Lecture of FreeFEM++

Update:2019/3/24

Minimization Problem

- Define the functional
  - $L(u) = ..$
- Take the Fréchet derivative w.r.t u
  - $L(u - u') = L(u) - L(u)[u']..$
- Set the variation $L(u)[u'] = 0$
  - $L(u - u') = L(u)$

Minimization problem 1

$$L(u) = \frac{1}{2}\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right) d\Omega - \int_{\Omega} f u d\Omega$$

Minimization problem 1

$$L(u) = \frac{1}{2}\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right)\, d\Omega - \int_{\Omega} f u d\Omega$$

$$L(u - u')$$
$$= \frac{1}{2}\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right)\, d\Omega - \int_{\Omega} f u d\Omega$$
$$- \left\{\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y}\right)\, d\Omega - \int_{\Omega} f u' d\Omega\right\} + \frac{1}{2}\int_{\Omega}\left(\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\right)\, d\Omega$$

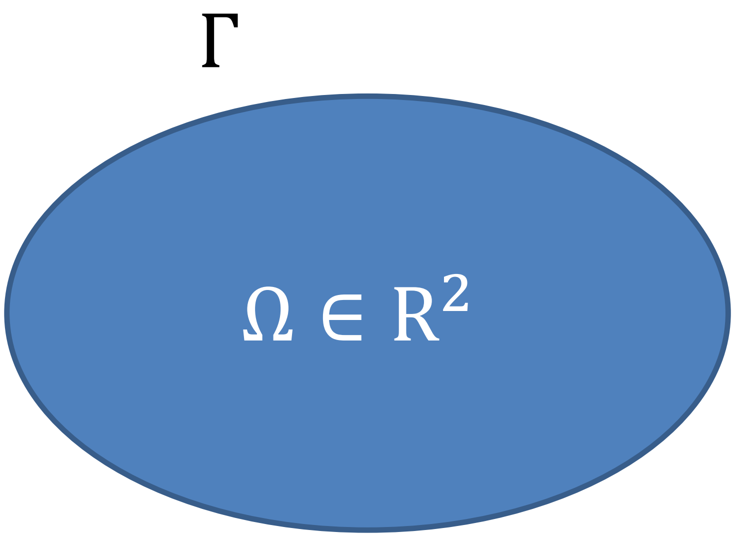
Minimization problem 1

$$L(u) = \frac{1}{2}\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right) d\Omega - \int_\Omega fu d\Omega$$

$$L(u - u')$$
$$= \frac{1}{2}\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right) d\Omega - \int_\Omega fu d\Omega$$
$$- \left\{\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega - \int_\Omega fu' d\Omega\right\} + \frac{1}{2}\int_\Omega \left(\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega$$

$$= L(u)$$
$$- \left\{\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega - \int_\Omega fu' d\Omega\right\} + \frac{1}{2}\int_\Omega \left(\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega$$

Reminder:Strong form and Weak form of Poisson eq.
ポアソン方程式の強形式と弱形式



Strong form

$$-\Delta u - f = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \Gamma$$

Weak form

$$\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\, d\Omega - \int_{\Omega} f v d\Omega = 0$$

Minimization problem 1

$$L(u) = \frac{1}{2}\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right) d\Omega - \int_{\Omega} fu d\Omega$$

$$L(u-u')$$
$$= \frac{1}{2}\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right) d\Omega - \int_{\Omega} fu d\Omega$$
$$- \left\{\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega - \int_{\Omega} fu' d\Omega\right\} + \frac{1}{2}\int_{\Omega}\left(\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega$$

$$= L(u)$$
$$- \left\{\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega - \int_{\Omega} fu' d\Omega\right\} + \frac{1}{2}\int_{\Omega}\left(\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega$$

Set the equation be 0 because of the weak form

Minimization problem 1

$$L(u) = \frac{1}{2}\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right) d\Omega - \int_{\Omega} fu d\Omega$$

$$L(u-u')$$
$$= \frac{1}{2}\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right) d\Omega - \int_{\Omega} fu d\Omega$$
$$- \left\{\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega - \int_{\Omega} fu' d\Omega\right\} + \frac{1}{2}\int_{\Omega}\left(\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega$$

$$= L(u)$$
$$- \left\{\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega - \int_{\Omega} fu' d\Omega\right\} + \frac{1}{2}\int_{\Omega}\left(\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega$$

