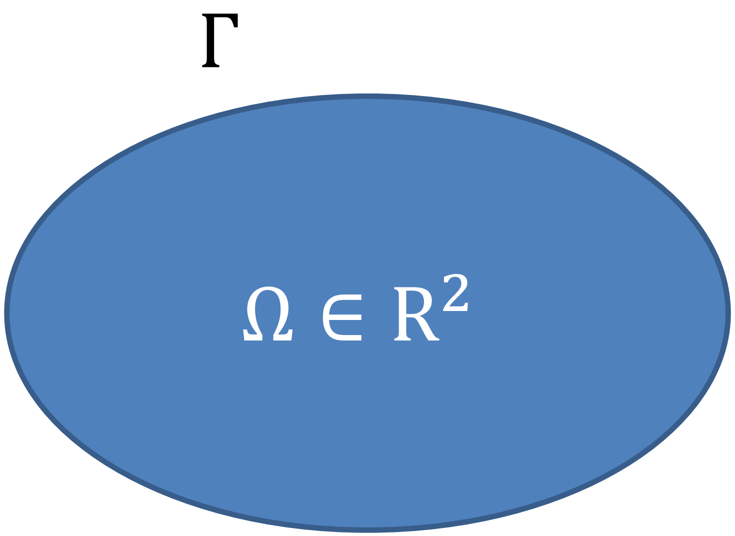
Lecture of FreeFEM++

Update:2017/11/30

Strong form of Poisson eq.
ポアソン方程式の強形式



$$-\Delta u - f = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \Gamma$$

1. 関数uのtrial functionをvとし,
2. 強形式とvとの積をとり,
3. 空間積分を行い,
4. 部分積分を行い,
5. 境界条件を代入する.

1. Let trial function for u be v,
2. Multiplying the strong form by v,
3. Integrating over the domain,
4. Using integration by parts,
5. Substituting boundary conditions.

Weak form of Poisson eq.
ポアソン方程式の弱形式

$$\int_\Omega (-\Delta u - f) v \, d\Omega = 0$$

$$\Rightarrow -\int_\Omega \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \right) v \, d\Omega - \int_\Omega f v d\Omega = 0$$

Weak form of Poisson eq.
ポアソン方程式の弱形式

$$\int_\Omega (-\Delta u - f) v \, d\Omega = 0$$

$$\Rightarrow -\int_\Omega \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \right) v \, d\Omega - \int_\Omega f v d\Omega = 0$$

$$\Rightarrow -\int_\Omega \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} v\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} v\right) \right) d\Omega + \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega - \int_\Omega f v d\Omega = 0$$

By using integration by parts

左辺第1項を部分積分

Weak form of Poisson eq.
ポアソン方程式の弱形式

$$\int_{\Omega} (-\Delta u - f) v \, d\Omega = 0$$

$$\Rightarrow -\int_{\Omega} \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \right) v \, d\Omega - \int_{\Omega} f v d\Omega = 0$$

$$\Rightarrow -\int_{\Omega} \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} v\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} v\right) \right) d\Omega + \int_{\Omega} \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega - \int_{\Omega} f v d\Omega = 0$$

$$\Rightarrow -\int_{\Omega} \nabla \cdot \left(\frac{\partial u}{\partial x} v, \frac{\partial u}{\partial y} v \right) d\Omega + \int_{\Omega} \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega - \int_{\Omega} f v d\Omega = 0$$

By rewriting with the divergence

発散を使って記述し直す

Weak form of Poisson eq.
ポアソン方程式の弱形式

$$\int_\Omega (-\Delta u - f)v \, d\Omega = 0$$

$$\Rightarrow -\int_\Omega \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \right) v \, d\Omega - \int_\Omega f v d\Omega = 0$$

$$\Rightarrow -\int_\Omega \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} v\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} v\right) \right) d\Omega + \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega - \int_\Omega f v d\Omega = 0$$

$$\Rightarrow -\int_\Omega \nabla \cdot \left(\frac{\partial u}{\partial x} v, \frac{\partial u}{\partial y} v \right) d\Omega + \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega - \int_\Omega f v d\Omega = 0$$

$$\Rightarrow -\int_\Gamma \left(\frac{\partial u}{\partial n} \right) v d\gamma + \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega - \int_\Omega f v d\Omega = 0$$

By divergence theorem

発散定理を使って空間積分を
境界積分に記述し直す

境界条件

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right)\, d\Omega - \int_\Gamma \left(\frac{\partial u}{\partial n} \right) v d\gamma - \int_\Omega f v d\Omega = 0$$

- ディィリクレ条件(Dirichlet condition)
  - 境界上で，trial function vは0なので，

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right)\, d\Omega - \int_\Omega f v d\Omega = 0$$

  - FreeFEM++のソースコードに具体的なディィリクレ条件を記入
- ノイマン条件 (Neumann condition)
  - 境界積分に$\frac{\partial u}{\partial n} = g$ を代入すると，

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right)\, d\Omega - \int_\Gamma g\, v d\gamma - \int_\Omega f v d\Omega = 0$$

Boundary Conditions

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) \, d\Omega - \int_\Gamma \left(\frac{\partial u}{\partial n} \right) v d\gamma - \int_\Omega f v d\Omega = 0$$

