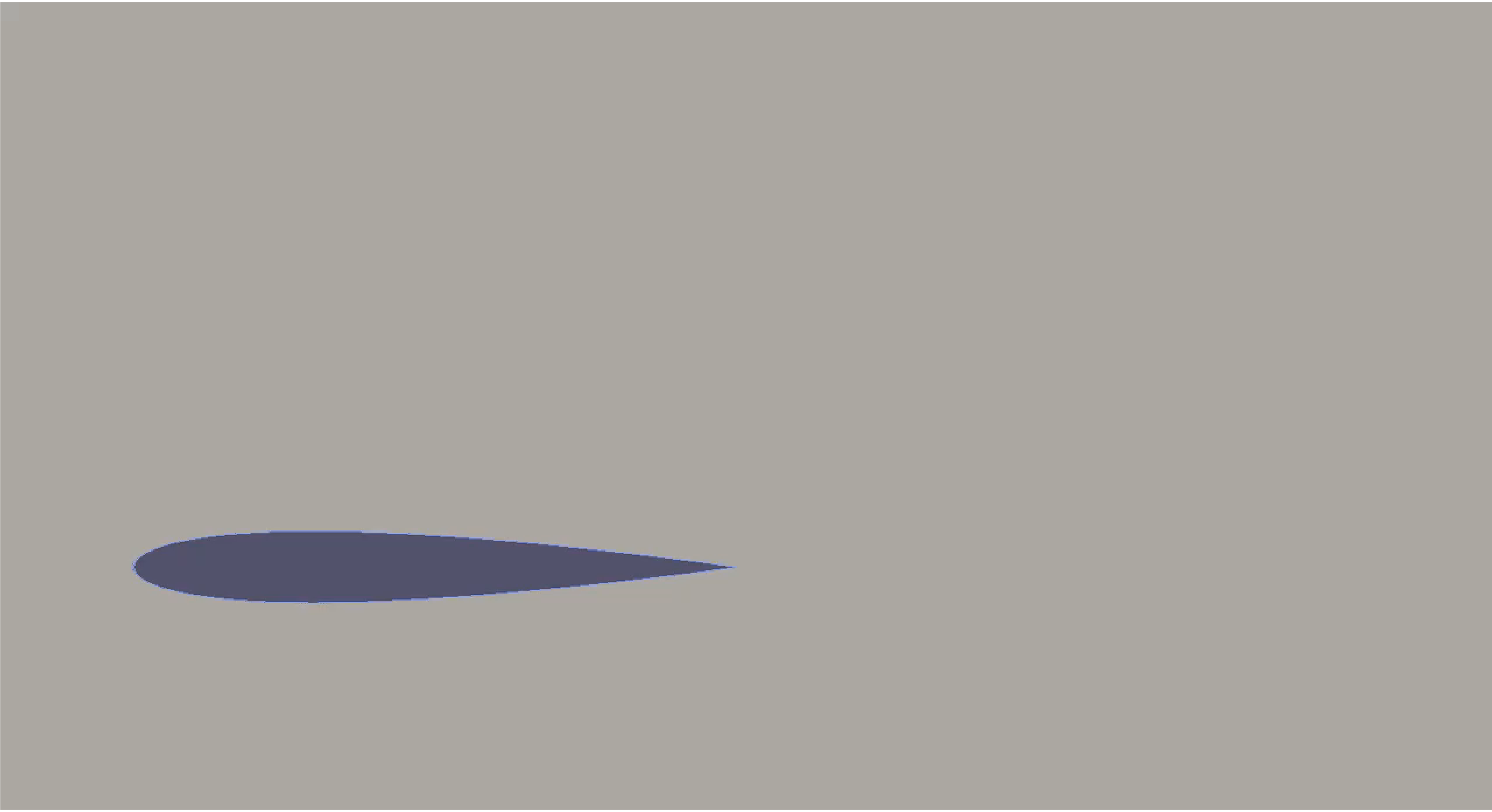
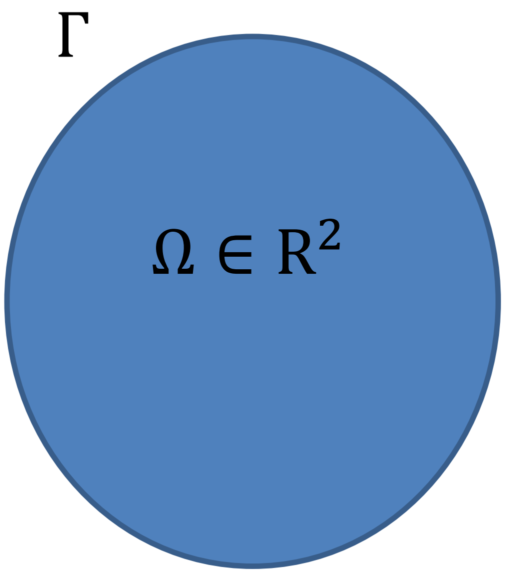
Lecture of FreeFEM++