Neglect because of $|u'| \ll |u|$

Minimization problem 1

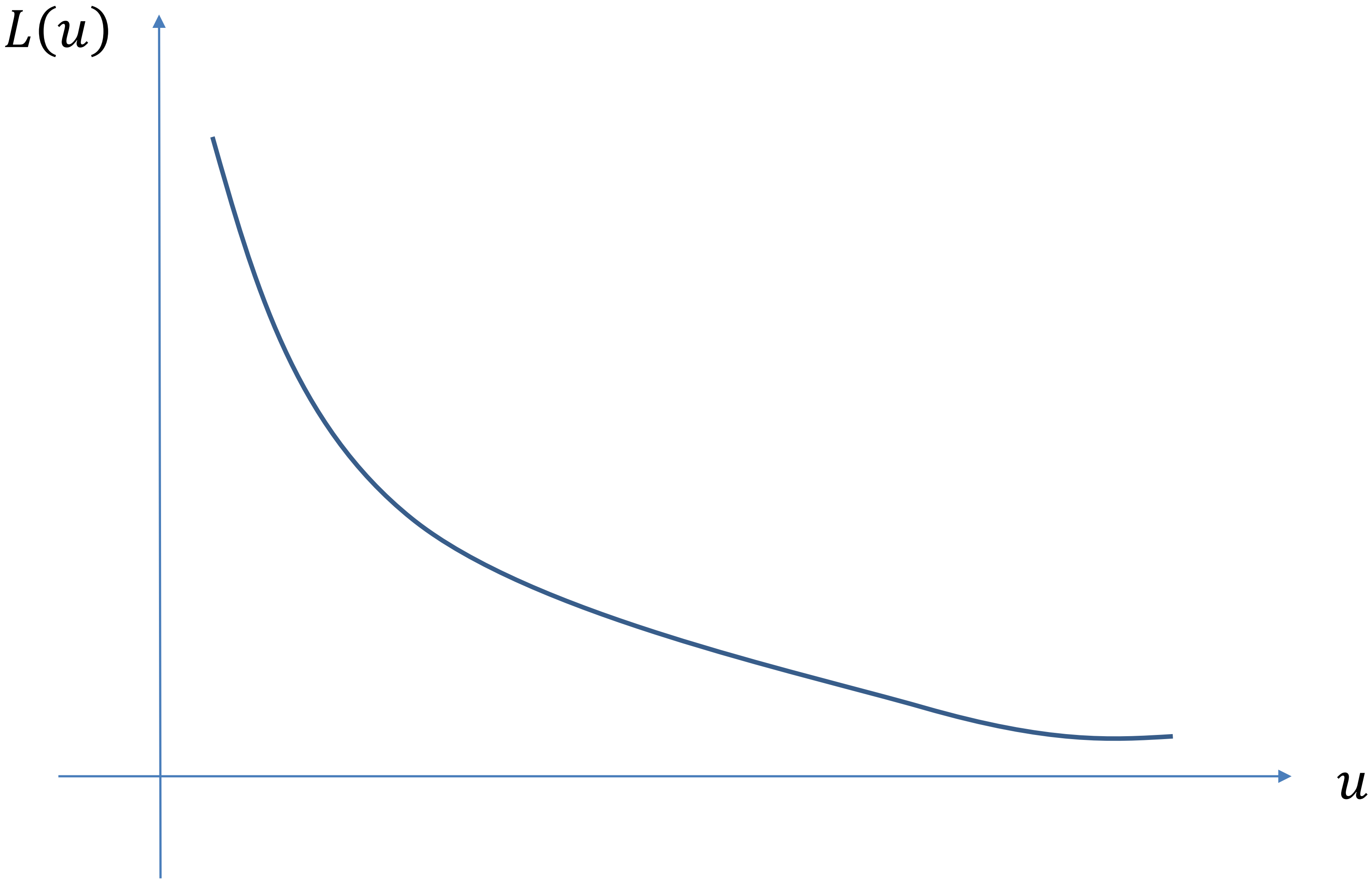
$$L(u) = \frac{1}{2}\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right) d\Omega - \int_{\Omega} fud\Omega$$

$$\begin{aligned} &L(u-u') \\ &= \frac{1}{2}\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right) d\Omega - \int_{\Omega} fud\Omega \\ &-\left\{\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega - \int_{\Omega} fu'd\Omega\right\} + \frac{1}{2}\int_{\Omega}\left(\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega \end{aligned}$$

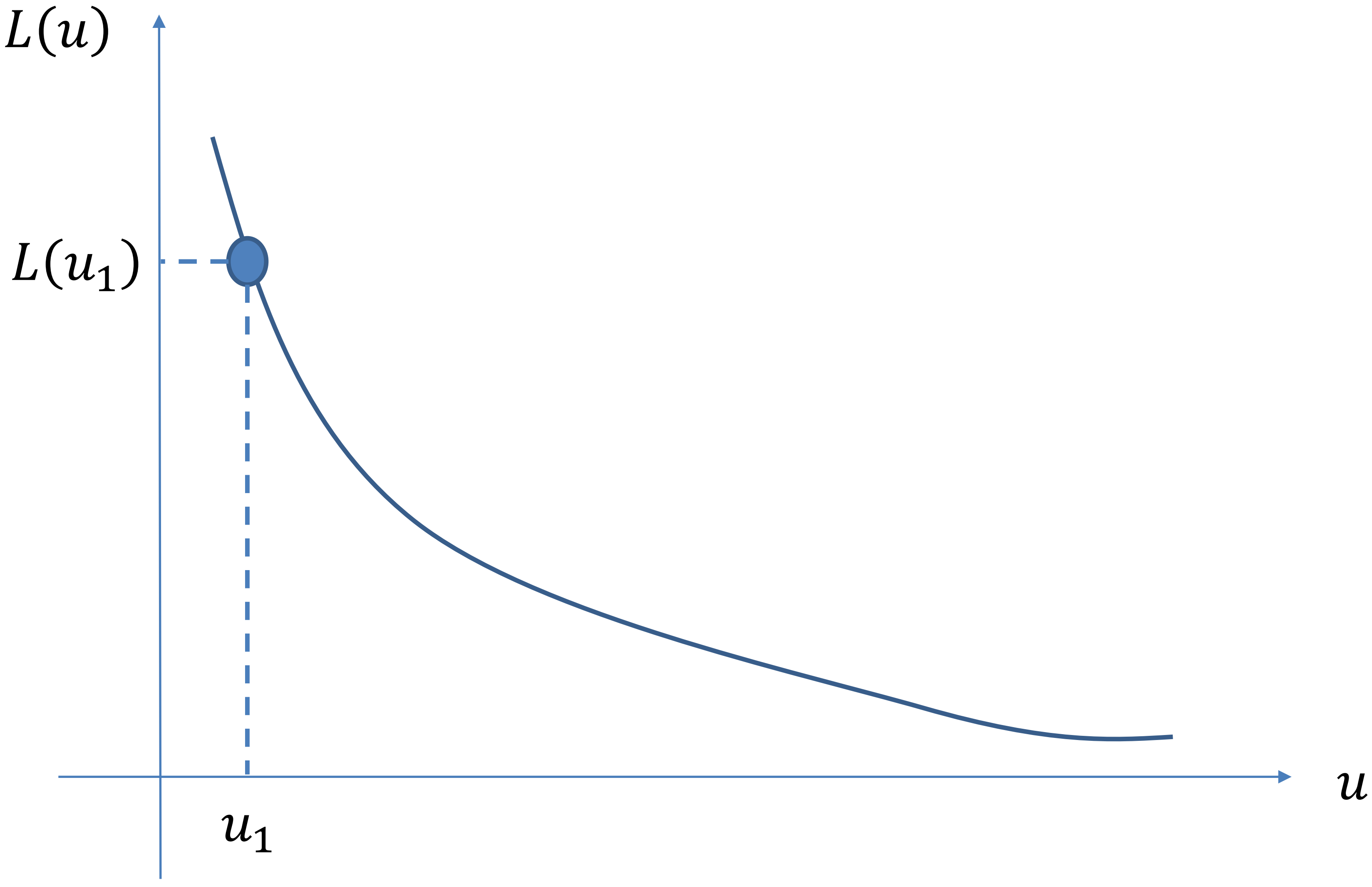
$$\begin{aligned} &= L(u) \\ &-\left\{\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega - \int_{\Omega} fu'd\Omega\right\} + \frac{1}{2}\int_{\Omega}\left(\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega \end{aligned}$$