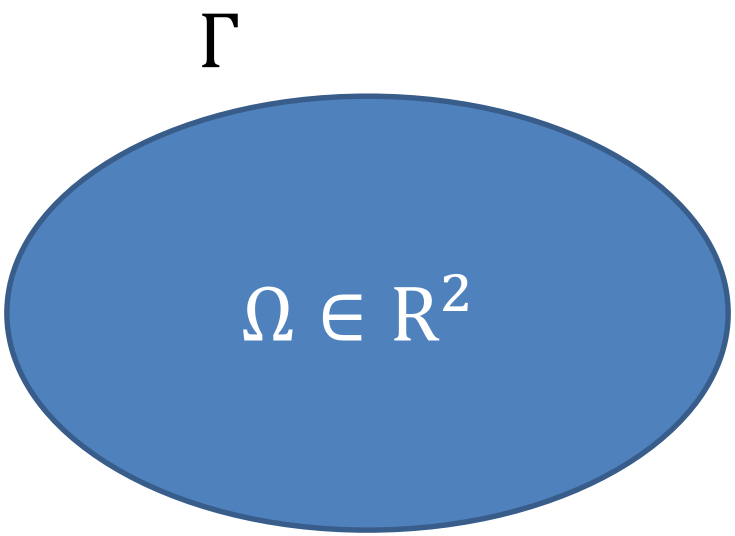
- In the case of Dirichlet condition
  - Let the trial function v be 0 on the boundary, we have

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) \, d\Omega - \int_\Omega f v d\Omega = 0$$

  - But commands for the Dirichlet condition should be written in the source code in FreeFEM++.
- In the case of Neumann condition
  - Substitute $\frac{\partial u}{\partial n} = g$ into boundary integration term, we have

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) \, d\Omega - \int_\Gamma g \, v d\gamma - \int_\Omega f v d\Omega = 0$$