Update:2017/11/30

Diffusion equation: diffusion.edp
拡散方程式

$$\frac{\partial f}{\partial t} - \Delta f = 0 \text{ in } \Omega$$



Initial condition（初期条件）

$$f = f_0 \text{ in } \Omega$$

where（その際，）

$$f_0 = \exp\left[\begin{array}{c} -10\{(x-0.3)^2 \\ +(y-0.3)^2\} \end{array} \right]$$

Boundary condition（境界条件）

$$f = 0 \text{ on } \Gamma$$

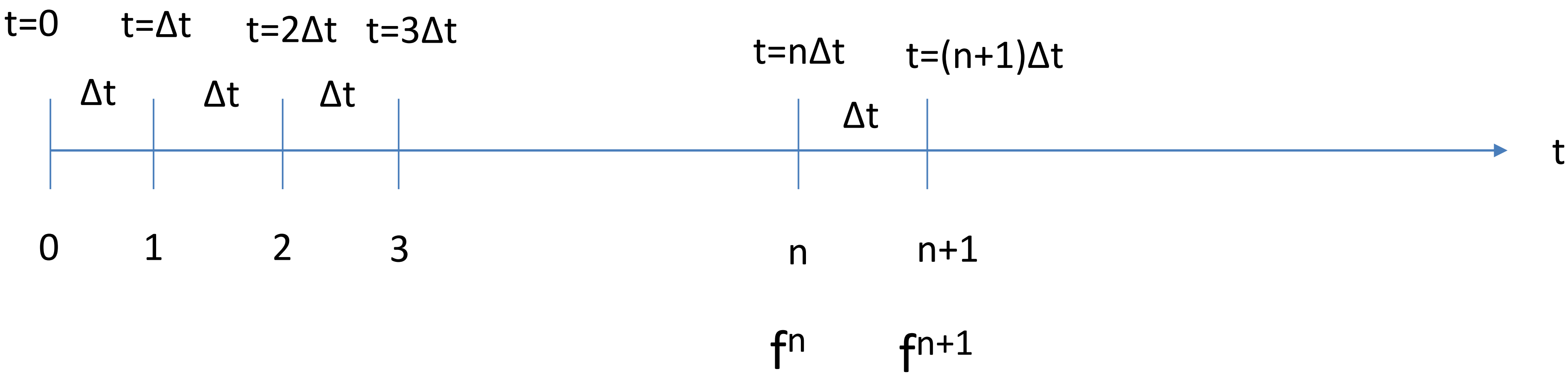
Diffusion equation: diffusion.edp

$$\frac{\partial f}{\partial t} - \Delta f = 0$$

Discretizing diffusion equations

in the time direction t at regular interval Δt

時間方向にΔt間隔で拡散方程式の差分をとる



Slides	Source Code
Δt	dt
t	(n+1)*dt
f^{n+1}	f
f^{n}	fo

Diffusion equation: diffusion.edp

$$\frac{\partial f}{\partial t} - \Delta f = 0$$

$$\Longrightarrow \frac{f^{n+1} - f^{n}}{\Delta t} - \Delta f^{n+1} = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1} - f^{n}}{\Delta t} - \Delta f^{n+1} \right) v d\Omega = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1}}{\Delta t} v - \frac{f^{n}}{\Delta t} v + \nabla f^{n+1} \cdot \nabla v \right) d\Omega - \int_{\Gamma} \frac{\partial f^{n+1}}{\partial n} v d\Gamma = 0$$