$\Longrightarrow L(u-u') = L(u) = L(w)$: minimization of the functional

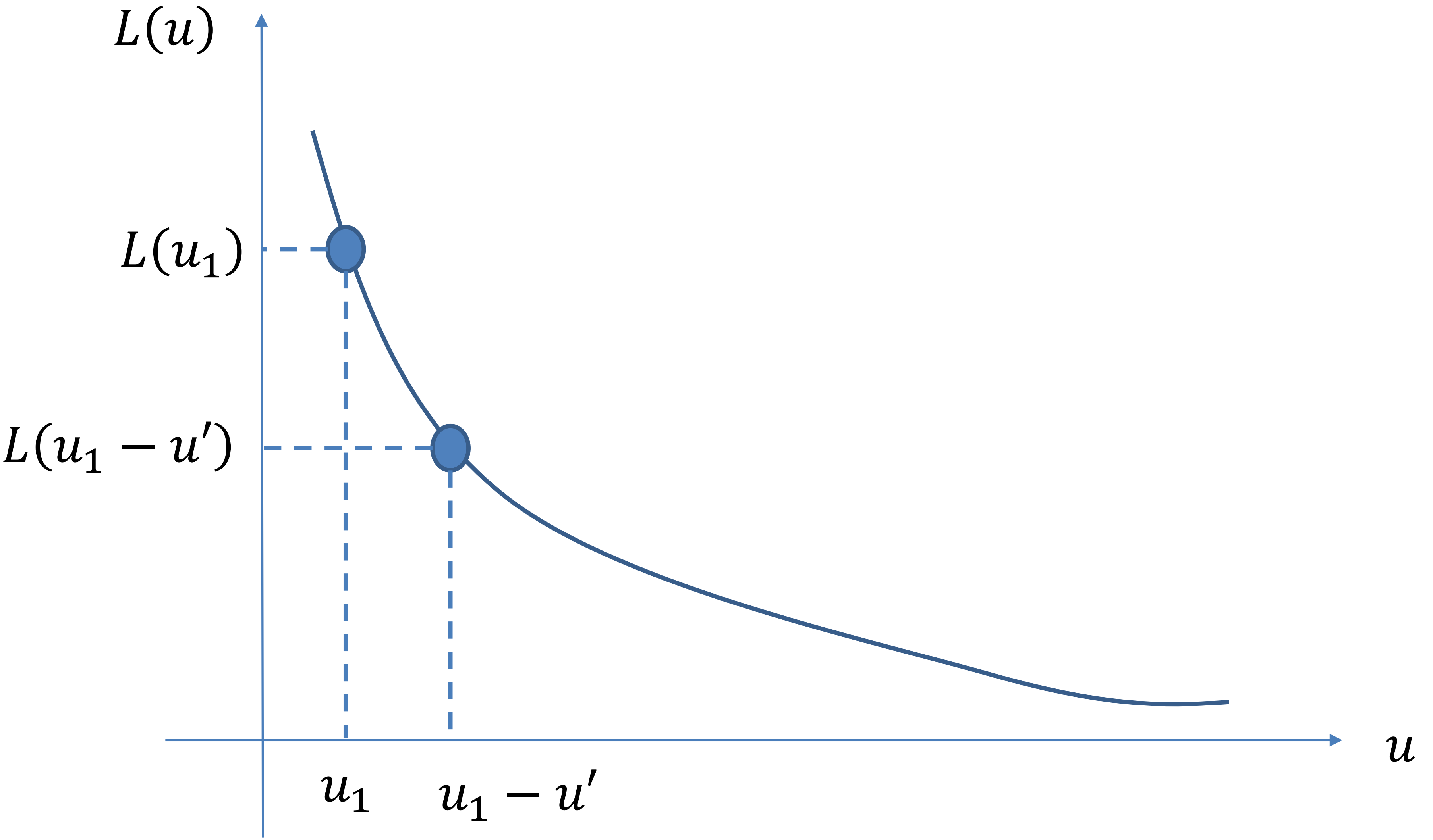
Schematic picture



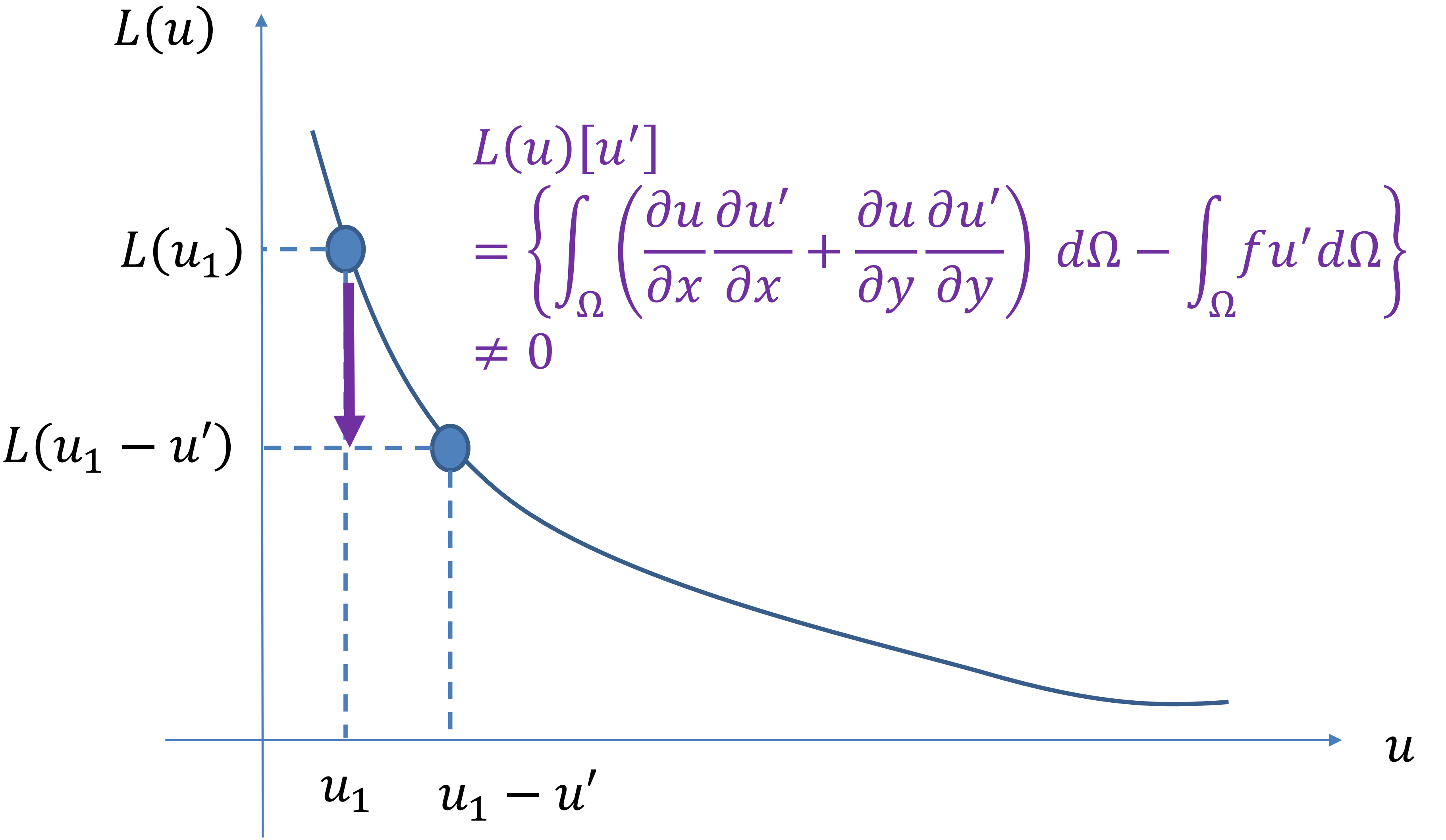
Schematic picture



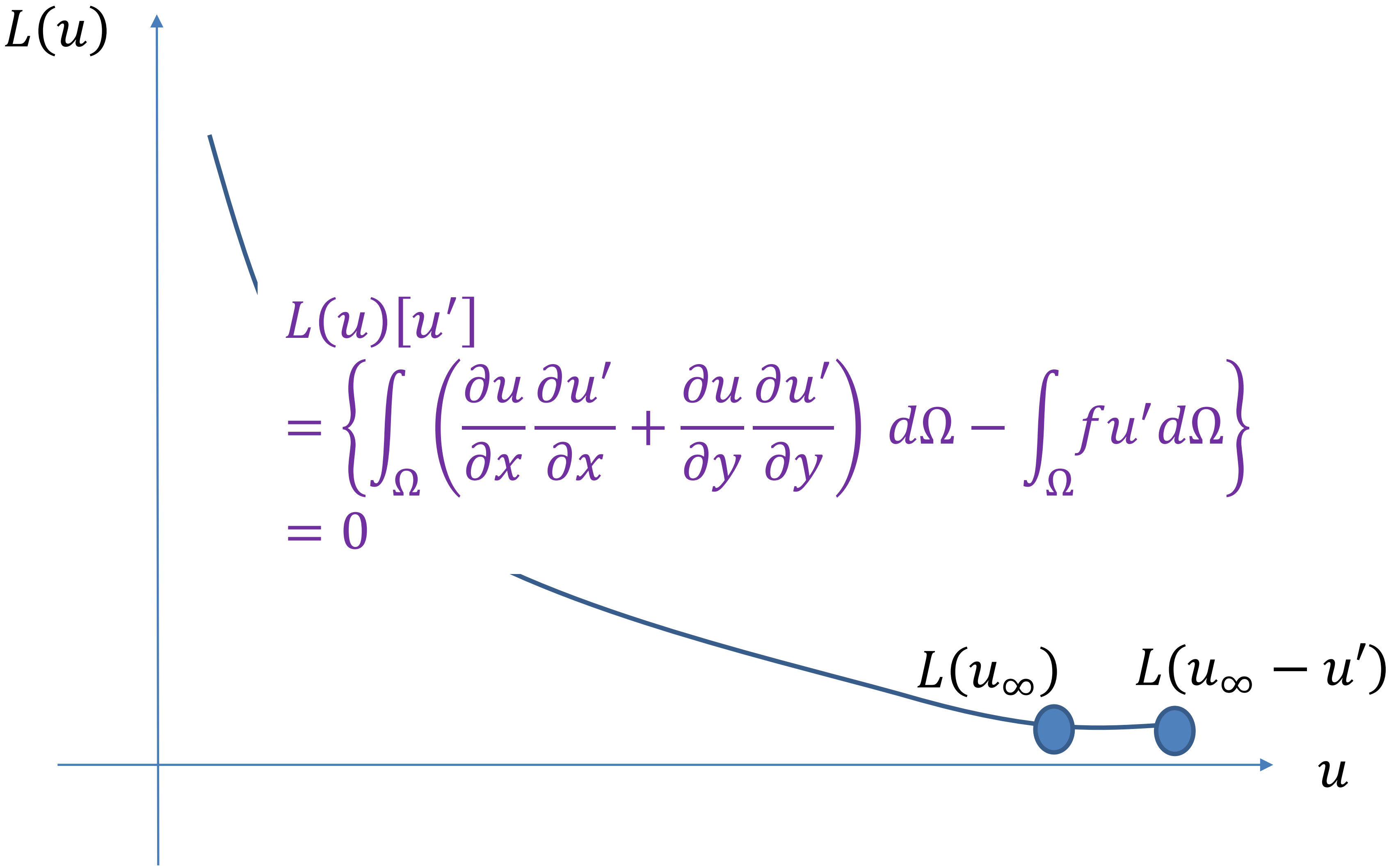
Schematic picture



Schematic picture

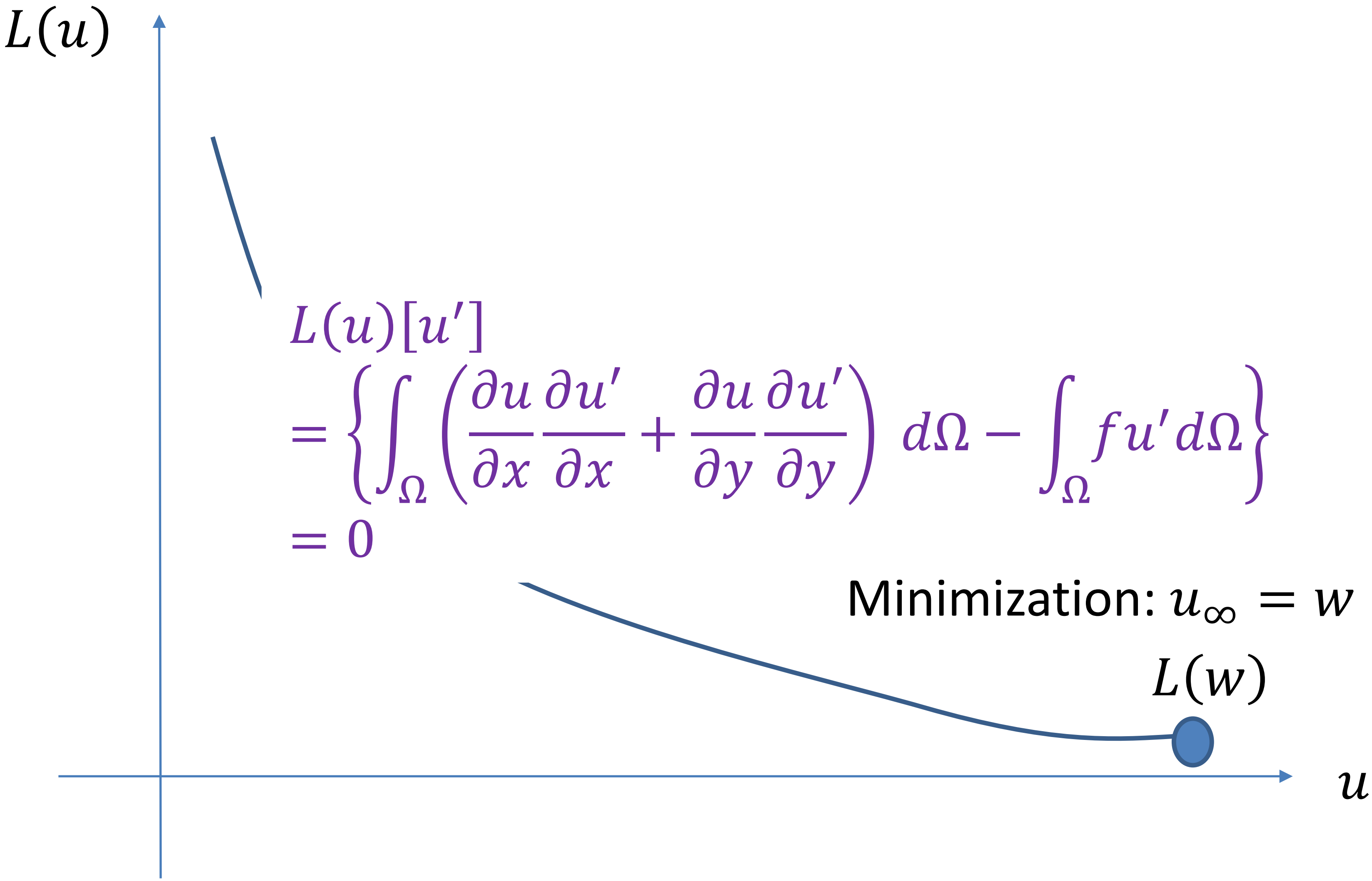


Schematic picture



$$L(u)[u'] = \left\{ \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y} \right) d\Omega - \int_\Omega f u' d\Omega \right\} = 0$$