Strong form and Weak form of Poisson eq.
ポアソン方程式の強形式と弱形式



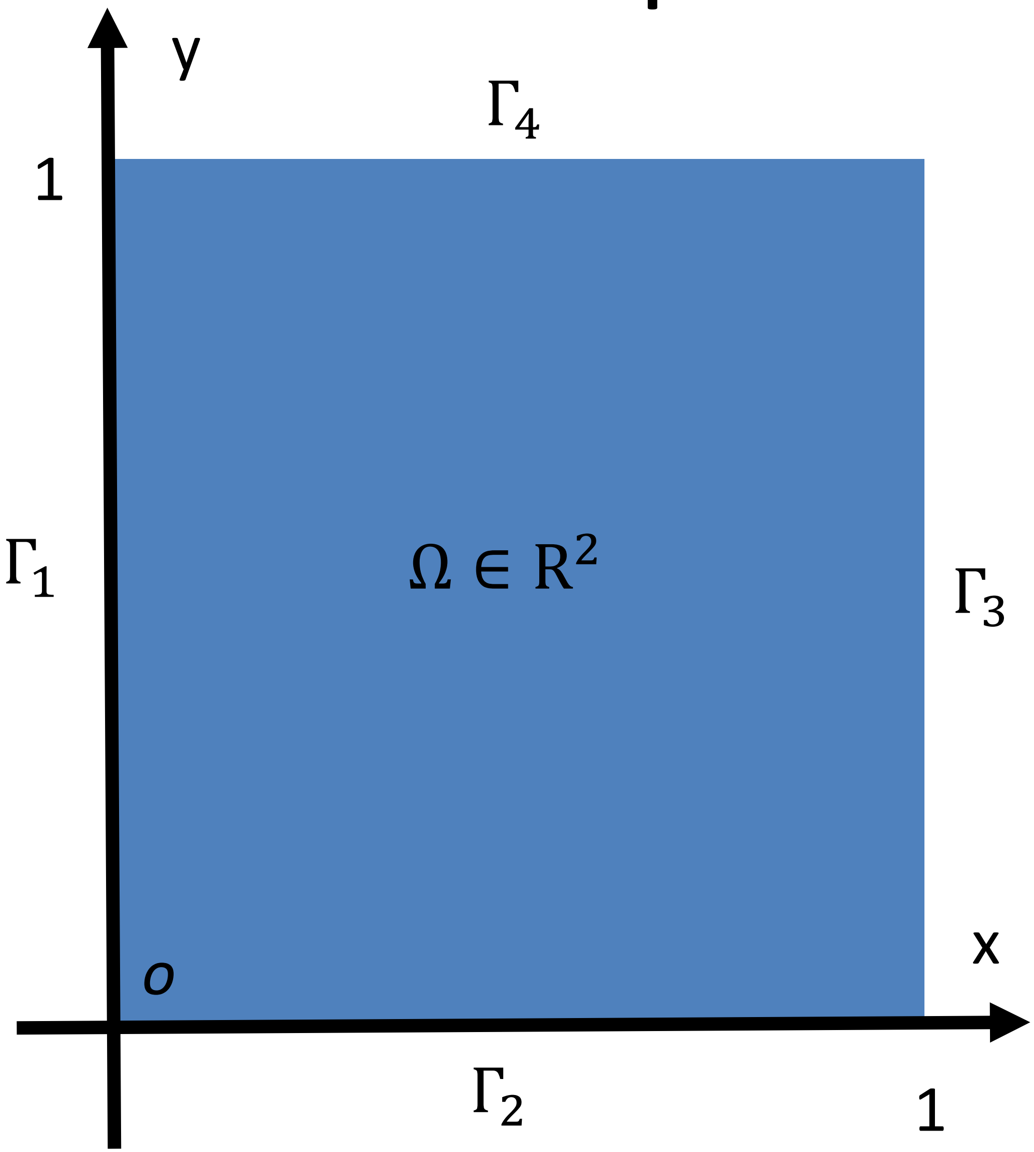
Strong form

$$-\Delta u - f = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \Gamma$$

Weak form

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega - \int_\Omega f v d\Omega = 0$$

Poisson equation: Laplace.edp



$$-\Delta u = 0 \text{ in } \Omega$$
$$u = 1 \text{ on } \Gamma_1$$
$$u = 0 \text{ on } \Gamma_2$$
$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_3$$
$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_4$$

Weak form of Laplace.edp

$$\int_\Omega (-\Delta u) v \, d\Omega = 0$$

$$\Rightarrow -\int_\Omega \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \right) v \, d\Omega = 0$$

$$\Rightarrow -\int_\Omega \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} v\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} v\right) \right) d\Omega + \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega = 0$$

$$\Rightarrow -\int_\Omega \nabla \cdot \left(\frac{\partial u}{\partial x} v, \frac{\partial u}{\partial y} v \right) d\Omega + \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega = 0$$

$$\Rightarrow -\int_\Gamma \left(\frac{\partial u}{\partial n} \right) v d\gamma + \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega = 0$$

$$\Rightarrow \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega = 0$$

Poisson equation: Laplace.edp

```
fespace Vh(Th,P2);
Vh uh,vh;

problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0) ;
laplace;
plot(uh);
```

Poisson equation: Laplace.edp

Commands for definitions of finite element space

有限要素空間を定義するためのコマンド

fespace Vh(Th,P2);

Vh uh,vh;

Poisson equation: Laplace.edp

The name of finite element space,
you can use any words.

任意に決めることが可能な有限要素空間の名前

fespace Vh(Th,P2);

Vh uh,vh;

Poisson equation: Laplace.edp

mesh Th=buildmesh(a0(10)+a1(10)+a2(10)+a3(10));

fespace Vh(Th,P2);

Vh uh,vh;

Poisson equation: Laplace.edp

Mesh

Th=buildmesh(a0(10)+a1(10)+a2(10)+a3(10));

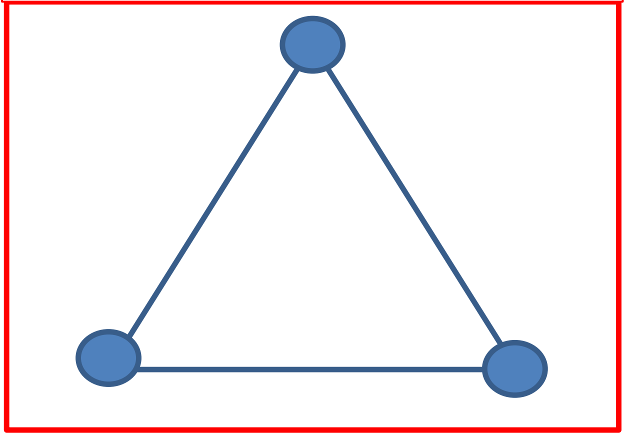
fespace Vh(Th,P1 or P2);

Vh uh,vh;

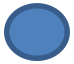
Poisson equation: Laplace.edp

Mesh
Th=buildmesh(a0(10)+a1(10)+a2(10)+a3(10));

P1 element（P1要素）



fespace Vh(Th,P1);

Variables are calculated
on 3 points  of one triangle
頂点の上で変数が計算される.

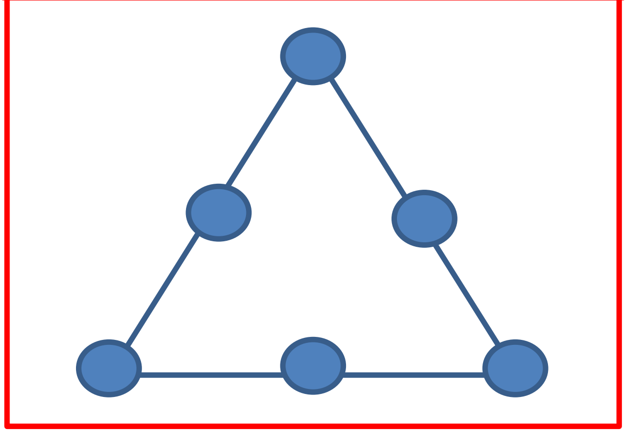
Vh uh,vh;

Poisson equation: Laplace.edp

Mesh

Th=buildmesh(a0(10)+a1(10)+a2(10)+a3(10));

P2 element（P2要素）



fespace Vh(Th,P2);

Variables are calculated

on 6 points of one triangle

頂点と線分の中心で変数が計算される.