By taking integration by parts

Diffusion equation: diffusion.edp

$$\frac{\partial f}{\partial t} - \Delta f = 0$$

$$\Longrightarrow \frac{f^{n+1} - f^{n}}{\Delta t} - \Delta f^{n+1} = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1} - f^{n}}{\Delta t} - \Delta f^{n+1} \right) v d\Omega = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1}}{\Delta t} v - \frac{f^{n}}{\Delta t} v + \nabla f^{n+1} \cdot \nabla v \right) d\Omega - \int_{\Gamma} \frac{\partial f^{n+1}}{\partial n} v d\Gamma = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1}}{\Delta t} v - \frac{f^{n}}{\Delta t} v + \nabla f^{n+1} \cdot \nabla v \right) d\Omega = 0$$

By Dirichlet condition

Commands “for”

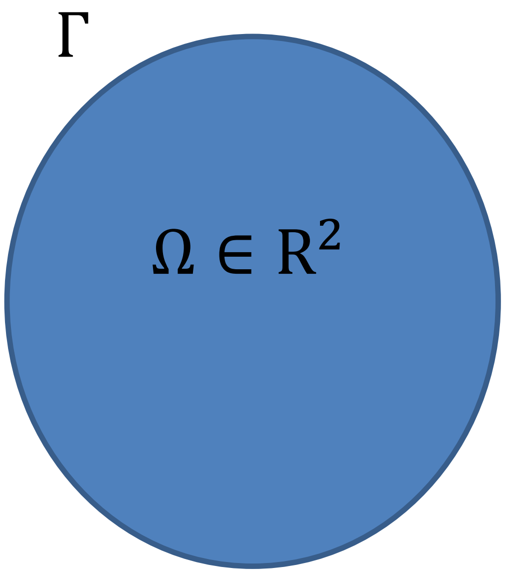
```
int n;
for ( n=0; n< 100 ; n=n+1)
{
    fo=f;
    A;
    plot(f);
};
```

“for” command repeat these operations from n=0 to n=99,
By adding 1 to n.

- fo=f; ← Substitute f into fo
- A; ← Calculate problem A
- plot(f); ← Output f

Convection equation: convection.edp
移流方程式

$$\frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f = 0 \text{ in } \Omega$$



where

$$\boldsymbol{u} = (y, -x)$$

Initial condition

$$f = f_0 \text{ in } \Omega$$

where

$$f_0 = \exp\left[\begin{matrix} -10\{(x-0.3)^2 \\ +(y-0.3)^2\} \end{matrix}\right]$$

Boundary condition

$$f = 0 \text{ on } \Gamma$$

Convection equation: convection.edp

$$\frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f = 0$$

$$\Longrightarrow \frac{f^{n+1} - f^n}{\Delta t} + (\boldsymbol{u} \cdot \nabla) f^n = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{f^{n+1} - f^n}{\Delta t} + (\boldsymbol{u} \cdot \nabla) f^n \right) v d\Omega = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{f^{n+1}}{\Delta t} v - \frac{f^n}{\Delta t} v + \{(\boldsymbol{u} \cdot \nabla) f^n\} v \right) d\Omega = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{f^{n+1}}{\Delta t} v - \frac{v}{\Delta t} \{f^n - \Delta t (\boldsymbol{u} \cdot \nabla) f^n\} \right) d\Omega = 0$$

characteristics finite element scheme
特性曲線法

$$\{f^n - \Delta t(\boldsymbol{u} \cdot \nabla) f^n\}$$
$$= convect(\boldsymbol{u}, -\Delta t, f^n)$$