Schematic picture



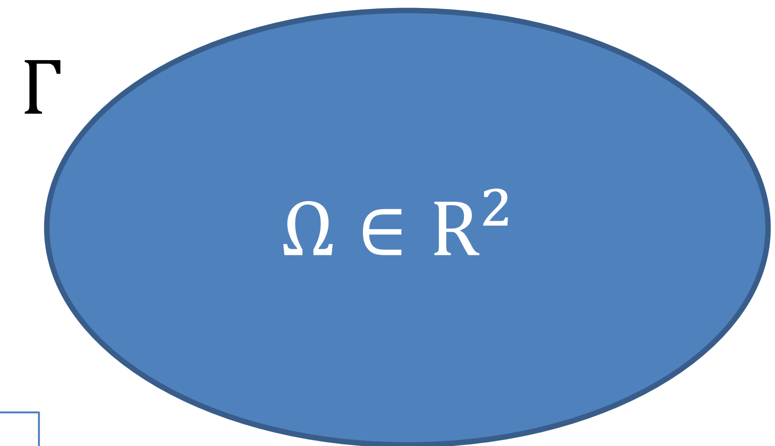
$$L(u)[u'] = \left\{ \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y} \right) d\Omega - \int_\Omega f u' d\Omega \right\} = 0$$

Minimization: $u_\infty = w$

Summarize:
numerical calculation of Poisson equation

Solving the BVP

$$-\Delta u - f = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \Gamma$$



Min. Problem of the functional $L(u)$

$$L(u) = \frac{1}{2}\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right) d\Omega - \int_{\Omega} f u d\Omega$$

Weak form (variation)

$$L(u)[u'] = \int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega - \int_{\Omega} f u' d\Omega = 0$$

Optimization Problem

- Basic description:
  - Find w: the minimum value of u

  ,where u denotes the design parameter.