Vh uh,vh;

Poisson equation: Laplace.edp

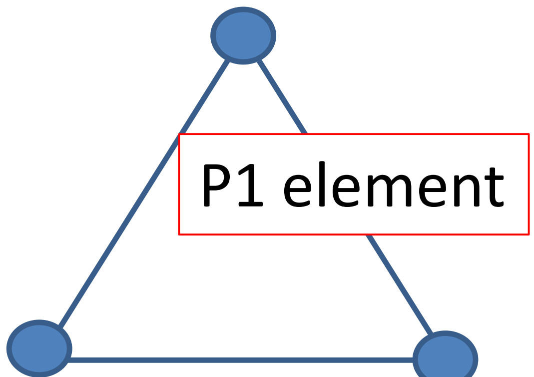
Mesh
Th=buildmesh(a0(10)+a1(10)+a2(10)+a3(10));

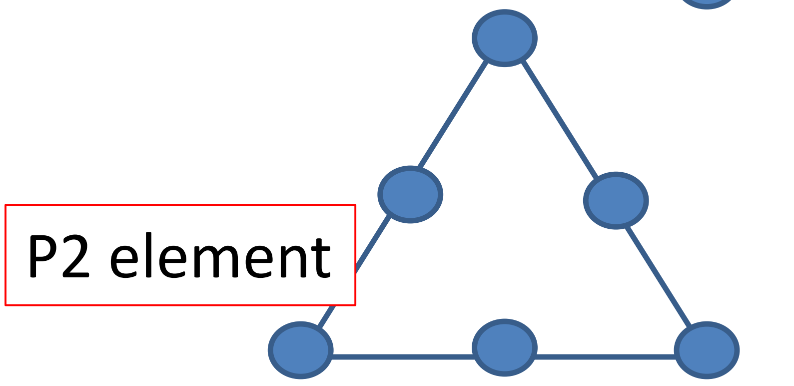
fespace Vh(Th,P1 or P2);

P1: low cost and low accuracy
（低い計算コスト，低い計算精度）

P2: high cost and high accuracy
（高い計算コスト，高い計算精度）

Vh uh,vh;





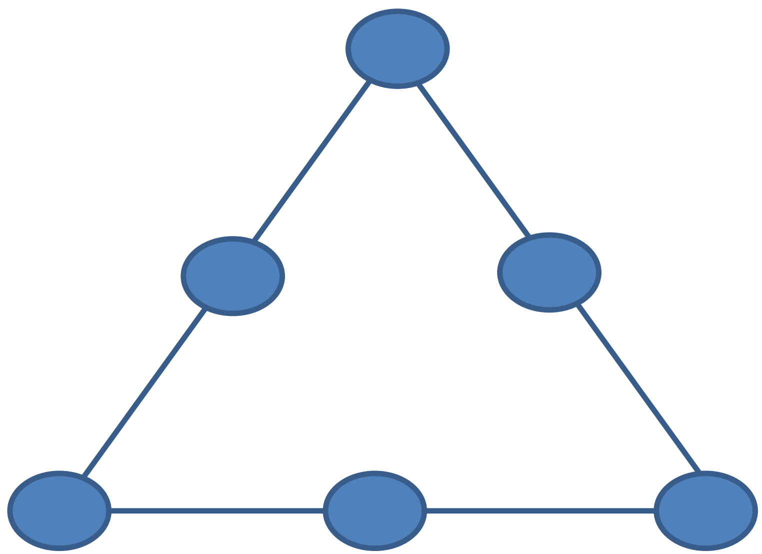
Poisson equation: Laplace.edp

Mesh

Th=buildmesh(a0(10)+a1(10)+a2(10)+a3(10));

fespace Vh(Th,P2);

Vh uh,vh;



Poisson equation: Laplace.edp

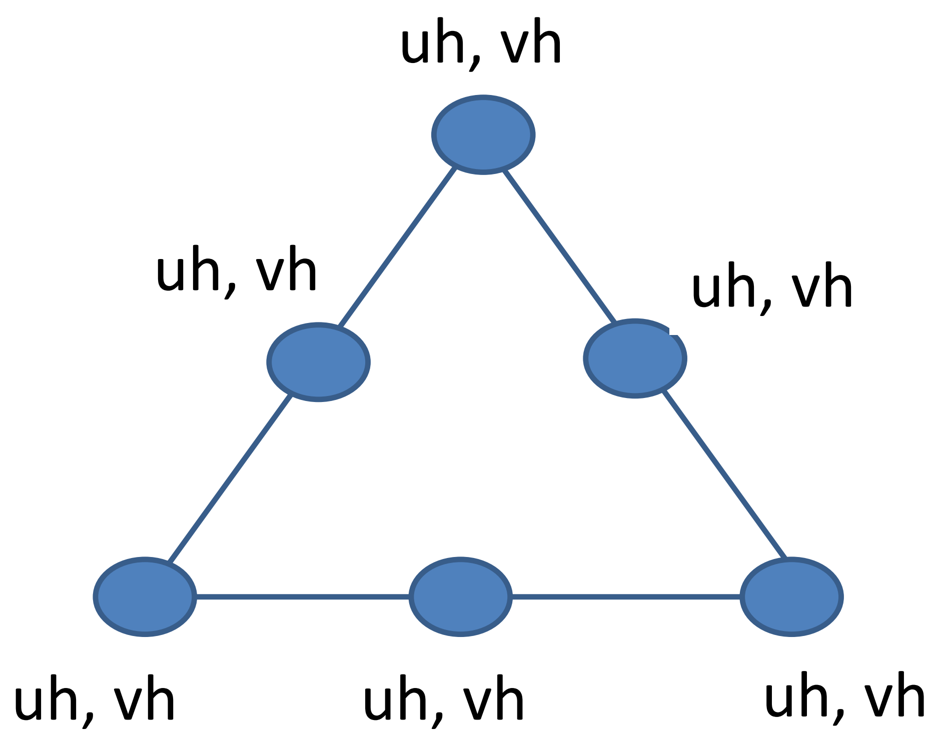
Mesh
Th=buildmesh(a0(10)+a1(10)+a2(10)+a3(10));