Convection equation : convection.edp

$$\frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f = 0$$

$$\Longrightarrow \frac{f^{n+1} - f^{n}}{\Delta t} + (\boldsymbol{u} \cdot \nabla) f^{n} = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1} - f^{n}}{\Delta t} + (\boldsymbol{u} \cdot \nabla) f^{n} \right) w\mathrm{v}\Omega = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1}}{\Delta t} v - \frac{f^{n}}{\Delta t} v + \{(\boldsymbol{u} \cdot \nabla) f^{n}\} v \right) d\Omega = 0$$

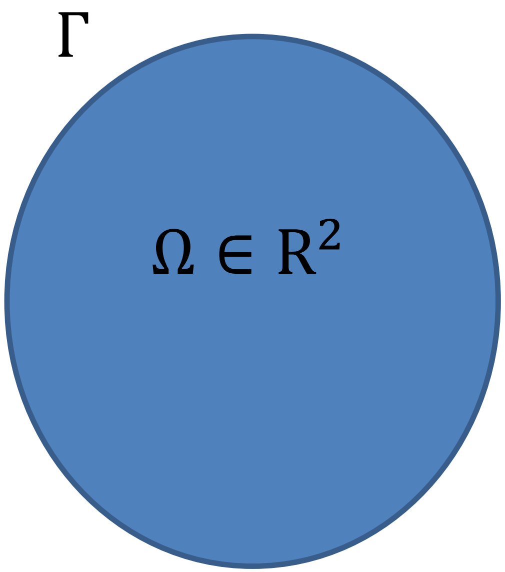
$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1}}{\Delta t} v - \frac{v}{\Delta t} \{f^{n} - \Delta t (\boldsymbol{u} \cdot \nabla) f^{n}\} \right) d\Omega = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1}}{\Delta t} v - \frac{v}{\Delta t} convect(\boldsymbol{u}, -\Delta t, f^{n}) \right) d\Omega = 0$$

$$\Longrightarrow f^{n+1} = convect(\boldsymbol{u}, -\Delta t, f^{n})$$

Convection Diffusion equation: convection_diffusion.edp

$$\frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f - \Delta f = 0 \text{ in } \Omega$$



where

$$\boldsymbol{u} = (y, -x)$$

Initial condition

$$f = f_0 \text{ in } \Omega$$

where

$$f_0 = \exp\left[\begin{matrix} -10\{(x-0.3)^2 \\ +(y-0.3)^2\} \end{matrix}\right]$$

Boundary condition

$$f = 0 \text{ on } \Gamma$$

Convection diffusion equation: convection_diffusion.edp

$$\frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f - \Delta f = 0$$

$$\Rightarrow \frac{f^{n+1} - f^{n}}{\Delta t} + (\boldsymbol{u} \cdot \nabla) f^{n} - \Delta f^{n+1} = 0$$

$$\Rightarrow \int_{\Omega} \left(\frac{f^{n+1} - f^{n}}{\Delta t} + (\boldsymbol{u} \cdot \nabla) f^{n} - \Delta f^{n+1} \right) v d\Omega = 0$$

$$\Rightarrow \int_{\Omega} \left(\frac{f^{n+1}}{\Delta t} v - \frac{f^{n}}{\Delta t} v + \{ (\boldsymbol{u^n} \cdot \nabla) f^{n} \} v + \nabla f^{n+1} \cdot \nabla v \right) d\Omega - \int_{\Gamma} \frac{\partial f^{n+1}}{\partial n} v \, d\Gamma = 0$$

By taking integration by parts

Convection diffusion equation: convection_diffusion.edp

$$\frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f - \Delta f = 0$$

$$\Longrightarrow \frac{f^{n+1} - f^{n}}{\Delta t} + (\boldsymbol{u} \cdot \nabla) f^{n} - \Delta f^{n+1} = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1} - f^{n}}{\Delta t} + (\boldsymbol{u} \cdot \nabla) f^{n} - \Delta f^{n+1} \right) v d\Omega = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1}}{\Delta t} v - \frac{f^{n}}{\Delta t} v + \{ (\boldsymbol{u}^{\boldsymbol{n}} \cdot \nabla) f^{n} \} v + \nabla f^{n+1} \cdot \nabla v \right) d\Omega - \int_{\Gamma} \frac{\partial f^{n+1}}{\partial n} v \, d\Gamma = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1}}{\Delta t} v - \frac{v}{\Delta t} convect(\boldsymbol{u}, -\Delta t, f^{n}) + \nabla f^{n+1} \cdot \nabla v \right) d\Omega - \int_{\Gamma} \frac{\partial f^{n+1}}{\partial n} v d\Gamma = 0$$