Optimization Problem

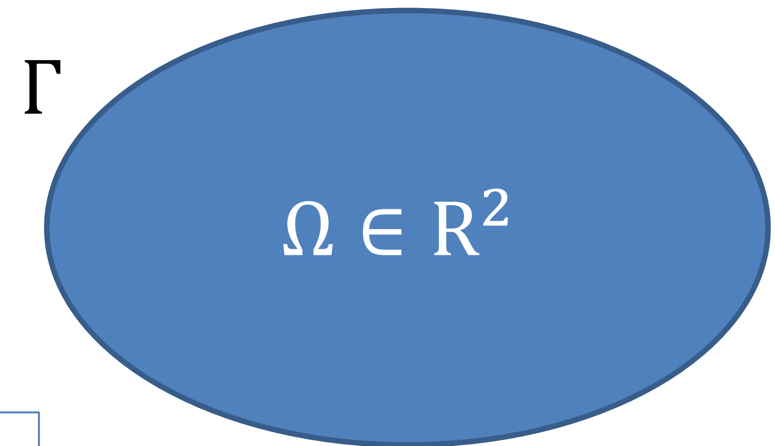
- Basic description:
  - Find w: the minimum value of u

  ,where u denotes the design parameter.
  - Minimize $F(u)$: the cost function

Reminder:
numerical calculation of Poisson equation

Solving the BVP

$$-\Delta u - f = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \Gamma$$



↓

Min. Problem of the functional $L(u)$

$$L(u) = F(u) = \frac{1}{2}\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right) d\Omega - \int_{\Omega} f u \, d\Omega$$

↓

Weak form (variation)

$$L(u)[u'] = \int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y}\right) d\Omega - \int_{\Omega} f u' \, d\Omega = 0$$

Optimization Problem

- Basic description:
  - Find w: the minimum value of u

  ,where u denotes the design parameter.
  - Minimize $F(u)$: the cost function

- The functional is written as
  - $L(u) = F(u)$

Constrained Optimization Problem

- Basic description:
  - Find w: the minimum value of u
  ,where u denotes the design parameter.
  - Minimize $F(u)$: the cost function
  - Subject to $G(u, v)$: the constraint function

- The functional is written as
  - $L(u, v) = F(u) - G(u, v)$

Constrained Optimization Problem

- Basic description:
  - Find $F(u) = \frac{1}{2}\int_\Omega u^2 \, d\Omega$
  - Subject to $G(u, v) = \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) d\Omega - \int_\Omega f v d\Omega$

- The functional is written as
  - $L(u, v) = F(u) - G(u, v)$
    $= \frac{1}{2}\int_\Omega u^2 \, d\Omega - \left\{\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) d\Omega - \int_\Omega f v d\Omega\right\}$

Constrained Optimization Problem

$$L(u-u', v-v')$$

$$= L(u,v) + \left\{ \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v'}{\partial y} \right) d\Omega - \int_\Omega f v' d\Omega \right\} + \left\{ -\int_\Omega u u' d\Omega + \int_\Omega \left(\frac{\partial u'}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u'}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega \right\}$$

$$= L(u,v) + L(u,v)[v'] + L(u,v)[u']$$

$L(u,v)[v'] = 0 \Longrightarrow \Delta u + f = 0$: the main problem

$L(u,v)[u'] = 0 \Longrightarrow \Delta v + u = 0$: the adjoint problem

Topology Optimization

- Derive strong forms for the main and the adjoint problem for the following constrained minimization problem,
  - The cost function:

$$F(\theta, u) = \int_\Omega \psi(\phi^\gamma(\theta)) \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y} \right) d\Omega$$

  - The constraint function:

$$G(\theta, u, v) = \int_\Omega \psi(\phi^\gamma(\theta)) \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega - \int_\Omega f v d\Omega$$

for real valued parameters $\alpha_1, \alpha_2, \gamma$

$$\phi(\theta) = 0.5\{\tanh(\theta) + 1\},$$

$$\psi(\phi^\gamma) = \alpha_1(1 - \phi^\gamma) + \alpha_2 \phi^\gamma$$

Variation w.r.t. u, v

$$L(\theta, u, v) = F(\theta, u) - G(\theta, u, v)$$

$$L(\theta - \theta', u - u', v - v')$$
$$= L(\theta, u, v) + L(\theta, u, v)[\theta'] + L(\theta, u, v)[u'] + L(\theta, u, v)[v']$$

The main problem:

$$L(\theta, u, v)[v'] = \int_\Omega \psi(\phi^\gamma(\theta)) \left(\frac{\partial u}{\partial x}\frac{\partial v'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v'}{\partial y} \right) d\Omega - \int_\Omega f v' d\Omega$$

The adjoint problem:

$$L(\theta, u, v)[u']$$
$$= 2\int_\Omega \psi(\phi^\gamma(\theta)) \left(\frac{\partial u}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u'}{\partial y} \right) d\Omega - \int_\Omega \psi(\phi^\gamma(\theta)) \left(\frac{\partial u'}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u'}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega$$

Variation w.r.t. θ

- Take the Gâteaux derivative w.r.t θ

$$\phi(\theta - \theta') = \phi(\theta) - \frac{\partial \phi}{\partial \theta}\theta' ..$$

$$\Rightarrow \phi^{\gamma}(\theta - \theta') = \phi^{\gamma}(\theta) - \gamma \phi^{\gamma-1}\frac{\partial \phi}{\partial \theta}\theta' ..$$

