\{(a_1(f), c_{-1}(f), c_1(f)) \in \mathbb{C}^3 : f \in \mathcal{C}o_p\}$ is contained in

$$\Big\{(\alpha_1,\gamma_{-1},\gamma_1) \in \mathbb{C}^3 : \gamma_{-1} \neq 0,\ \ \mu = (1-p^2)^2\frac{\gamma_1}{\gamma_{-1}} \in \overline{\mathbb{D}}$$
$$\text{and } \alpha_1 \in -\frac{\gamma_{-1}}{p^2}\exp\left(-Q_2(p,\overline{\mathbb{D}},\mu)\right)\Big\}.$$

The reverse inclusion relation follows from the fact that the function

$$f(z) = -\int_p^z \frac{\gamma_1}{(\zeta-p)^2}\exp\left(-Q_2\left(\frac{p-z}{1-pz}, w, \mu\right)\right) d\zeta$$

satisfies $f \in \mathcal{C}o_p$, $c_{-1}(f) = \gamma_{-1}$, $(1-p^2)^2\frac{c_1(f)}{c_{-1}(f)} = \mu$ and $a_1(f) = -p^{-2}\gamma_{-1}\exp(-Q_2(p,w,\mu))$.

This implies (i) and (ii) are direct consequences of the second part of Proposition 5.5. □

From [20] we know furthermore, that $f \in \mathcal{C}o_p$ and its residue can also be expressed by

$$f'(z) = \frac{a_1p^2}{(z-p)^2(1-zp)^2}\exp\left(2\int_p^{\frac{z-p}{1-pz}} \frac{p}{1-p\zeta} - \frac{\psi(\zeta)}{1-\zeta\psi(\zeta)}d\zeta\right)$$
$$c_{-1}(f) = \frac{a_1p^2}{(1-p^2)^2}\exp\left(2\int_0^p \frac{\psi(x)}{1-x\psi(x)} - \frac{p}{1-px}dx\right)$$

with $\psi : \mathbb{D} \to \mathbb{D}$ holomorphic in $\mathbb{D}$ and $\psi(0) = 0$. The functions satisfying sharpness can be constructed by $\psi_w(z) = w\,z$, $w \in \overline{\mathbb{D}}$, leading to

$$f'_w(z) = \frac{a_1p^2}{(z-p)^2(1-pz)^2}\left(\frac{w-p^2}{p^2w-1}z^2 + \frac{1-w}{p^2w-1}2pz + 1\right).$$

Considering the analytic function

$$A_w(z) = f_w(z) - \frac{c_{-1}(f_w)}{z-p}$$

we obtain

$$A'_w(z) = \frac{w}{p^2w-1}\cdot\frac{a_1p^2}{(1-pz)^2},$$
$$A_w^{(n)}(z) = \frac{w}{p^2w-1}\cdot\frac{n!a_1p^{n+1}}{(1-pz)^{n+1}}$$

and therefore

$$A_w^{(n)}(p) = \frac{w}{p^2w-1} \cdot \frac{a_1 n! p^{n+1}}{(1-p^2)^{n+1}}.$$

Since $c_n(A_w) = c_n(f_w)$ for $n \in \mathbb{N}$ and $\{\frac{w}{p^2w-1} : w \in \mathbb{D}\}$ is a disk, we have

$$\begin{aligned}
\left\{\frac{c_n(f_w)}{a_1(f_w)}\right\} &= \left\{\frac{w}{p^2w-1} \cdot \frac{p^{n+1}}{(1-p^2)^{n+1}} : w \in \overline{\mathbb{D}}\right\} \\
&= \overline{\mathbb{D}}\left(\frac{-p^{n+3}}{(1-p^2)^{n+2}(1+p^2)}, \frac{p^{n+1}}{(1-p^2)^{n+2}(1+p^2)}\right) \\
&\subset \left\{\frac{c_n(f)}{a_1(f)} : f \in \mathcal{C}o_p\right\}.
\end{aligned}$$

Thus the following statement holds.

Corollary 5.10. *For $0 < p < 1$ and $n \in \mathbb{N}$*

$$\overline{\mathbb{D}}\left(\frac{-p^{n+3}}{(1-p^2)^{n+2}(1+p^2)}, \frac{p^{n+1}}{(1-p^2)^{n+2}(1+p^2)}\right) \subset \left\{\frac{c_n(f)}{a_1(f)} : f \in \mathcal{C}o_p\right\}.$$

We also conjecture, that the opposite inclusion holds. This would describe a generalization of Theorem B without the restriction of p.

However, it was suggested by Prof. Wirths, that this might not be true in the general case. The further analysis of this problem will be one of the many future tasks in this field.

Acknowledgments

After this lengthy mathematical discussion, I would like to use this place to say a few personal words.

First of all, I am deeply grateful to Prof. Toshiyuki Sugawa for his support and valuable suggestions. The conversations were, and still are, always a joy, both mathematically and socially. My gratitude also goes to Prof. Hiroshi Yanagihara for his insights and his kindness and to Prof. Karl-Joachim Wirths for the access to his publications and alerting the problem considering the last conjecture. Furthermore I am grateful to Prof. Eric Schippers for his hospitality during my stay in Canada at the University of Manitoba, where I had the opportunity to exchange many thoughts in various fields of mathematics. I also want to thank Prof. Kazuyuki Tsurumi for the access to some rare references. For originally pointing me to the subject of concave functions as well as for his patience during my time of the Diplom-Thesis in Würzburg, I am deeply indebted to Prof. Stephan Ruscheweyh.