fespace Vh(Th,P2);

Vh uh,vh;

The name of variables
You can use any words

任意に決めることが可能な
変数の名前



Poisson equation: Laplace.edp

Commands for definition of problems

problemを定義するためのコマンド

```
problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0) ;
laplace;
```

Poisson equation: Laplace.edp

The name of the problem
You can use any words.

任意に決めることが可能なproblemの名前

```
problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0) ;
laplace;
```

Poisson equation: Laplace.edp

Variables that you need to calculate

計算に必要な変数を記述

```
problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0) ;
laplace;
```

Poisson equation: Laplace.edp

Numerical Scheme : LU decompositions

数値計算スキーム

```
problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0) ;
laplace;
```

Poisson equation: Laplace.edp

Weak form in FreeFEM++

FreeFEM++における弱形式の記述

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) \, d\Omega$$

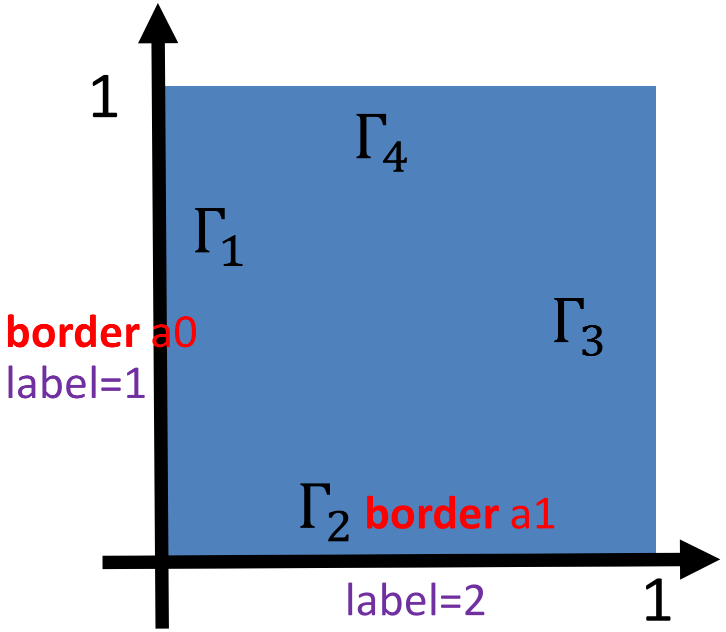
```
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
```

Poisson equation: Laplace.edp (Dirichlet conditions)

```
problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0) ;
```

Dirichlet condition

$$u = 1 \text{ on } \Gamma_1$$
$$u = 0 \text{ on } \Gamma_2$$




```
border a0(t=1,0){ x=0; y=t; label=1;}
border a1(t=0,1){ x=t; y=0; label=2;}
```

Poisson equation: Laplace.edp Neumann conditions

```
problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0) ;
```

$$\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\,d\Omega-\int_{\Gamma}\left(\frac{\partial u}{\partial n}\right)v d\gamma=0$$

$\frac{\partial u}{\partial n}=0 \text{ on } \Gamma_3$

$\frac{\partial u}{\partial n}=0 \text{ on } \Gamma_4$

Substitute $\frac{\partial u}{\partial n}=0$ into the second term

$\frac{\partial u}{\partial n}=0$ を境界積分項に代入する.

Neumann conditionsを使いたい場合

$$\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\,d\Omega-\int_{\Gamma} g\,v d\gamma=0$$

- 以下のように境界積分をソースコードに記述

  +int1d(Th, *label number*)(- g *vh)

ex) 境界border a1で g =2とすると, 以下のようになる

  +int1d(Th, 2)(-2*vh)

```
border a1(t=0,1){ x=t; y=0; label=2;}
```

If you want to use Neumann conditions…

$$\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\,d\Omega-\int_{\Gamma}g\,v d\gamma=0$$

- Please type boundary integration as follows;

  +int1d(Th, *label number*)(- g *vh)

ex)

on the boundary border a1, for g =2, you should type

+int1d(Th, 2)(-2*vh)

border a1(t=0,1){ x=t; y=0; label=2;}

Poisson equation: Laplace.edp

```
problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0);
```