- Variation w.r.t. ψ

$$\psi(\phi^{\gamma}) = \alpha_1(1 - \phi^{\gamma}) + \alpha_2\phi^{\gamma}$$

$$\psi\big(\phi^{\gamma}(\theta - \theta')\big)$$

$$= \psi\left(\phi^{\gamma}(\theta) - \gamma\phi^{\gamma-1}\frac{\partial \phi}{\partial \theta}\theta'\right)$$

$$= \alpha_1\left(1 - \phi^{\gamma}(\theta) + \gamma\phi^{\gamma-1}\frac{\partial \phi}{\partial \theta}\theta'\right) + \alpha_2\left(\phi^{\gamma}(\theta) - \gamma\phi^{\gamma-1}\frac{\partial \phi}{\partial \theta}\theta'\right)$$

$$= \{\alpha_1(1 - \phi^{\gamma}) + \alpha_2\phi^{\gamma}\} + \alpha_1\gamma\phi^{\gamma-1}\frac{\partial \phi}{\partial \theta}\theta' - \alpha_2\gamma\phi^{\gamma-1}\frac{\partial \phi}{\partial \theta}\theta'$$

$$= \psi(\phi^{\gamma}(\theta)) - (\alpha_2 - \alpha_1)\gamma\phi^{\gamma-1}\frac{\partial \phi}{\partial \theta}\theta'$$

The sensitivity Analysis

The sensitivity

$$L(\theta,u,v)[\theta']$$
$$= -\int_{\Omega} (\alpha_2-\alpha_1)\gamma\phi^{\gamma-1}\frac{\partial\phi}{\partial\theta}\theta'\left\{\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right)-\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\right\}d\Omega$$
$$= -\int_{\Omega} (\alpha_2-\alpha_1)\gamma\phi^{\gamma-1}\frac{\partial\phi}{\partial\theta}\theta' S d\Omega$$

$$S = \left\{\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right)-\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\right\}$$

$$\frac{\partial\phi}{\partial\theta} = 0.5\mathrm{sech}(\theta) = \frac{1}{\exp\theta+\exp(-\theta)}$$

The sensitivity Analysis

- The main problem : $L(\phi, u, v)[v'] = 0 \Longrightarrow \psi \Delta u = f$
  - The main variable u is obtained.

- The adjoint problem : $L(\phi, u, v)[u'] = 0 \Longrightarrow \Delta v = 2\Delta u$
  - The adjoint variable v is obtained.

- The sensitivity is obtained substituting the main and the adjoint variables;

$$-(\alpha_2 - \alpha_1)\gamma\phi^{\gamma-1}\frac{\partial\phi}{\partial\theta}\left\{\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right) - \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\right\}$$

Topology Optimization

- Firstly, we should normalize the sensitivity

$$\int_{\Omega} \nabla\Theta \cdot \nabla y \, d\Omega = \int_{\Omega} \left\{ -(\alpha_2 - \alpha_1)\gamma\phi^{\gamma-1} \frac{\partial\phi}{\partial\theta} \right\} S \, y d\Omega$$

- Volume preserving is

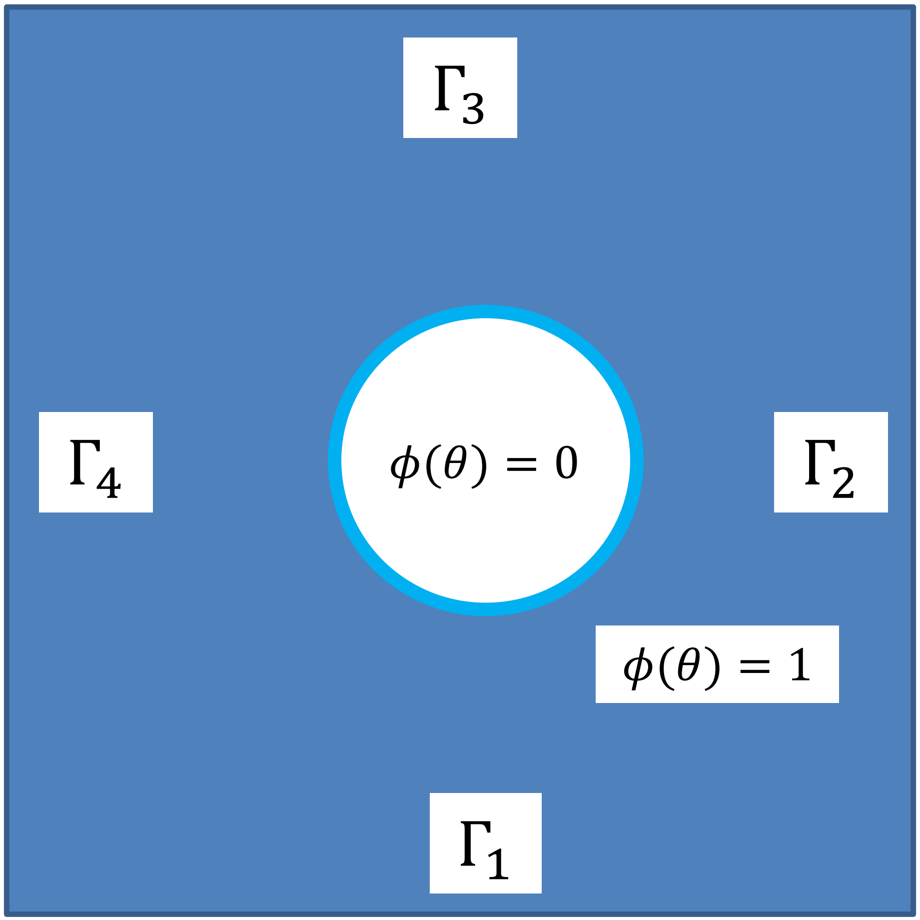
$$\int_{\Omega} \Theta \, d\Omega = A \Rightarrow \int_{\Omega} \Theta \, d\Omega - A = 0$$

$$\Rightarrow \int_{\Omega} \left(\Theta - \frac{A}{|\Omega|} \right) d\Omega = 0 \Rightarrow \Theta_n = \Theta - \frac{A}{|\Omega|}, \text{where } |\Omega| = \int_{\Omega} d\Omega$$