Additionally, I am thankful to Dr. Ikkei Hotta for his suggestions and for many interesting conversations in both Germany and Japan. I am also really grateful to all the members (past and present) of our work-group, Ms. Norie Gouke, Mr. Naoki Hiranuma, Ms. Yan Jiang, Ms. Mai Kodaka, Mr. Ryota Kurosaki, Ms. Ming Li, Mr. Hirokazu Shimauchi, Ms. Lijie Sun and Dr. Tanran Zhang (listed in alphabetical order), who made the development of this work not as dry and tedious as it otherwise might have been. Further, I want to express my gratitude to the staff of our division, Ms. Sumie Narasaka, Ms. Makiko Owase and Ms. Chisato Karino, for their support and insight. Last but not least, I wish to thank my family for their support.

This work was supported by Grant-in-Aid for JSPS Fellows No. $26 \cdot 2855$.

Bibliography

[1] F.G. Avkhadiev, Ch. Pommerenke and K.-J. Wirths, *Sharp inequalities for the coefficients of concave schlicht functions*, Comment. Math. Helv. **81** (2006), 801-807.

[2] F.G. Avkhadiev and K.-J. Wirths, *Convex holes produce lower bounds for coefficients*, Complex Variables **47** (2002), 553-563.

[3] F.G. Avkhadiev and K.-J. Wirths, *Concave schlicht functions with bounded opening angle at infinity*, Lobachevskii J. of Math. **17** (2005), 3-10.

[4] F.G. Avkhadiev and K.-J. Wirths, *A proof of the Livingston conjecture*, Forum Math. **19** (2007), 149-157.

[5] L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreis vermitteln*, Sitz. Ber. Preuss. Akad. Wiss. **38**, (1916), 940-955.

[6] B. Bhowmik, S. Ponnusamy and K.-J. Wirths, *Domains of variability of Laurent coefficients and the convex hull for the family of concave univalent functions*, Kodai Math. Journal **30** (2007), 385-393

[7] L. Cruz and Ch. Pommerenke, *On concave univalent functions*, Complex Var. and Elliptic Equ. **52** (2007), 153-159.

[8] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137-152

[9] P. Duren, *Univalent Functions*, Springer, New York, 1983.

[10] P. Duren, *Theory of H^p Spaces*, Dover, New York, 2000.

[11] P. Garabedian and M. Schiffer, *A Coefficient Inequality for Schlicht Functions*, Ann. Math. **61** (1955), 116-136.

[12] P. Garabedian and M. Schiffer, *A proof of the Bieberbach conjecture for the fourth coefficient*, J. Rational Mech. Anal. **4** (1955), 427-465.

[13] A.W. Goodman, *Functions typically-real and meromorphic in the unit circle*, Trans. Amer. Math. Soc., **81** (1956), 92-105.

[14] A. W. Goodman, *Univalent Functions*, Mariner Press, Tampa 1983.

[15] J.A. Jenkins, *On a conjecture of Goodman concerning meromorphic univalent functions*, Michigan Math. J. **9** (1962), 25-27.

[16] A.E. Livingston, *Convex meromorphic mappings*, Ann. Polo. Math. **59** (1994), 275-291.

[17] C. Löwner, *Untersuchungen uber schlichte konforme Abbildungen des Einheitskreises I*, Math. Ann. **176**, (1923), 61-94.

[18] J. Miller, *Convex meromorphic mappings and related functions*, Proc. Amer. Math. Soc. **25** (1970), 220-228.

[19] J. Miller, *Convex and starlike meromorphic functions*, Proc. Amer. Math. Soc. **80** (1980), 607-613.

[20] R. Ohno, *Characterizations for concave functions and integral representations*, Topics in Finite or Infinite Dimensional Complex Analysis, Tohoku University Press, Sendai (2013), 203-216.

[21] R. Ohno, *An extension of necessary and sufficient conditions for concave functions*, submitted.

[22] R. Ohno and H. Yanagihara, *On a coefficient body for concave functions*, Comp. Methods Funct. Theory **13**(2) (2013), 237-251.

[23] M. Ozawa, *On the Bieberbach conjecture for the sixth coefficient*, Kodai Math. Sem. Rep. **29** (1968), 97-128.

[24] R. Pederson, *A proof of the Bieberbach conjecture for the sixth coefficient*, Arch Rational Mech. Anal. **31** (1968), 331-351

[25] R. Pederson and M. Schiffer, *A proof of the Bieberbach conjecture for the fifth coefficient*, Arch. Rational Mech. Anal. **45** (1972), 161-193.

[26] J. Pfaltzgraff and B. Pinchuk, *A variational method for classes of meromorphic functions*, J. Analyse Math. **24** (1971), 101-150.

[27] G. Pólya and I.J. Schoenberg, *Remarks on de la Vallée Pousin means and convex vonformal maps of the circle*, Pacific J. Math., **8** (1958), 295-334.

[28] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer, Berlin, 1992.

[29] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Goettingen, 1975.

[30] St. Ruschweyh and T. Sheil-Small, *Hadamard products of scholicht functions and the Pólya Schoenberg conjecture*, Comment. Math. Helv. **48** (1973), 119-135.

[31] D.C. Spencer, *Some problems in conformal mappings*, Bull. Soc. Math. France **53** (1947), 417-439.

[32] T. J. Suffridge, *Some remarks on convex maps of the unit disk*, Duke Math. J. 37 (1970), 775-777.

[33] K.-J. Wirths, *On the residuum of concave univalent functions*, Serdica Math. J. **32** (2006), 209-214.

[34] H. Yanagihara, *Regions of variability for functions of bounded derivatives*, Kodai Math. J. **28** (2005), 452-462.