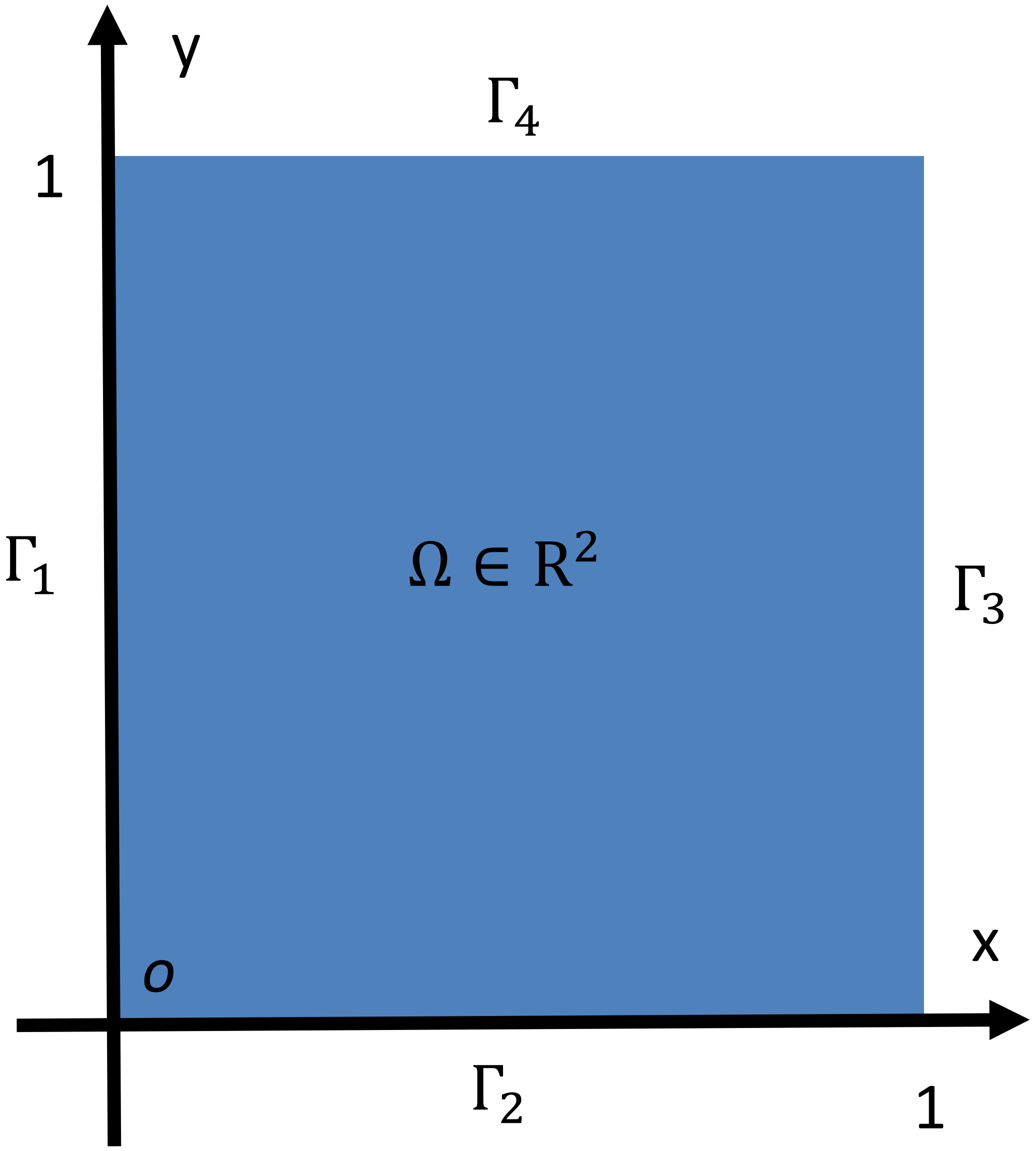
Dirichlet condition

```
laplace;
```

Numerical Calculation

数値計算

Problem3



$$-\Delta u - 1 = 0 \text{ in } \Omega$$

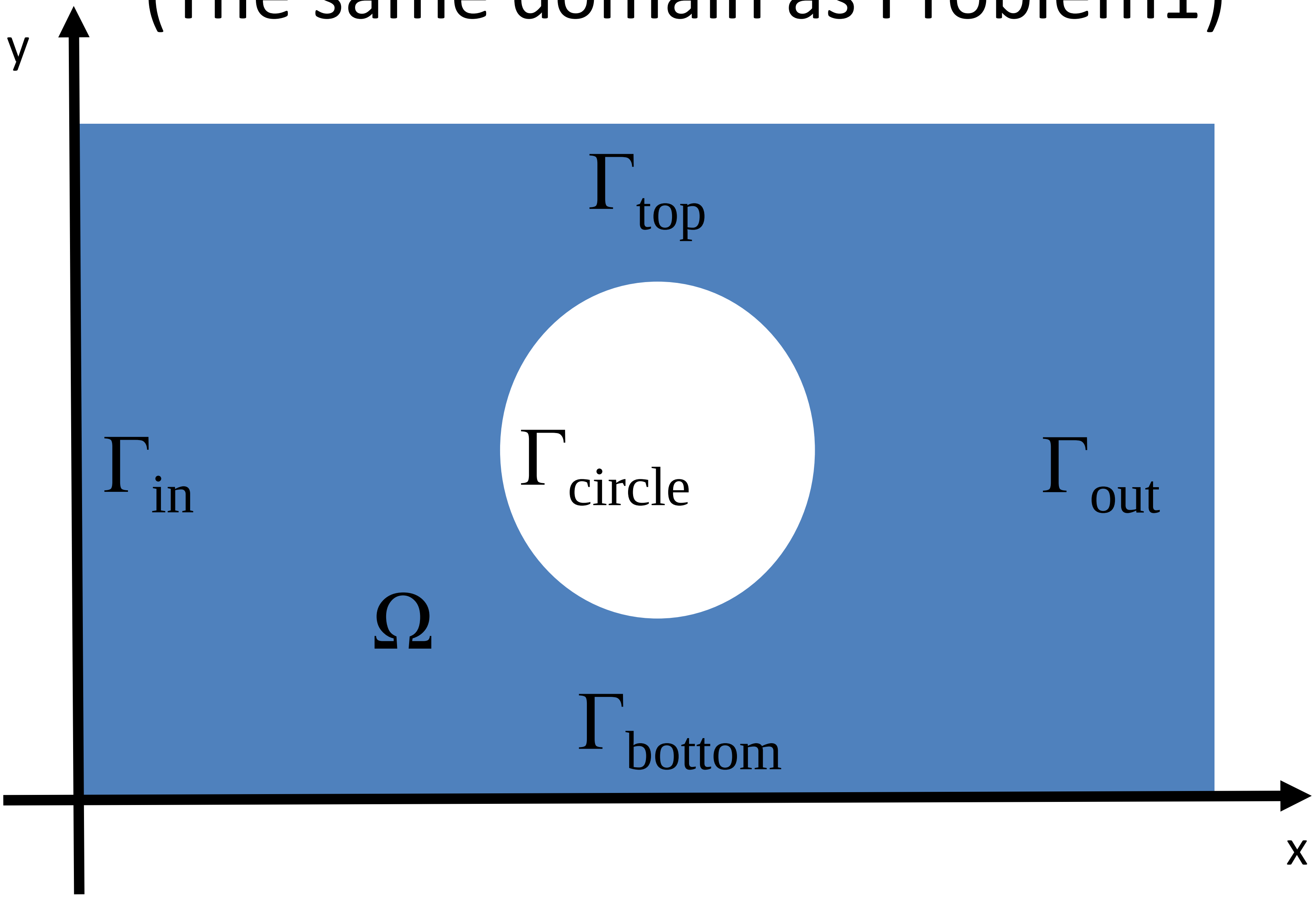
$$u = y * y - y \text{ on } \Gamma_1$$

$$u = 0 \text{ on } \Gamma_2$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_3$$

$$\frac{\partial u}{\partial n} = 1 \text{ on } \Gamma_4$$

Problem4
(The same domain as Problem1)



Problem4
(The same domain as Problem1)

$$-\Delta u - \sin(x) * \cos(y) = 0 \text{ in } \Omega$$

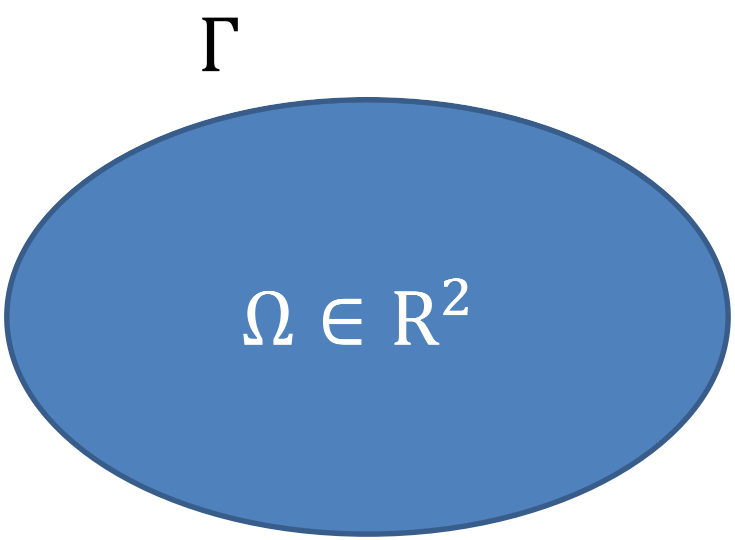
$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{\text{top}} \cup \Gamma_{\text{bottom}}$$

$$u = 1 \text{ on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$$

$$\frac{\partial u}{\partial n} = 1 \text{ on } \Gamma_{\text{circle}}$$

Stokes equations(Home work)

- Let trial function for $\boldsymbol{u}$ and p be $\boldsymbol{v}$ and q, where $\boldsymbol{u} = (u_x, u_y)$, $\boldsymbol{v} = (v_x, v_y)$.
  $\boldsymbol{u}, p$に対するTrial function を$\boldsymbol{v}, q$とし，その際 $\boldsymbol{u} = (u_x, u_y)$, $\boldsymbol{v} = (v_x, v_y)$とする.
- Describe the following equation in weak form.
  以下の式を弱形式で記述せよ.