- We calculate the equation for the step size δ,

$$\theta^{n+1} = \theta^{n} - \delta\Theta_n$$

Laplace_topo.edp



$$-\psi\big(\phi(\theta)\big)\Delta u = 0 \text{ in } \Omega$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on} \Gamma_1 \cup \Gamma_3$$

$$u = -y(y-1) \text{ on } \Gamma_4$$

$$u = 0 \text{ on } \Gamma_2$$

$$\psi(\phi) = \alpha_1(1-\phi^\gamma) + \alpha_2\phi^\gamma$$

$$\phi(\theta) = 0.5\{\tanh(\theta) + 1\}$$

$$\theta = (x-0.5)^2 + (y-0.5)^2 - 0.2^2$$

$$\alpha_1 = 0.1,$$
$$\alpha_2 = 0.01,$$
$$\gamma = 10$$

Problem 19
use “Laplace_topo.edp”

Derive strong forms for the main and the adjoint problem for the following constrained minimization problem,

- The cost function:

$$F(u) = \int_{\Omega} \frac{1}{\mathrm{Re}} \nabla \boldsymbol{u}^{\mathrm{T}} : \nabla \boldsymbol{u}^{\mathrm{T}} d\Omega$$

- The constraint function(Stokes eq. with external force $\psi \boldsymbol{u}$):

$$G(\theta, u, v) = \int_{\Omega} \frac{1}{\mathrm{Re}} \nabla \boldsymbol{u}^{\mathrm{T}} : \nabla \boldsymbol{v}^{\mathrm{T}} d\Omega - \int_{\Omega} \psi \boldsymbol{u} \cdot \boldsymbol{v} d\Omega + \int_{\Omega} \{(\nabla \cdot \boldsymbol{u}) q + (\nabla \cdot \boldsymbol{v}) p\} d\Omega$$

for real valued parameters β, ψ_1

$\phi = 0.5\{\tanh(\beta\theta) + 1\}$ and $\psi = \psi_1 \frac{\alpha(1-\phi)}{\alpha+\phi}$

Problem 20

Derive strong forms for the main and the adjoint problem for the following constrained minimization problem,

- The cost function:

$$F(u) = \int_\Omega \frac{1}{\mathrm{Re}} \nabla \boldsymbol{u}^{\mathrm{T}} : \nabla \boldsymbol{u}^{\mathrm{T}} d\Omega$$

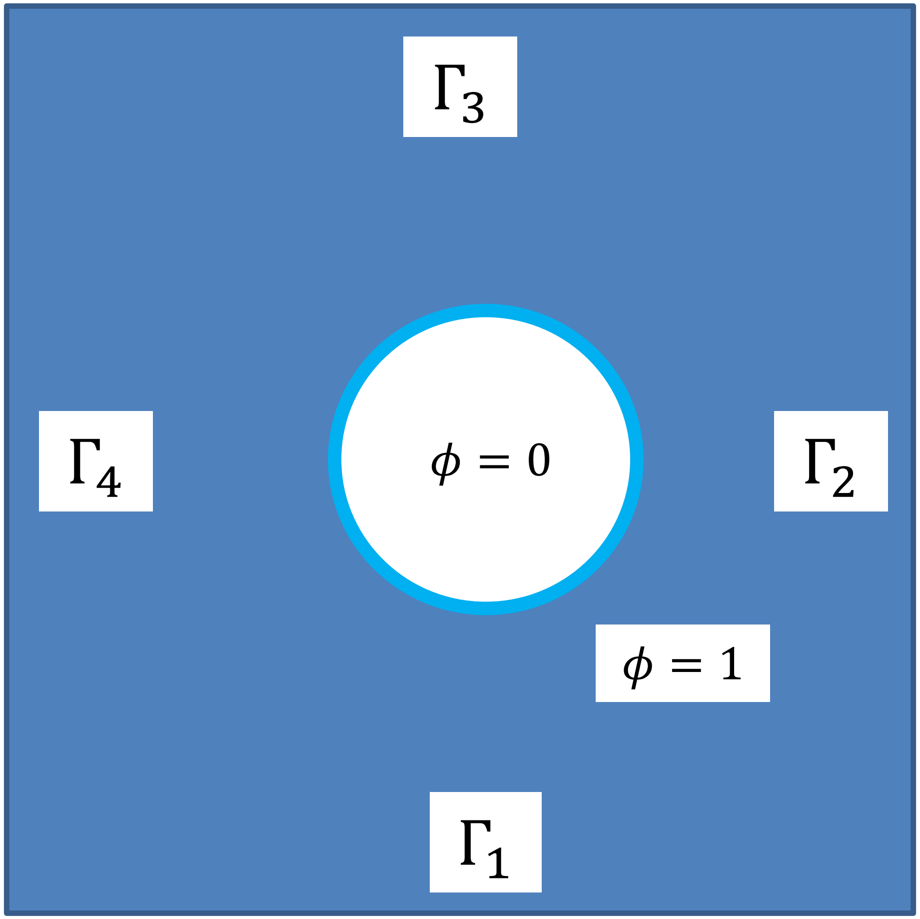
- The constraint function(Stokes eq. with external force $\psi \boldsymbol{u}$):

$$G(\theta, u, v) = \int_\Omega \frac{1}{\mathrm{Re}} \nabla \boldsymbol{u}^{\mathrm{T}} : \nabla \boldsymbol{v}^{\mathrm{T}} d\Omega - \int_\Omega \psi \boldsymbol{u} \cdot \boldsymbol{v} d\Omega + \int_\Omega \{(\nabla \cdot \boldsymbol{u}) q + (\nabla \cdot \boldsymbol{v}) p\} d\Omega$$

for real valued parameters β, ψ_1

$\phi = 0.5\{\tanh(\beta\theta) + 1\}$ and $\psi = \psi_1 \frac{\alpha(1-\phi)}{\alpha+\phi}$

Cont. Problem 20



$$u = 1, v = 0 \text{ on } \Gamma_3$$
$$\boldsymbol{u} = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_2$$

$$\psi = \psi_1 \frac{\alpha(1-\phi)}{\alpha + \phi}$$

$$\phi = 0.5\{\tanh(\beta\theta) + 1\}$$

$$\theta = (x - 0.5)^2 + (y - 0.5)^2 - 0.25^2$$

$$\psi_1 = 100, \beta = 100$$