By using “convect” command

Convection diffusion equation: convection_diffusion.edp

$$\frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f - \Delta f = 0$$

$$\Longrightarrow \frac{f^{n+1} - f^{n}}{\Delta t} + (\boldsymbol{u} \cdot \nabla) f^{n} - \Delta f^{n+1} = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1} - f^{n}}{\Delta t} + (\boldsymbol{u} \cdot \nabla) f^{n} - \Delta f^{n+1} \right) v d\Omega = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1}}{\Delta t} v - \frac{f^{n}}{\Delta t} v + \{ (\boldsymbol{u}^{\boldsymbol{n}} \cdot \nabla) f^{n} \} v + \nabla f^{n+1} \cdot \nabla v \right) d\Omega - \int_{\Gamma} \frac{\partial f^{n+1}}{\partial n} v \, d\Gamma = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1}}{\Delta t} v - \frac{v}{\Delta t} convect(\boldsymbol{u}, -\Delta t, f^{n}) + \nabla f^{n+1} \cdot \nabla v \right) d\Omega - \int_{\Gamma} \frac{\partial f^{n+1}}{\partial n} v d\Gamma = 0$$

$$\Longrightarrow \int_{\Omega} \left(\frac{f^{n+1}}{\Delta t} v - \frac{v}{\Delta t} convect(\boldsymbol{u}, -\Delta t, f^{n}) + \nabla f^{n+1} \cdot \nabla v \right) d\Omega = 0$$

By boundary condition

Problem6
(The same domain as Problem1)

$$\frac{\partial u}{\partial t} - \Delta u = 0 \text{ in } \Omega$$

Initial condition:

$$u = \exp\{-10\{(x-1)^2 + (y-1.5)^2\}\}$$

Boundary conditions:

$$u = 0 \text{ on } \Gamma_{\mathrm{top}}$$
$$u = 0 \text{ on } \Gamma_{\mathrm{bottom}}$$
$$\frac{\partial u}{\partial n} = 0 \text{ on} \Gamma_{\mathrm{in}}$$
$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{\mathrm{out}}$$
$$u = 0 \text{ on } \Gamma_{\mathrm{circle}}$$

Problem7
(The same domain as Problem1)

$$\frac{\partial u}{\partial t} + (\boldsymbol{f} \cdot \nabla)u - \Delta u = 0 \text{ in } \Omega$$

where $\boldsymbol{f} = (-y(y-5), 0)$

Initial condition:

$$u = \exp\{-10\{(x-1)^2 + (y-1.5)^2\}\}$$

Boundary conditions:

$$u = 0 \text{ on } \Gamma_{\text{top}}$$

$$u = 0 \text{ on } \Gamma_{\text{bottom}}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on} \Gamma_{\text{in}}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{\text{out}}$$

$$u = 0 \text{ on } \Gamma_{\text{circle}}$$

Special Problem

$$\frac{\partial f}{\partial t} - \phi \Delta f = 0 \text{ in } \Omega$$

where $\boldsymbol{u} = (y, -x)$

Initial condition $f = f_0$ in Ω,

where $f_0 = \exp[-10\{(x - 0.3)^2 + (y - 0.3)^2\}]$,

Boundary condition $f = 0$ on Γ,

The parameter ϕ is defined as

$$\phi = \alpha_1(1 - \theta) + \alpha_2\theta, \theta = \frac{1}{2}(\tanh(\beta\boldsymbol{X}) + 1)$$

where $\alpha_1 = 0.0001, \alpha_2 = 0.01, \beta = 50,$

$\boldsymbol{X} = (x - 0.4)^2 + (y + 0.3)^2 - 0.3^2$