$$\nabla p - \frac{1}{\mathrm{Re}} \Delta \boldsymbol{u} = 0 \text{ in } \Omega$$

$$\boldsymbol{\nabla} \cdot \ \boldsymbol{u} = 0 \text{ in } \Omega$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma$$

$$\nabla p - \frac{1}{\mathrm{Re}}\Delta \boldsymbol{u} = 0, \nabla \cdot \boldsymbol{u} = 0$$

$$\Longrightarrow \int_{\Omega} \left\{ \left(\nabla p - \frac{1}{\mathrm{Re}}\Delta \boldsymbol{u} \right) \cdot \boldsymbol{v} - (\nabla \cdot \boldsymbol{u}) q \right\} d\Omega = 0$$

$$\Longrightarrow \int_{\Omega} \begin{bmatrix} \left\{ \frac{\partial p}{\partial x} - \frac{1}{\mathrm{Re}} \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) \right\} v_x \\ + \left\{ \frac{\partial p}{\partial y} - \frac{1}{\mathrm{Re}} \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right) \right\} v_y \\ -(\nabla \cdot \boldsymbol{u}) q \end{bmatrix} d\Omega = 0$$

$$\Longrightarrow \int_{\Omega} \begin{bmatrix} \dfrac{\partial p}{\partial x} v_x + \dfrac{\partial p}{\partial y} v_y \\ + \dfrac{1}{\mathrm{Re}} \left(\dfrac{\partial u_x}{\partial x} \dfrac{\partial v_x}{\partial x} + \dfrac{\partial u_x}{\partial y} \dfrac{\partial v_x}{\partial y} \right) - \dfrac{1}{\mathrm{Re}} \nabla \cdot \left(\dfrac{\partial u_x}{\partial x} v_x, \dfrac{\partial u_x}{\partial y} v_x \right) \\ + \dfrac{1}{\mathrm{Re}} \left(\dfrac{\partial u_y}{\partial x} \dfrac{\partial v_y}{\partial x} + \dfrac{\partial u_y}{\partial y} \dfrac{\partial v_y}{\partial y} \right) - \dfrac{1}{\mathrm{Re}} \nabla \cdot \left(\dfrac{\partial u_y}{\partial x} v_y, \dfrac{\partial u_y}{\partial y} v_y \right) \\ -(\nabla \cdot \boldsymbol{u}) q \end{bmatrix} d\Omega = 0$$

$$\Longrightarrow \int_{\Omega} \begin{bmatrix} \dfrac{\partial p}{\partial x} v_x + \dfrac{\partial p}{\partial y} v_y \\ + \dfrac{1}{\mathrm{Re}} \left(\dfrac{\partial u_x}{\partial x} \dfrac{\partial v_x}{\partial x} + \dfrac{\partial u_x}{\partial y} \dfrac{\partial v_x}{\partial y} \right) \\ + \dfrac{1}{\mathrm{Re}} \left(\dfrac{\partial u_y}{\partial x} \dfrac{\partial v_y}{\partial x} + \dfrac{\partial u_y}{\partial y} \dfrac{\partial v_y}{\partial y} \right) \\ -(\nabla \cdot \boldsymbol{u}) q \end{bmatrix} d\Omega - \int_{\Gamma} \dfrac{1}{\mathrm{Re}} \left(\dfrac{\partial u_x}{\partial n} v_x + \dfrac{\partial u_y}{\partial n} v_y \right) d\gamma = 0$$

$$
\Rightarrow \int_{\Omega} \begin{bmatrix} +\dfrac{\partial}{\partial x}(pv_x) - \dfrac{\partial v_x}{\partial x}p \\ +\dfrac{\partial}{\partial y}(pv_y) - \dfrac{\partial v_y}{\partial y}p \\ +\dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial u_x}{\partial x}\dfrac{\partial v_x}{\partial x} + \dfrac{\partial u_x}{\partial y}\dfrac{\partial v_x}{\partial y}\right) \\ +\dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial u_y}{\partial x}\dfrac{\partial v_y}{\partial x} + \dfrac{\partial u_y}{\partial y}\dfrac{\partial v_y}{\partial y}\right) \\ -(\nabla \cdot \boldsymbol{u})q \end{bmatrix} d\Omega - \int_{\Gamma} \frac{1}{\mathrm{Re}}\left(\frac{\partial u_x}{\partial n}v_x + \frac{\partial u_y}{\partial n}v_y\right) d\gamma = 0
$$

$$
\Rightarrow \int_{\Omega} \begin{bmatrix} -(\nabla \cdot \boldsymbol{v})p \\ +\dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial u_x}{\partial x}\dfrac{\partial v_x}{\partial x} + \dfrac{\partial u_x}{\partial y}\dfrac{\partial v_x}{\partial y}\right) \\ +\dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial u_y}{\partial x}\dfrac{\partial v_y}{\partial x} + \dfrac{\partial u_y}{\partial y}\dfrac{\partial v_y}{\partial y}\right) \\ -(\nabla \cdot \boldsymbol{u})q \end{bmatrix} d\Omega
$$

$$
+ \int_{\Gamma} \left\{ (pv_x + pv_y)\boldsymbol{n} - \frac{1}{\mathrm{Re}}\left(\frac{\partial u_x}{\partial n}v_x + \frac{\partial u_y}{\partial n}v_y\right) \right\} d\gamma = 0
$$